Practical Fault-Tolerant Data Aggregation

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Abstract. During Financial Cryptography 2012 Chan et al. presented a novel privacy-protection fault-tolerant data aggregation protocol. Comparing to previous work, their scheme guaranteed provable privacy of individuals and could work even if some number of users refused to participate.
In our paper we demonstrate that despite its merits, their method provides unacceptably low accuracy of aggregated data for a wide range of assumed parameters and cannot be used in majority of real-life systems. To show this we use both precise analytic and experimental methods.
Additionally, we present a precise data aggregation protocol that provides provable level of security even facing massive failures of nodes. Moreover, the protocol requires significantly less computation (limited exploiting of heavy cryptography) than most of currently known fault tolerant aggregation protocols and offers better security guarantees that make it suitable for systems of limited resources (including sensor networks). To obtain our result we relax however the model and allow some limited communication between the nodes.

Key words: Data aggregation, differential privacy, fault tolerance

1 Introduction

Aggregation of data is a fundamental problem that has been approached from different perspectives. Recently there were many papers published, that presented methods of data aggregation that preserve privacy of individual users. More precisely, the goal of the protocol is to reveal some general aggregated statistics (like an average value of incomes) while keeping income of each individual secret, even if the aggregator is untrusted (e.g., tries to learn input of individual users). The general notion is to design a protocol that allows the aggregator to learn a perturbed sum, but no intermediate results.

In [1] Shi et al. have introduced a new approach to aggregation of information in distributed systems based on combining cryptographic techniques and typical "methods of differential privacy" , that was originally used for protecting privacy of individuals in statistical data bases after some data was revealed. The privacy is usually realized by adding some carefully prepared noise to the aggregated values. Similar approach has been independently proposed in [2].

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Those papers put a new light on the problem of privacy preserving data aggregation – the authors constructed a protocol that can be very useful, however its applicability is limited to some narrow class of scenarios due to a few shortcomings. One of them is the fact that **all** of the members of a group of users have to cooperate to compute the aggregated data. Thus, this approach is not appropriate for a dynamic, real-life systems (e.g. mobile sensor networks), even though it seems to be a perfect solution for fixed, small system of devices, where a series of data is generated periodically for a long time and the number of failures is always small (e.g. collecting measurements of electricity consumption in a neighborhood).

Another protocol, called Binary Protocol, has been presented in [3], wherein authors presented the first privacy preserving aggregation protocol that is, to some extent, fault tolerant. In our paper we focus on showing some shortcomings of the solution from [3] (by pointing out the extent to which it is fault tolerant) as well as present our approach to privacy preserving and fault tolerant data aggregation.

1.1 Our Contribution and Organization of the Paper

Below, we explicitly state the paper's contribution. Further on, in Subsect. 2 we briefly describe the model assumed in our paper and provide some notation used throughout it as well as introduce some definitions we use further on. In Sect. 3 we recall the Binary Protocol by Chan et al. presented in [3], followed by discussion of its disadvantages in Sect. 4. In Sect. 5 we present and analyze our protocol addressing some of the Binary Protocol's issues. Sect. 6 is devoted to recalling some of the previous work related to the problem addressed in the paper. Finally, in Sect. 7 we conclude and indicate some possible future work.

The contribution of our paper is twofold.

- We show that the fault tolerant protocol [3] (called Binary Protocol) offers very low level of accuracy of aggregated data even for small number of faults for any reasonable size of the network. This holds despite very good asymptotic guarantees.
- On the positive side we construct a modified protocol that offers much better accuracy and significantly lower computational requirements. We assume however a weaker security model and we allow some some limited, very local communication between users.

2 Definitions and Tools

Let us call some definitions and facts that will be used throughout this paper.

Definition 1. *(Symmetric Geometric Distribution). Let $\alpha > 1$. We denote by $Geom(\alpha)$ the symmetric geometric distribution that takes integer values such that the probability mass function at k is $\frac{\alpha-1}{\alpha+1} \cdot \alpha^{-|k|}$.*

Fact 1 *Let $\epsilon > 0$. Let u, v be integers such that $|u-v| \leq \Delta$. Let r be a random variable having distribution $Geom(\exp(\frac{\epsilon}{\Delta}))$. Then for any integer k*

$$Pr[v + k] \leq \exp(\epsilon) \Pr[u + k].$$

Definition 2. *(Diluted Geometric Distribution). Let* $\alpha > 1$ *and* $0 < \beta \leq 1$. *A random variable has* β*-diluted Geometric distribution* $Geom^{\beta}(\alpha)$ *if with probability* β *it sampled from* $Geom(\alpha)$*, and with probability* $1 - \beta$ *is set to* 0.

In the same manner as in [3], we use *computational differential privacy* as a measure of privacy protection. This notion has been introduced (in a similar form) in [4] and is in fact a computational counterpart of differential privacy from [5].

Definition 3. *(Computational Differential Privacy Against Compromise(cf. [3])) Suppose the users are compromised by some underlying randomized process* $\mathcal{C}$*, and we use* C *to denote the information obtained by the adversary from the compromised users. Let* $\varepsilon, \delta > 0$. *A (randomized) protocol* Π *preserves computational* (ε, δ)*-differential privacy (against the compromising process* $\mathcal{C}$*) if there exists a negligible function* $\eta : \mathbf{N} \rightarrow \mathbf{R}^{+}$ *such that for all* $\lambda \in \mathbf{N}$*, for all* $i \in \{1, 2, \ldots, n\}$*, for all vectors* $x, y \in \{0, 1\}^{n}$ *that differ only at position i, for all probabilistic polynomial-time Turing machines* $\mathcal{A}$*, for any output* $b \in \{0, 1\}$*,*

$$\Pr_{\mathcal{C}_i}\left[\mathcal{A}\left(\Pi\left(\lambda, x\right), C\right) = b\right] \leq e^{\varepsilon} \Pr_{\mathcal{C}_i}\left[\mathcal{A}\left(\Pi\left(\lambda, x\right), C\right) = b\right] + \delta + \eta\left(\lambda\right) \ ,$$

where the probability is taken over the randomness of $\mathcal{A}$*,* Π *and* $\mathcal{C}_i$*, which denotes the underlying compromising process conditioning on the event that user* i *is uncompromised.*

In a similar manner to regular differential privacy, we say that protocol Π preserves computational ε-differential privacy if it preserves computational $(\varepsilon, 0)$-differential privacy.

The intuition behind this definition is as follows. Every party has some bit b. From observing some processing of data, it is not feasible for any computationally bounded adversary to learn too much about b. This should hold with probability at least $1 - \delta$.

3 Protocol by Chan et al. – Description

3.1 Description of the Protocol

In the paper [3] authors propose a fault tolerant, privacy preserving data aggregation protocol which they named Binary Protocol. The purpose of the protocol is to allow some untrusted Aggregator $\mathbf{AGG}$, to learn the sum of values v_i, $1 \leq i \leq n$, where each value is stored in one of n nodes. The idea is based on earlier work [1], in particular the Block Aggregation protocol.

The general notion behind Block Aggregation is to generate a random number r_i for each of n nodes as well as an additional r_0 given to the Aggregator, such that $\sum_{i=0}^{n} r_i = 0$. Those random numbers are used as a blinding of the original values v_i. Namely, each node transmits $\mathrm{Enc}_{\mathrm{sk_i}}\left(\mathrm{v_i} + \mathrm{r_i}\right)$, using some homomorphic encryption $\mathrm{Enc}\left(\right)$. One may easily note that as long as each node transmits its value, $\mathbf{AGG}$ may add r_0, so all the blindings may cancel out. The problem that occurred with Block Aggregation is that whenever a single node fails to deliver their share (and what is really important – their

r_i), the blindings do not cancel out, hence provide a significant aggregation error.
Binary Protocol presented in [3] addresses the incompleteness of the data by arranging the nodes in a virtual binary tree. One may visualize each node as a leaf of a binary tree, with all the tree-nodes up to the root being virtual. The Aggregator is identified with an additional tree-node, which is located "above" the root and is connected only to the tree-root. In order to simulate the tree structure, the nodes and $\mathbf{AGG}$ are equipped with both, appropriate secret keys and blindings for each of the tree-layer, where layer is equivalent to the depth the tree-node is at, i.e., the first layer consists of root, second layer consists of two direct children of the root, ... and the $\lceil \log n \rceil + 1^{\text{st}}$ layer consists of the leaves. Finally, each node performs Block Aggregation protocol for each of the layers, i.e., they generate their block $\mathrm{Enc}_{\mathrm{sk_i}}\left(\mathrm{v_i} + \mathrm{r_i}\right)$ for the $\lceil \log n \rceil + 1^{\text{st}}$ layer and their shares for larger blocks of higher layers. In each of the layers, the blinding r_i is taken from a different distribution, namely β parameter for diluted geometric distribution is derived as follows: $\beta = \min\left(\frac{1}{|B|} \ln \frac{1}{\delta_0}, 1\right)$, where $|B|$ is the number of tree-nodes in the layer and $\delta_0 > 0$ is a privacy parameter. One may note that, the more tree-nodes in the layer, the blinding becomes sparser.
If all nodes present their shares the problem is reduced to the original Block Aggregation. Namely, the Aggregator may decrypt the root-layer block, obtaining the sum of all the v_is with the blinding canceled out. However, if at least one node $\mathbf{N}_i$ fails, all the blocks containing $\mathbf{N}_i$ will suffer the same issues as Block Aggregation with a missing node. Namely, large, uncanceled random disturbance. In order to provide the aggregation of the working nodes, the authors allow the Aggregator to find such a covering of the tree from the blocks of different layers that all the working nodes are covered, none of the failed nodes is included and that $\mathbf{AGG}$ is able to recover the result.
Binary Protocol provides security under computational differential privacy model and results in $\mathrm{O}\left(n \log n\right)$ communicates exchanged in the network and guarantees $\tilde{\mathrm{O}}\left((\log n)^{\frac{3}{2}}\right)$ error, but hides significant constants. Nevertheless in a practical setting, those results are less satisfying than one would expect, the issues concerning the privacy and the resulting error are raised in Sect. 4.

4 Analysis of Chan et al.'s Protocol – The Magnitude of Error

In this section we will show that the error magnitude in Binary Protocol is significant in many realistic scenarios. Note that in [3] the authors assumed that each user has data $x_i \in \{0, 1\}$, which means that the range of the sum of aggregated data is $[0, n]$. Thus, error of magnitude γn shall be regarded quite large already for moderate γ. They have also shown that the magnitude of error is $o(n)$ asymptotically. However, in practical applications we are also interested in performance of this protocol for realistic values of n. We will show that for many reasonable values of the number of users n and number of failures κ the error is large (γn for some constant γ) with significant probability.

We aim to show a precise magnitude of error in the Binary Protocol. To achieve this, we will use some subtler method than these presented by the authors of [3]. To support our analytic results we show results of simulations. Note that in [3] the authors described only simulations without failures, even though their protocol is specifically

designed to handle failed users. Due to space limitation, most of the technical proofs are moved to Appendix.

4.1 Analytical Approach

The size of error depends on the number of failed users and the way they are distributed amongst all participants. Let us fix n as the number of participants. Like the authors of [3], we assume for simplicity that n is a power of 2. Our reasoning can be however generalized for every n. We also assume that κ users have failed. We assume that these failed users are uniformly distributed amongst all participants, which seems to be reasonable in real-life scenarios. The error generated during the Binary Protocol is the sum of all noises in the aggregated blocks. In our analysis, first we show an exact formula for the expected value of the number of noises added by individual nodes. The exact formula is given in the following theorem.

Theorem 1. *Let Y be a random variable which denotes the number of noises added during the Binary Protocol also let $\kappa > 0$ and fix n as the number of participants. Then, the expected value of random variable Y is given by the following formula:*

$$EY = n - \kappa + n \cdot \sum_{i=1}^{\log_2(n)-1} \left(\frac{\binom{n-\frac{n}{2^i}}{\kappa}}{\binom{n}{\kappa}} \cdot (\beta_i - \beta_{i+1}) \right),$$

where β_i are the same as in [3].

Proof of this theorem is moved to Appendix. It consists of a non-trivial observation and some straightforward combinatorial and probabilistic techniques. Now we show a lower bound for this value for limited range of n. We present it in the following

Lemma 1. *Let $2^4 \leqslant n \leqslant 2^{21}$ and $\delta = 0.05$, then EY has a following lower bound:*

$$EY \geqslant n - \kappa - n \cdot \left(e^{-\frac{8\kappa}{n}} + \frac{\ln(\frac{\log_2(n)+1}{\delta})}{8} \cdot \left(e^{-\frac{16\kappa}{n}} - e^{-\frac{8\kappa}{n}} \right) \right).$$

Proof is based on straightforward algebraic manipulations and a simple observation and is also moved to Appendix. Note that if $n < 2^4$ then every remaining user has to add noise, so there is no need to give a lower bound in that case, because then the number of noisy inputs is exactly $n - \kappa$. Note also that even though we fixed a specific δ, it is the same δ which was broadly used in [3], and furthermore, our reasoning in proof of this lemma can easily be modified for different δ.

We can use this bound to obtain a following

Corollary 1. *For $n \leqslant 2^{10}$, $\delta = 0.05$ and $\kappa = \log_2(n)$, we have*

$$EY \geqslant 0.1n\,.$$

Also, if $\kappa = \frac{n}{2^6}$, then for $2^6 \leqslant n \leqslant 2^{12}$ we have

$$EY \geqslant 0.16n\,.$$

This comes immediately from Lemma 1 and an obvious observation that $\frac{EY}{n}$ is decreasing. After plugging the greatest value of n that is allowed by assumptions we obtain these bounds.

Having an exact formula and also a lower bound for the expected number of noises generated, we can calculate the error. Let us assume that we have m noises generated. Recall that each of them comes from symmetric geometric distribution $Geom(\alpha)$ with $\alpha > 1$, which is comprehensively described both in [1] and [3]. We denote the sum of all noises as Z. One can easily see that $EZ = 0$ due to symmetry of distribution. However, $E|Z|$ might be, and we will show that it often is, quite large.

Theorem 2. *Consider Binary Protocol with fixed α and let m denote the number of noises generated. Then let Z be a random variable which denotes the value of generated noises. We have*

$$E|Z| = \int_0^\infty \frac{4 \cdot a \cdot m \cdot \sin t \cdot (\alpha - 1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha \cos t + 1)^{m+1}} dt.$$

We present the proof of this theorem in Appendix. It is based on techniques comprehensively described in [6]. We also show a lower bound for $E|Z|$ in a following

Lemma 2. *For fixed n and ϵ, which is a privacy parameter, and provided that $m = \gamma n$, for $\gamma \in [0, 1]$ we have*

$$E|Z| \geqslant c_{n,\epsilon} \cdot \sqrt{\gamma} \cdot \frac{\log_2(n) \cdot \sqrt{n}}{\epsilon\sqrt{\pi}} - 0.1\ ,$$

where $c_{n,\epsilon}$ is a constant, which is at least 1.4 for moderate values of n and ϵ.

Proof of this lemma consists of quite intricated bounding of the integral from Theorem 2 and is presented in the Appendix.

Having all useful theorems and lemmas we can obtain a following

Corollary 2. *Consider Binary Protocol model from [3], for $\delta = 0.05$, $\epsilon = 0.5$, $n \leqslant 2^{10}$ and $\kappa = \log_2(n)$. Let $|Z|$ be the absolute value of all noises aggregated during this protocol. We have*

$$E|Z| \geqslant 0.15 \cdot n.$$

Moreover, if we take $\kappa = \frac{n}{2^6}$ and $2^6 \leqslant n \leqslant 2^{12}$ we have

$$E|Z| \geqslant 0.12 \cdot n.$$

This is an immediate result from Lemma. 2, we can see that $\frac{E|Z|}{n}$ is a decreasing function, so it is enough to plug $n = 2^{10}$ into lower bound for $E|Z|$ for the first part of the corollary and $n = 2^{12}$ for the second part of the corollary.

This clearly shows that even if we consider the lower bound for the number of noises and their magnitude, the Binary Protocol is far from perfect for many realistic scenarios. Even worse conclusions will be drawn in Subsect. 4.2, where we use the exact formulas given in Theorems 1 and 2 to numerically analyze the errors generated in this protocol.

4.2 Experimental Approach

In Subsect. 4.1 we gave both exact formulas and lower bounds for the number of noises generated and their sum. Note that the lower bounds are not very tight for many n. In this subsection we will show that the errors generated are even larger in some cases. We will use the exact formulas to precisely calculate the errors numerically. First let us consider the case where $n \leqslant 2^{10}$, $\kappa = \lfloor \log_2(n) \rfloor$, and privacy parameters are $\epsilon = 0.5$, $\delta = 0.05$. See Fig. 1. It clearly shows that the error magnitude in Binary Protocol is, in fact, significantly greater than the lower bound we gave in Cor. 2. Now let $2^6 \leqslant n \leqslant 2^{12}$, $\kappa = \frac{n}{2^6}$ and privacy parameters stays the same. See Fig. 2. Again we can see that the error magnitude is unacceptably high, greater than $0.2n$. Note that the noise is independent from the data, so such error could be very problematic, especially if the sum of the real data is small. The noise could be greater than the data itself.

We can also check how great the errors will be for constant value of $\kappa = 5$. See Fig. 3.

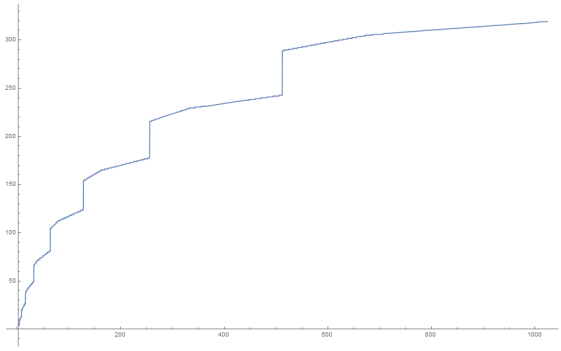

Fig. 1: Error magnitude in Binary Protocol with $\epsilon = 0.5$, $\delta = 0.05$ and $\kappa = \lfloor \log_2(n) \rfloor$.

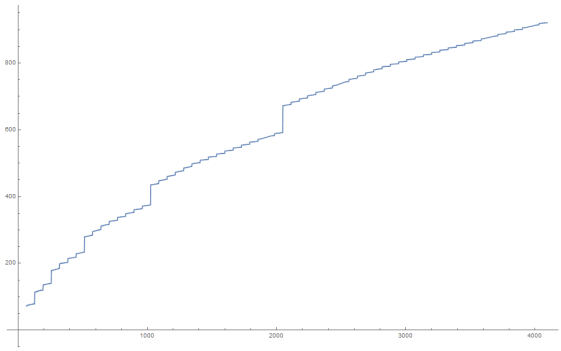

Fig. 2: Error magnitude in Binary Protocol with $\epsilon = 0.5$, $\delta = 0.05$ and $\kappa = \left\lfloor \frac{n}{2^6} \right\rfloor$.

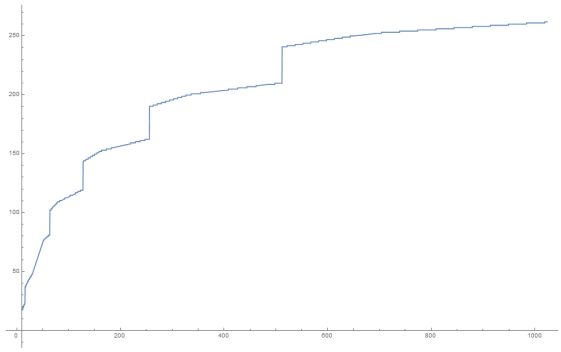

Fig. 3: Error magnitude in Binary Protocol with $\epsilon = 0.5$, $\delta = 0.05$ and $\kappa = 5$.

4.3 Some Other Shortcomings

Note that in [3], but also in numerous other papers concerning data aggregation with untrusted aggregator, we have a constant privacy parameter δ (i.e. $\delta = 0.05$). This significantly decreases the amount of noises generated, but is fundamentally incorrect in terms of classic differential privacy standards. Such approach is in some sense equivalent to randomly choosing δ fraction of the database and revealing their data to everyone. In fact, the magnitude of δ should be $O(\frac{1}{n})$, where n is the number of users. This is necessary, to ensure that the probability of leakage is negligible. More information about this can be found in [7].

5 Precise Aggregation Algorithm with Local Communication

In this part we present an alternative protocol PAALEC (Precise Aggregation Algorithm with Local Communication) that in some scenarios offers much better accuracy of aggregated data when failures occur, while preserving high level of users' privacy protection. In fact our protocol works in a substantially different way and for slightly modified model. Thus, despite its performance and accuracy that outperforms the original protocol of Chan at al., they are not fully comparable.

First of all, we assume that users may communicate (also in order to bypass the lower bound pointed out in [8]). Let us stress that the communication is limited to some small circle of "neighbors". The idea behind the presented construction is to take advantage of some natural structures emerging in distributed systems (e.g. social networks) wherein, apart from logical connections between each user and a server/aggregator there are also some direct links between users. Clearly, such model is not adequate for some real-life problems discussed in [3], for example in sensor fields with unidirectional communication. Thus there are applications where the original protocol from [3] is the only one possible.

5.1 Modified Model

We assume that the network consists of n users - $V = \{v_1, v_2, \ldots, v_n\}$ as well as the aggregator $\mathbf{AGG}$ and a set of $k < n$ *local aggregators* $\mathbf{Agg}_1, \ldots, \mathbf{Agg}_k$. Please note that the local aggregators may be separate entities but without any significant changes they may be selected from the set of regular users V. The only issue with this approach is that we have to ensure that the local aggregator is either selected during the aggregation round or it cannot fail during **a single** execution of aggregation process. We assume that each user is assigned to **exactly one** local aggregator. We denote the set of nodes assigned to the local aggregator $\mathbf{Agg}_i$ by V_i. An example of the network's topology is depicted in Fig. 4.

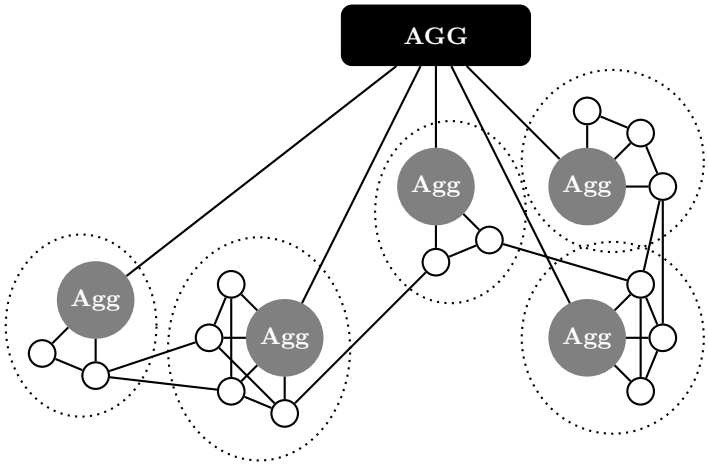

Fig. 4: Example of a clusterized network with global aggregator ($\mathbf{AGG}$) and local aggregators ($\mathbf{Agg}$) marked.

We can derive a graph $G = (V,\ E)$ from the network structure, where V are all the nodes and the set of edges is created based on the ability to establish communication (e.g., transmission range in a sensor network, friendship relation in a social network). Namely, the edge $\{v,\ v'\} \in E$ if and only if v and v' are *neighbors* and can communicate via a private channel. In our protocol we assume that each node can perform some basic cryptographic operations and has access to a source of randomness. By $N(v)$ we denote a set of such vertices v' of G that the edge $\{v,\ v'\} \in E$. Security of the protocol described in section depends on the structure of graph G, and how many parties the adversary can corrupt. Discussion on security of the protocol can be found in 5.4.

5.2 Building Blocks

Similarly to previous papers, for obtaining high level of data privacy we combine cryptographic techniques with data perturbation methods typical for research concentrated on differential privacy of databases.

The first technique we use in our protocol is a homomorphic encryption scheme based on original ElGamal construction enriched by some extra techniques introduced in [9]. More precisely, encrypted messages can be "aggregated" and re-encrypted. Moreover one can "add" an extra encryption layer to a given ciphertext, in such way that the message can be decrypted only using both respective keys. Clearly this operation preserves the homomorphic property.

Let $\mathbf{G}$ be a group such that the Diffie-Hellman problem is hard. Let g be a generator of $\mathbf{G}$. Let $\mathrm{sk}, \mathrm{sk}'$ be a some private keys and $g^{\mathrm{sk}}, g^{\mathrm{sk}'}$ are respective public keys.

Encryption of '1' A pair $\mathrm{Enc}_{\mathrm{sk}}(1) = (g^r, g^{r\cdot \mathrm{sk}})$ for a random $r \in \mathbf{G}$ is an encryption of 1.

Re-encryption Ciphertext representing 1 can be re-encrypted. Namely, one can get another ciphertext representing one, **without private key**. Namely having $\mathrm{Enc}_{\mathrm{sk}}(1) = (g^r, g^{r\cdot \mathrm{sk}})$ one can choose r' and compute $Re(\mathrm{Enc}_{\mathrm{sk}}(1)) = g^{r\cdot r'}, g^{r\cdot r'\cdot \mathrm{sk}})$ that represents 1 as well.

Adding layer of encryption Having a ciphertext $\mathrm{Enc}_{\mathrm{sk}}(1) = (g^r, g^{r\cdot \mathrm{sk}})$ a party having private key sk' can "add encryption layer" to a ciphertext obtaining

$$\mathrm{Enc}_{\mathrm{sk}+\mathrm{sk}'}(1) = ((g^r)^{r'}, (g^{r\cdot \mathrm{sk}})^{r'} \cdot (g^r)^{r'\mathrm{sk}'}) = (g^{r\cdot r'}, g^{r\cdot r'\cdot(\mathrm{sk}+\mathrm{sk}')}).$$

Filling the ciphertext Having $\mathrm{Enc}_{\mathrm{sk}}(1) = (g^r, g^{r\cdot \mathrm{sk}})$ one can compute

$$\mathrm{Enc}_{\mathrm{sk}+\mathrm{sk}'}(\mathrm{C}) = (g^r, g^{r\cdot \mathrm{sk}} \cdot C).$$

Partial decryption Having $\mathrm{Enc}_{\mathrm{sk}}(\mathrm{C}) = (g^{r\cdot r'}, g^{r\cdot r'\cdot(\mathrm{sk}+\mathrm{sk}')}C)$ and a private key sk' one can "remove one layer of encryption" -i.e., obtain

$$\mathrm{Enc}_{\mathrm{sk}}(\mathrm{C}) = (g^{r\cdot r'}, \frac{g^{r\cdot r'\cdot(\mathrm{sk}+\mathrm{sk}')}C}{(g^{r\cdot r'})^{\mathrm{sk}'}}) = (g^{r\cdot r'}, g^{r\cdot r'\cdot \mathrm{sk}}C).$$

For the sake of clarity we skip some technical details (i.e., choice of the group size, generators etc.) as well as full security discussion of this encryption scheme. Note that these are quite standard techniques used in many papers including [9,10].

Similarly to previous papers (including [3,1]) we utilize the following method: if we know that each user $v \in V$ has a value from an interval of moderate size $\xi_v \in [0, \Delta]$ then the sum of values of all ξ_v's cannot exceed $n\Delta$. Thus one can find a discreet logarithm for $g^{\sum_{v\in V}\xi_v}$ even if finding a discreet logarithm of g^r is not feasible if r is a random element of $\mathbf{G}$. Using Pollard's Rho method this can be completed in average time $O(\sqrt{n\Delta})$.

5.3 Protocol Description

During the protocol, we assume that the aggregator $\mathbf{AGG}$ has a private key sk, moreover each of the local aggregators $\mathbf{Agg}_i$ has its own private key sk_i. We also assume that there is a public parameter g, that is a generator of some finite group $\mathbf{G}$, in which Diffie-Hellman problem is hard. By $\mathrm{Enc}_{\mathrm{sk}}(\mathrm{c})$ we denote the encryption structure introduced in Section 5.2.

Let us assume that each user v has a private value ξ_v from the range $(0, \Delta)$. The final aim is to provide $\mathbf{AGG}$ the sum $\sum_{v \in V} \xi_v$ perturbed in such way that the privacy (expressed in terms of differential privacy) of all $v \in V$ is preserved. Clearly, the privacy can be endangered both by reveling the output as well as by collecting information about the aggregation process.

Setup

- $\mathbf{AGG}$ broadcasts to the local aggregators $\mathrm{Enc}_{\mathrm{sk}}(1)$.
- Each of the local aggregators $\mathbf{Agg}_i$ constructs $\mathrm{Enc}_{\mathrm{sk}+\mathrm{sk}'}(1)$ and publishes it for all users from V_i.

The setup phase is performed only once during network's lifetime. Moreover if needed, each $\mathbf{Agg}_i$ may provide a non-interactive proof that the operations were performed correctly and honestly [11,12].

Aggregation

Algorithm for node v

- For each node $v' \in N(v)$ generate a random value $x_{v'}^{v} \in \mathbf{G}$.
- Using a private channel send each value $x_{v'}^{v}$ to the appropriate neighbor v'.
- Having received all $x_{v}^{v'}$ from each of the neighbors, select random r_v from $Geom^{\beta}(\alpha)$ and calculate

$$c_v = \sum_{v' \in N(v)} x_{v}^{v'} - \sum_{v' \in N(v)} x_{v'}^{v} + r_v + \xi_v.$$

- Compute $Re(\mathrm{Enc}_{\mathrm{sk}+\mathrm{sk_i}}(\mathrm{g}^{\mathrm{c_v}}))$ and send it to $\mathbf{Agg}_i$.

An example of node's communication is shown in Fig. 5.

Algorithm for local aggregator $\mathbf{Agg}_i$

- Having received $\mathrm{Enc}_{\mathrm{sk}+\mathrm{sk_i}}(\mathrm{g}^{\mathrm{c_v}})$ from all nodes from V_i, compute

$$\mathrm{Enc}_{\mathrm{sk}}(\mathrm{g}^{\mathrm{c_v}}) = \left(g^{r_i}, \frac{g^{r_i(\mathrm{sk}+\mathrm{sk}_i)+c_v}}{g^{r_i \cdot \mathrm{sk}_i}} \right).$$

  This operations result in obtaining

$$\mathrm{Enc}_{\mathrm{sk}}(\mathrm{g}^{\mathrm{c_{v_1}}}) = (g^{r_{v_1}}, g^{r_{v_1} \cdot \mathrm{sk}+c_{v_1}}), \ldots, \mathrm{Enc}_{\mathrm{sk}}(\mathrm{g}^{\mathrm{c_{v_l}}}) = (g^{r_{v_l}}, g^{r_{v_l} \cdot \mathrm{sk}+c_{v_l}})$$

  for $l = |V_i|$.
- Compute

$$\mathrm{Enc}_{\mathrm{sk}}\left(\mathrm{g}^{\mathrm{c_{v_1}}+\cdots+\mathrm{c_{v_l}}}\right) = \left(\prod_{i=1}^{l} g^{r_i}, \prod_{i=1}^{l} g^{r_i \mathrm{sk}+c_{v_i}} \right) = \left(g^{\sum_{i=1}^{l} r_i}, g^{(\sum_{i=1}^{l} r_i)\mathrm{sk}+\sum_{i=1}^{l} c_{v_i}} \right).$$

- Send the value $\mathrm{Enc}_{\mathrm{sk}}\left(\mathrm{g}^{\mathrm{c_{v_1}}+\cdots+\mathrm{c_{v_l}}}\right)$ to the aggregator $\mathbf{AGG}$.

Final aggregation

- Having received the aggregated values from each V_i, for each of those values $\mathbf{AGG}$ calculate $y_i = g^{\sum_{v \in V_i} c_v}$, using its private key sk for each $i = 1, \ldots, k$. Then compute

$$y = \prod_{i}^{k} y_i = \prod_{i} g^{\sum_{v \in V_i} c_{v_i}} = g^{\sum_{v \in V} c_{v_i}}.$$

- Then $\mathbf{AGG}$ compute discrete logarithm of y as a final (perturbed) value being a sum of all $\sum_{v \in V} \xi_v$.

Note that the protocol depends on two security parameters β and α. They strongly depend on the topology of the underlying graph. We discuss this issue in the next subsection.

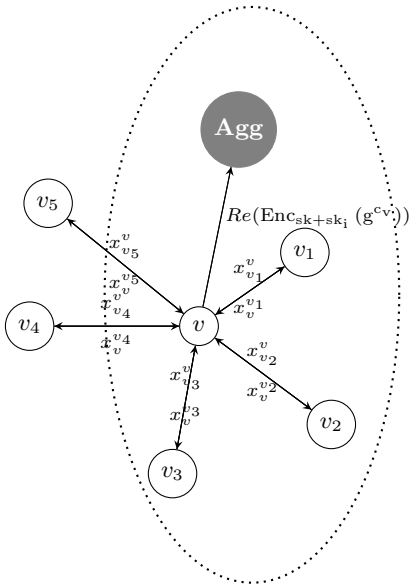

Fig. 5: An example of communication in a single aggregation round from a perspective of node v. The dotted line marks the set of nodes assigned to a single local aggregator $\mathbf{Agg}$.

5.4 Comparison and Analysis

In this section we outline the analysis of the presented aggregation protocol with respect to correctness, level of privacy provided and error of the result obtained by the

aggregator. The analysis is slightly more complicated since the parameters of the protocol strongly depend on the underlying network. We argue however that they offer very good properties for wide classes of networks.

Correctness First, let us look at the result obtained by the aggregator $\mathbf{AGG}$ in the last step of the protocol. This is a discrete logarithm of $g^{\sum_{v\in V} c_{v_i}}$. Let us observe that

$$\begin{aligned}\sum_{v\in V} c_v &= \sum_{v\in V}\left(\sum_{v'\in N(v)} x_v^{v'} - \sum_{v'\in N(v)} x_{v'}^{v} + r_v + \xi_v\right)\\ &= \sum_{v\in V}\sum_{v'\in N(v)} x_v^{v'} - \sum_{v\in V}\sum_{v'\in N(v)} x_{v'}^{v} + \sum_{v\in V}\xi_v + \sum_{v\in V} r_v\\ &= \sum_{v\in V}\xi_v + \sum_{v\in V} r_v.\end{aligned}$$

The value $\sum_{v\in V}\xi_v$ is the exact sum of values kept by nodes and sum of all the noises $\sum_{v\in V} r_v$. This leads to two conclusions. First, the result is correct. Second, retrieving the data using Pollard's Rho method (or even brute force method) is feasible since the absolute value of the first sum has to be smaller than $n\Delta$. Similarly, the error cannot be too big. We discuss the choice of parameters below.

Privacy protection We assume that the encryption scheme $\mathrm{Enc}_{\mathrm{sk}}()$ is *semantically secure*. In particular after re-encryption operation one cannot retrieve any information about the plaintext without the private key sk possibly except some negligible probability $\eta(\lambda)$ with respect to the key-length λ or some other security parameters. In particular, in our protocol, the local aggregator $\mathbf{AGG}_i$ cannot learn the contributions sent to $\mathbf{AGG}_j$ for $i \neq j$ without access to keys sk_j and sk.

For the simplicity of analysis of the privacy protection let us consider the simplest case when $k = 1$, i.e. there is only one aggregator. In such case we may assume $\mathbf{AGG}_1 = \mathbf{AGG}$. Let $V^H \subset V$ be the set of uncompromised users. Note that all neighboring users exchange a purely random values $x_v^{v'}$'s that finally cancel-out, however as long as they remain unknown to the adversary, they perfectly obfuscate the results sent to the aggregator (exactly in the same manner as the one-time pad cipher). This can be easily adopted to our protocol to get the following fact.

Fact 2 *Let us assume that the adversary can control* $\mathbf{AGG}$ *and a subset of users* $V^H \setminus V$. *Let* $\mathcal{S}$ *be a connected component of the subgraph of* $\mathcal{G} = (V, E)$ *induced by the subset* V^H. *Then, the adversary can learn nothing but* $\sum_{v\in\mathcal{S}}(\xi_v + r_v)$ *about the values* ξ_v*'s from the execution of PAALEC for any* $v \in V^H$.

Theorem 3. *Let us assume that PAALEC with parameter* $\alpha = \exp(\frac{\epsilon}{\Delta})$ *is executed in the network represented by a graph* $\mathcal{G} = (V, E)$ *and* $\mathcal{G}'$ *is a subgraph of* $\mathcal{G}$ *induced by the set of uncompromised users* V^H. *Moreover we assume that each user* v *contributes a value* $\xi_v \in [0, \Delta]$.

If in each connected component $\mathcal{S}$ of $\mathcal{G}'$ there is a user s, such that his added noise r is taken from $Geom(\exp(\frac{\epsilon}{\Delta}))$, then PAALEC preserves computational $(\varepsilon, 0)$-differential privacy.

Proof. Let $\Xi = \sum_{s \in S} \xi_s$ and let Ξ' be the same sum with changed a single value ξ_s. By the assumption about the range of the aggregated values we get $|\Xi' - \Xi| \leq \Delta$. Let r be a random variable taken from the symmetric geometric distribution $Geom(\exp(\frac{\epsilon}{\Delta}))$. From Fact 1 we know that $Pr[\Xi + r]$ may differ from $Pr[\Xi' + r]$ by at most a multiplicative factor $\exp(\epsilon)$. However, from Fact 2 we know that the adversary may learn nothing more than the sum of all values from the component $\mathcal{S}$. To complete the proof it is enough to recall that we assumed that probability of gaining some other knowledge e.g. by choosing weak parameters of the cipher is at most negligible function $\eta(\lambda)$.

From this theorem follows next corollary.

Corollary 3. *If PAALEC is executed on a graph such that a subgraph induced by the set of uncompromised users V^H is connected and with probability at least $1 - \delta$ at least one uncompromised users adds his value r from $Geom(\exp(\frac{\epsilon}{\Delta}))$ then PAALEC computationally preserves (ε, δ)-differential privacy.*

Translating into real terms Theorem3 with Corollary3 says if the connections between honest users are enough dense and we can somehow guarantee that at least one honest node adds the noise, the system is secure. The core of the problem is judge if a real-world networks a dense enough and what parameters of adding noise are sufficient. This problem is discussed in the next paragraph.

Accuracy

Theorem 4. *Let us consider a random network with n nodes. Each of possible $\binom{n}{2}$ connections (edges) is independently added to the network with probability $p \geq \frac{8 \log n}{n}$. Let $\mathcal{S}$ be a subgraph induced by a subset of at least $m \geq n/2$ randomly chosen nodes. Then $\mathcal{S}$ is connected with probability at least $1 - 1/n$.*

Note that the presented model boils down to the classic Erdos-Renyi model [**?**]. For the sake of completeness and to get explicit constants we present the proof in the appendix.

From Theorem 4 we learn that a all "typical" network of n nodes with random connections such that the average number of neighbors is $8 \log n = \Theta(\log n$ is dense enough even if the adversary is able to compromise as much as $n/2$ nodes.

If we have guaranteed at least $n/2$ honest (uncompromised and working) nodes one may note that the probability that none of them adds the noise is at least $(1 - \beta)^{n/2}$. To have $(1 - \beta)^{n/2} < \delta$ one needs to have β such that $\log(1 - \beta) < \frac{2 \log \delta}{n}$. Since $\log(1 + x) \leq x$ for $x > -1$ it is enough to use $\beta > -\frac{2 \log \delta}{n}$. It is easy to see that in this case we have a constant expected error.

Remarks and Extensions We proved that the proposed protocol guarantees a very good accuracy even facing a massive failures and compromising of nodes Half of nodes may failed or cooperate with the adversary (In fact this result can be generalized to any

constant fraction of users) . The analysis and the model can be relaxed/extended in many directions. One can instantly observe that the analysis can be extended for smaller δ for the price of moderate increasing of the expected noise. Note that the value of δ set to a celebrated magic constant 0.05 seems to be to big for practice. Indeed, this implies that one of 20 individuals may loose their privacy.

We believe that this approach can be useful for other graphs-including graphs representing social networks. Note that if a graph guarantees a some privacy then more dense graph (with some added edges) offers at least the same level of privacy. Thus it is enough if each users adds something like $\Theta(\log n)$ "randomly" chosen neighbors to protect the privacy.

Note that our protocol is not immune against an adversarial nodes that sends incoherent random data. To the best of our knowledge all protocols (including [3,1]) of this type are prone to so called *contaminating* attacks. To mitigate this problem one may apply orthogonal methods presented in [13].

6 Previous and Related Work

Data aggregation in distributed networks has been thoroughly studied due to practical importance of such protocols. Measuring and aggregating data and rising alarm are arguably three most important functionalities of distributed sensing networks, and with the increased number of personal mobile devices, the aggregation becomes of greatest interest among the three. Exemplary protocols that do not address security nor privacy may be found in [14,15], with the latter being often presented as a model aggregation algorithm.

There are several settings considering data aggregation. They differ in both, the abilities and constraints of the nodes performing the aggregation, as well as the issues that the algorithm addresses. Some of the adversities that may be addressed include data confidentiality (i.e., protecting the data from disclosure), privacy of the nodes (inability to learn exact values of each node), node failure and spontaneous node joining the network as well as data poisoning or contamination (i.e., injecting malicious data by the adversary that allows them to either significantly influence the outcome of the algorithm either learning more information about the execution that they would not gain when following the protocol honestly).

Our paper follows the model considered in [3], where the nodes have constrained abilities and their energy pool is limited. Authors present a privacy preserving aggregation protocol that assumes malicious aggregator, moreover they claim tolerance for node failure and join, hence addressing a majority of the issues. Similar problems that focus on narrower range of properties have been also studied in [1,2]. In [16,17] authors present some aggregation protocols that preserve privacy, however they do not consider dynamic changes inside of the network. The latter also considers data poisoning attacks, however the authors do not provide strict proofs. A different approach was presented in [18,19], where the authors present a framework for some aggregation functions and consider the confidentiality of the result, however leaving nodes' privacy out of scope of their papers. On the other hand, there is bulk of research that focuses on fault tolerance that leaves privacy and security issues either out of scope or just mentioned, not

keeping it as a priority. Examples of such work may be found in [20], where local aggregation and some additional amendment strategies are used in order to minimize the aggregation error. In [21] authors propose a modular algorithm that uses time divided in rounds in order to schedule fault-tolerant aggregation algorithm. Clusterization and a hierarchical approach has been used also in [22], where aside from fault tolerance and aggregation accuracy, battery preservation (i.e., energetic efficiency) is considered. In [8] the authors present an asymptotic lower bound on the error of the aggregation that preserves privacy, showing that in order to reduce the errors, one has to resign from perfect privacy and focus rather on computational variant of the privacy preservation.
An example of work on secure data aggregation in stronger models may be found in [23], where the authors consider data aggregation in a smart grid. Their protocol incorporates Paillier cryptosystem and using a specific function composition provides a Zero Knowledge Proof, that allows aggregating data without revealing the private information. Another approach to secure data aggregation in smart grids was presented in [24], where authors use Shamir's secret sharing technique. Another fruitful branch of the research on data aggregation considers data aggregation in vehicular ad hoc networks (VANET). The research in this field is motivated by the increasing number of „smart-cars" with internal computational unit that can communicate with both base station and directly with other units in nearby vehicles. One of the first works addressing this issue was [25,26,27]. A practical scenario for data aggregation in VANET has been presented in [28], where authors present a cluster approach to data aggregation in [29]. The security issue in VANET data-aggregation has been mentioned in [30,31]. A survey of the known protocols has been performed in [32]. One may note that usually the aggregation algorithms in VANET incorporate computation-heavy elements, e.g. fuzzy logic in [33] or computation heavy PKI in [34]. An example of generic modeling approach has been shown in [35].
One may note that retrieving encrypted or blinded data by one entity, that requires cooperation of others is similar to cryptographic secret-sharing. Some of the most important work on secret sharing may be found in [36,37], however in our paper we draw from the Universal Re-encryption method presented in [9].

7 Conclusions

In our paper we provided a precise analysis of accuracy of the data aggregation protocol presented by Chan et al. in [3]. We have shown that in many cases its accuracy may not be sufficient even if the number of faults is moderate. We constructed another fault tolerant, privacy preserving aggregation protocol that offers much better precision. In order to obtain this, we allowed a moderate communication between nodes, providing some modifications to the original model.

We believe that our approach and security model is justified in many real-life scenarios, however much research is left to be done in the field. First of all, our protocol as well as all other similar protocols we are aware of, is not immune against so called data poisoning attack. Indeed, a single malicious node can completely change the result of the aggregated data without being detected.

Another problem is finding more general statistics. Authors of aggregating schemes usually focus on simple summing of the values of all nodes in the network or limit the scope of their work to sum, product and average. In many cases we need however other statistics like minimum, maximum or the median. We suppose that finding more general statistics with guaranteed privacy of individuals is possible using methods explored in e-voting protocols. They however are very demanding in terms of required resources. From the theoretical point of view the important question is about the possible trade-offs between privacy protection, volume of communication and possible accuracy of the results of aggregation.

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Appendix

A Proof of Theorem 1.

Proof. Consider Binary Protocol model from [3]. We aim to give a precise formula for the expected value of the number of noises added in this protocol. For simplicity we assume that n is the power of 2. We assume that κ leaves have failed, and they are uniformly chosen from all n leaves. Let us begin with stating and proving the following

Lemma 3. *Consider Binary Protocol with fixed κ and n. We call a node an aggregating node, if it is used by the aggregator to obtain a sum of data from some subset of users. Let X_i^* be a random variable denoting the number of aggregating nodes on ith level of the tree. Then we have the following formula for $i \geqslant 1$*

$$EX_i^* = EX_i - 2EX_{i-1} = 2^i \cdot \left(\frac{\binom{n-\frac{n}{2^i}}{\kappa}}{\binom{n}{\kappa}} - \frac{\binom{n-\frac{n}{2^{i-1}}}{\kappa}}{\binom{n}{\kappa}} \right),$$

where X_i is a random variable, which denotes the number (on ith level) of segments corresponding to subsets of users with no failures.

Proof. First of all, we will call a segment in the Binary Protocol tree 'clean' if and only if there are no fails in this segment. Each node in the tree corresponds to a specific segment, according to Binary Protocol rules. See that on a certain tree level, all nodes correspond to segments of the same size, noted here by $|B_i|$. Throughout this reasoning we will call the 'root level' 0, children of the root are on level 1 and so on, up to level $log_2(n)$ which is the 'leaves level'.

Each user is aggregated in exactly one node, which belongs to some ith level and corresponds to a specific segment. This user generates a geometric noise with probability β_i, where:

$$\beta_i = \min\left(\frac{1}{|B_i|} \ln\left(\frac{log_2(n)+1}{\delta} \right), 1 \right).$$

We want to know an expected value of the number of noises generated throughout the whole protocol.

To do this, first we denote the number of 'clean' segments of size $|B_i|$ (corresponding to nodes on ith level of the tree) by a random variable X_i. See that $X_i \in \{0, 1, \ldots, 2^i\}$. Furthermore, we see that:

$$X_i = \sum_{j=1}^{2^i} X_{i,j},$$

where

$$X_{i,j} = \begin{cases} 1, & \text{if segment j on level i has no fails,} \\ 0, & \text{otherwise.} \end{cases}$$

This, and the fact that $EX_{i,j} = EX_{i,k}$ for every $j, k \in 0, \ldots, 2^i$, allows us to use linearity of expectation to calculate EX_i:

$$EX_i = E\sum_{j=1}^{2^i} X_{i,j} = \sum_{j=1}^{2^i} EX_{i,j} = 2^i EX_{i,1} = 2^i \cdot P(X_{i,1} = 1). \tag{1}$$

Now see that

$$P(X_{i,1} = 1) = \frac{\binom{n-|B_i|}{\kappa}}{\binom{n}{\kappa}},$$

and also $|B_i| = \frac{n}{2^i}$, thus plugging these to (1) we get

$$EX_i = 2^i \cdot \frac{\binom{n-\frac{n}{2^i}}{\kappa}}{\binom{n}{\kappa}}, \tag{2}$$

which can be easily calculated.

Now let us consider the number of segments which really aggregate the data. See that if a node is an aggregating node, that means that it corresponds to a clean segment, but its parent does not correspond to a clean segment. We denote the number of aggregating nodes on ith level by X_i^*, we can also see that:

$$X_i^* = X_i - 2X_{i-1},$$

where $i \in \{1, \ldots, \log_2(n)\}$. There are X_i clean nodes on ith level but we have to subtract all the clean nodes from higher level of the tree multiplicated by 2, because each of these clean nodes on a higher level is parent to two nodes on ith level, which are therefore not an aggregating nodes, because their parent is clean. That gives us

$$EX_i^* = EX_i - 2EX_{i-1} = 2^i \cdot \left(\frac{\binom{n-\frac{n}{2^i}}{\kappa}}{\binom{n}{\kappa}} - \frac{\binom{n-\frac{n}{2^{i-1}}}{\kappa}}{\binom{n}{\kappa}} \right),$$

which completes the proof of this lemma. □

Lemma 3 gives us an explicit formula for EX_i^*. Now, when we have a formula for the expected value of the number of aggregating nodes on each level, we can proceed to calculating the expected value of the number of geometric noises generated during the Binary Protocol.

Let Y_i be a random variable which denotes the number of noises generated on ith level of the tree. On ith level we aggregate X_i^* segments, each of these segments have $2^{\log_2(n)-i}$ users and each of these users generates geometric noise with probability β_i. Therefore we have $Y_i \sim Bin\left(2^{\log_2(n)-i} \cdot X_i^*, \beta_i\right)$, where $Bin(n, p)$ denotes binomial distribution. After observing this, we can see that

$$EY_i = EX_i^* \cdot 2^{\log_2(n)-i} \cdot \beta_i.$$

Every user is aggregated only on one level, so if we take a sum over all levels of the tree, we will get all the noises generated during the Binary Protocol. Let Y be a random variable that denotes the number of noises generated. We have

$$Y = \sum_{i=0}^{\log_2(n)} Y_i,$$

and we can also safely assume that if $\kappa > 0$, then $Y_0 = 0$, because if at least one user has failed, then we cannot possibly aggregate all users in the root of the tree. Furthermore, using linearity of expectation and well known expected value for Binomial distribution we have

$$EY = \sum_{i=1}^{\log_2(n)} EX_i^* \cdot 2^{\log_2(n)-i} \cdot \beta_i = \sum_{i=1}^{\log_2(n)} (EX_i - 2EX_{i-1}) \cdot 2^{\log_2(n)-i} \cdot \beta_i.$$

After simple algebraic manipulations we can get

$$\begin{aligned} EY &= \sum_{i=1}^{\log_2(n)} EX_i \cdot 2^{\log_2(n)-i} \cdot \beta_i - \sum_{i=1}^{\log_2(n)} 2EX_{i-1} \cdot 2^{\log_2(n)-i} \cdot \beta_i = \\ &= \sum_{i=1}^{\log_2(n)} EX_i \cdot 2^{\log_2(n)-i} \cdot \beta_i - \sum_{i=0}^{\log_2(n)-1} EX_i \cdot 2^{\log_2(n)-i} \cdot \beta_{i+1} = \\ &= EX_{\log_2(n)} \cdot \beta_{\log_2(n)} - n\beta_1 EX_0 + \sum_{i=1}^{\log_2(n)-1} EX_i \cdot 2^{\log_2(n)-i} \cdot (\beta_i - \beta_{i+1}). \end{aligned}$$

Also, as $\kappa > 0$, we have $X_0 = 0$ with probability 1. These facts yield the following result

$$\begin{aligned} EY &= EX_{\log_2(n)} + \sum_{i=1}^{\log_2(n)-1} EX_i \cdot 2^{\log_2(n)-i} \cdot (\beta_i - \beta_{i+1}) = \\ &= n \cdot \frac{\binom{n-1}{\kappa}}{\binom{n}{\kappa}} + \sum_{i=1}^{\log_2(n)-1} 2^i \cdot \frac{\binom{n-\frac{n}{2^i}}{\kappa}}{\binom{n}{\kappa}} \cdot 2^{\log_2(n)-i} \cdot (\beta_i - \beta_{i+1}) = \\ &= n - \kappa + n \cdot \sum_{i=1}^{\log_2(n)-1} \left(\frac{\binom{n-\frac{n}{2^i}}{\kappa}}{\binom{n}{\kappa}} \cdot (\beta_i - \beta_{i+1}) \right). \end{aligned}$$

This gives us a formula for calculating EY and completes the proof of this theorem. □

B Proof of Lemma 1.

Proof. We have $\delta = 0.05$ as in [3]. See that for $2^4 \leqslant n \leqslant 2^{21}$ we have

$$\beta_{log_2(n)} = \beta_{log_2(n)-1} = \beta_{log_2(n)-2} = 1.$$

This happens because for these levels we have $\frac{1}{|B_i|} \cdot \ln(\log(n)+1) > 1$. This means that users aggregated in segments of length 1 and 2 generate noise with probability 1. Furthermore, for $i \leqslant (\log_2(n)-3)$ we have $\beta_i < 1$. Also, for $i \leqslant (\log_2(n)-4)$ we have

$$\frac{\beta_{i+1}}{\beta_i} = \frac{|B_i|}{|B_{i+1}|} = 2.$$

Another observation is that we can get an upper bound for $\frac{\binom{n-\frac{n}{2^i}}{\kappa}}{\binom{n}{\kappa}}$ in a following way

$$\begin{aligned}
\frac{\binom{n-\frac{n}{2^i}}{\kappa}}{\binom{n}{\kappa}} &= \frac{(n-\frac{n}{2^i})! \cdot (n-\kappa)!}{(n-\frac{n}{2^i}-\kappa)! \cdot n!} = \\
&= \frac{(n \cdot \frac{2^i-1}{2^i}) \cdot (n \cdot \frac{2^i-1}{2^i} - 1) \cdot \ldots \cdot (n \cdot \frac{2^i-1}{2^i} - \kappa + 1)}{n \cdot (n-1) \cdot \ldots \cdot (n-\kappa+1)} = \\
&= \left(\frac{2^i-1}{2^i}\right)^{\kappa} \cdot \frac{n \cdot (n \cdot -\frac{2^i}{2^i-1}) \cdot \ldots \cdot (n-(\kappa-1) \cdot \frac{2^i}{2^i-1})}{n \cdot (n-1) \cdot \ldots \cdot (n-\kappa+1)} \leqslant \\
&\leqslant \left(\frac{2^i-1}{2^i}\right)^{\kappa} = \left(1-\frac{1}{2^i}\right)^{\kappa} = \left(\left(1-\frac{1}{2^i}\right)^{2^i}\right)^{\frac{\kappa}{2^i}} \leqslant e^{-\frac{\kappa}{2^i}},
\end{aligned}$$

where the last inequality comes from the fact that $(1-x) \leqslant e^{-x}$. We can use all these observations to obtain a lower bound. Let $\beta^* = \ln\left(\frac{log_2(n)+1}{\delta}\right)$. Then we have

$$\begin{aligned}
EY &= n-\kappa+n \cdot \sum_{i=1}^{\log_2(n)-1} \left(\frac{\binom{n-\frac{n}{2^i}}{\kappa}}{\binom{n}{\kappa}} \cdot (\beta_i - \beta_{i+1})\right) = \\
&= n-\kappa-n \cdot \left(\sum_{i=1}^{\log_2(n)-4} \left(\frac{\binom{n-\frac{n}{2^i}}{\kappa}}{\binom{n}{\kappa}} \cdot \beta_i\right) + \frac{\binom{n-8}{\kappa}}{\binom{n}{\kappa}} \cdot \left(1-\beta_{\log_2(n)-3}\right)\right) \geqslant \\
&\geqslant n-\kappa-n \cdot \left(\sum_{i=1}^{\log_2(n)-4} \left(e^{-\frac{\kappa}{2^i}} \cdot \beta_i\right) + e^{\frac{8\kappa}{n}} \cdot \left(1-\beta_{\log_2(n)-3}\right)\right) \geqslant \\
&\geqslant n-\kappa-n \cdot \left(\sum_{i=1}^{\log_2(n)-4} \left(e^{-\frac{\kappa}{2^{\log_2(n)-4}}} \cdot \beta_i\right) + e^{\frac{8\kappa}{n}} \cdot \left(1-\beta_{\log_2(n)-3}\right)\right) = \\
&= n-\kappa-n \cdot \left(e^{-\frac{16\kappa}{n}} \cdot \frac{\beta^*}{n} \cdot \sum_{i=1}^{\log_2(n)-4} \left(2^i\right) + e^{\frac{8\kappa}{n}} \cdot \left(1-\beta_{\log_2(n)-3}\right)\right) = \\
&= n-\kappa-n \cdot \left(e^{-\frac{16\kappa}{n}} \cdot \frac{\beta^*}{n} \cdot \left(\frac{n}{8}-2\right) + e^{\frac{8\kappa}{n}} \cdot \left(1-\frac{\beta^*}{8}\right)\right) \geqslant \\
&\geqslant n-\kappa-n \cdot \left(e^{-\frac{16\kappa}{n}} \cdot \frac{\beta^*}{8} + e^{\frac{8\kappa}{n}} \cdot \left(1-\frac{\beta^*}{8}\right)\right) = \\
&= n-\kappa-n \cdot \left(e^{-\frac{8\kappa}{n}} + \frac{\beta^*}{8} \cdot \left(e^{-\frac{16\kappa}{n}} - e^{-\frac{8\kappa}{n}}\right)\right).
\end{aligned}$$

Which gives our lower bound for EY and finishes the proof of this lemma. □

C Proof of Theorem 2.

Proof. We are interested in the absolute sum of m noises, to estimate the error in Binary Protocol. First, let Z be a random variable that denote the sum of noises. See that

$$Z = \sum_{i=1}^{m} Z_i,$$

where Z_i is a random variable with distribution Geom(α), where $\alpha = e^{\frac{\epsilon}{\log_2(n)+1}}$.

Let φ_{Z_i} denotes the characteristic function of Z_i. We have

$$\varphi_{Z_i} = \frac{(\alpha-1)^2}{\alpha^2 - \alpha(e^t + e^{-t}) + 1} = \frac{(\alpha-1)^2}{\alpha^2 - 2\alpha\cos t + 1}.$$

Let φ_Z denote the characteristic function of Z. As Z_i are i.i.d. random variables, we get

$$\varphi_Z = (\varphi_{Z_1})^m = \left(\frac{(\alpha-1)^2}{\alpha^2 - 2\alpha\cos t + 1}\right)^m.$$

We will use techniques comprehensively described in [6] to calculate expected value of $|Z|$. From Corollary 2 and Remark 3 in [6] we have a following formula

$$\varphi_{X_+}(t) = Ee^{itZ_+} = \frac{1}{2}[1 + f(t)] + \frac{1}{2\pi i}\int_{-\infty}^{\infty}[f(t+u) - f(u)]\frac{du}{u},$$

where Z_+ denotes $\max(0, Z)$, $f(t)$ denotes characteristic function of Z and the integral is understood in the principal value sense (see [6]). Now see that

$$|Z| = Z_+ + Z_- = Z_+ + (-Z_+) = 2Z_+,$$

which is true for symmetric Z. Fortunately, this is the case here. Furthermore, we have

$$E|Z| = 2EZ_+ = 2\frac{\varphi'_{Z_+}(0)}{i}. \tag{3}$$

We have to calculate the derivative of $\varphi_{Z_+}(t)$ at 0. It can be done in the following way

$$\begin{aligned}\varphi'_{Z_+}(0) &= \frac{f'(0)}{2} + \frac{d}{dt}\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}[f(t+u) - f(u)]\frac{du}{u}\right)(0)\\ &= \frac{1}{2\pi i}\left(\int_{-\infty}^{\infty}[f'(t+u)]\frac{du}{u}\right)(0) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}[f'(u)]\frac{du}{u}.\end{aligned} \tag{4}$$

It is not hard too see, that $f'(0) = 0$, because Z is symmetric. Moreover, as the integral is finite, because EZ is finite so $E|Z|$ also has to be finite, then we can use Lebesgue

theorem to derive function inside the integral. We can easily derive $f(t)$ which yields the following

$$f'(t) = \frac{-2 \cdot a \cdot m \cdot \sin t \cdot (\alpha - 1)^{2m}}{(\alpha^2 - 2\alpha \cos t + 1)^{m+1}}. \tag{5}$$

Combining (3), (4) and (5) we obtain the following formula for $E|Z|$

$$E|Z| = \int_{-\infty}^{\infty} \frac{2 \cdot a \cdot m \cdot \sin t \cdot (\alpha - 1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha \cos t + 1)^{m+1}} dt.$$

We can observe that it is an even function. That leaves us with

$$E|Z| = \int_{0}^{\infty} \frac{4 \cdot a \cdot m \cdot \sin t \cdot (\alpha - 1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha \cos t + 1)^{m+1}} dt,$$

which completes the proof of this theorem. □

D Proof of Lemma 2.

Proof. We have

$$E|Z| = \int_{0}^{\infty} \frac{4 \cdot a \cdot m \cdot \sin t \cdot (\alpha - 1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha \cos t + 1)^{m+1}} dt.$$

Now, seeing that this function is almost periodic (except from the $\frac{1}{t}$ part), we can consider splitting the integral into $[2k\pi, 2(k+1)\pi]$ intervals and try to find a good lower bound for this integral, as this will suffice to show that the errors in [3] are unacceptably high. Therefore, we have

$$E|Z| = \sum_{k=0}^{\infty} \left(\int_{2k\pi}^{2(k+1)\pi} \frac{4 \cdot a \cdot m \cdot \sin t \cdot (\alpha - 1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha \cos t + 1)^{m+1}} dt \right).$$

Consider any of these integrals, for $k > 0$. We have

$$\int_{2k\pi}^{2(k+1)\pi} \frac{4 \cdot \alpha \cdot m \cdot \sin t \cdot (\alpha - 1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha \cos t + 1)^{m+1}} dt \geqslant 0. \tag{6}$$

This is true, because the function on this interval is "almost odd" around point $2k\pi + \pi$, which is the middle of this interval. By saying "almost odd around a point" we mean that if it was not for the $\frac{1}{t}$ factor, the integral would be clearly symmetric in terms of absolute values on this integral. Knowing that it is positive on $[2k\pi, 2k\pi + \pi]$ and negative on $[2k\pi + \pi, 2(k+1)\pi]$, which comes from the fact that divisor is always positive so the sign depends only on the sine function, we would have our integral being equal to

0 on this interval. Unfortunately, because of that $\frac{1}{t}$ factor that is clearly not true. It is, however, not hard to see that the function is greater on the interval $[2k\pi, 2k\pi+\pi]$, where the function is positive, than on the $[2k\pi+\pi, 2(k+1)\pi]$ interval, where it is negative. This is true because apart from that $\frac{1}{t}$ factor, the absolute values of the function are symmetric around $2k\pi+\pi$, and the $\frac{1}{t}$ factor is greater for smaller t. This yields (6). This fact is clearly true for all of these intervals and we will use it for all $k > 0$, so that leaves us with

$$E|Z| = \sum_{k=0}^{\infty} \left(\int_{2k\pi}^{2(k+1)\pi} \frac{4 \cdot a \cdot m \cdot \sin t \cdot (\alpha-1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt \right)$$
$$\geqslant \int_{0}^{2\pi} \frac{4 \cdot \alpha \cdot m \cdot \sin t \cdot (\alpha-1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt.$$

Plotting this function shows that almost all of the mass is concentrated around 0, especially for α close to 1.Even though use the lower bound coming from the "symmetry", however there is no point using it on the whole interval, because we would obtain trivial inequality $E|Z| \geqslant 0$. We have to be a bit more careful with this. Clearly, we can use this observation about symmetry for any interval of type $[\pi - x, \pi + x]$, for $x \leqslant \pi$. This yields the following

$$E|Z| \geqslant \int_{0}^{2\pi} \frac{4 \cdot a \cdot m \cdot \sin t \cdot (\alpha-1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt$$
$$\geqslant \int_{0}^{\eta_{\alpha,m}} \frac{4 \cdot a \cdot m \cdot \sin t \cdot (\alpha-1)^{2m}}{t\pi \, (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt + \int_{2\pi-\eta_{\alpha,m}}^{2\pi} \frac{4 \cdot \alpha \cdot m \cdot \sin t \cdot (\alpha-1)^{2m}}{t\pi \, (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt.$$

Now see that if $\eta_{\alpha,m} < \frac{\pi}{2}$, we can bound the first integral in a following way

$$\int_{0}^{\eta_{\alpha,m}} \frac{4\alpha m \cdot \sin t \cdot (\alpha-1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt \geqslant \int_{0}^{\eta_{\alpha,m}} \frac{4\alpha m \cdot \cos t \cdot (\alpha-1)^{2m}}{\pi \cdot (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt, \quad (7)$$

which follows from the fact that $x \leqslant \tan x$ for $x \in [0, \frac{\pi}{2})$. Furthermore

$$\int_{2\pi-\eta_{\alpha,m}}^{2\pi} \frac{4\alpha m \cdot \sin t \cdot (\alpha-1)^{2m}}{t \cdot \pi \cdot (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt \geqslant \int_{2\pi-\eta_{\alpha,m}}^{2\pi} \frac{4\alpha m \sin t \, (\alpha-1)^{2m}}{t \cdot \pi \cdot (\alpha-1)^{2m+2}} dt, \quad (8)$$

which comes from plugging 1 instead of $\cos t$, so that the divisor is smaller which makes the function greater in terms of absolute value, but as the function is negative on this interval, it yields a lower bound. The function from (7) has an explicit anti-derivative. On the other hand, in (8) we have, in fact, an integral of $\frac{\sin t}{t}$ multiplied by a constant

depending on α and m. There also still remains a problem of choosing $\eta_{\alpha,m}$. First we can observe that, for small enough $\eta_{\alpha,m}$ we have

$$\int_{2\pi-\eta_{\alpha,m}}^{2\pi} \frac{\sin t}{t} dt \geqslant -\frac{\eta_{\alpha,m}^2}{10}.$$

Obviously it is true for $\eta_{\alpha,m} = 0$. Let $Si(x)$ denote the antiderivative of $\frac{\sin x}{x}$. After derivating left side we obtain

$$(Si(2\pi) - Si(2\pi - \eta_{\alpha,m}))' = -Si'(2\pi - \eta_{\alpha,m}) = \frac{\sin(2\pi - \eta_{\alpha,m})}{2\pi - \eta_{\alpha,m}} = \\ = -\frac{\sin(\eta_{\alpha,m})}{2\pi - \eta_{\alpha,m}} \geqslant -\frac{\eta_{\alpha,m}}{2\pi - \eta_{\alpha,m}}.$$

Derivating the right side gives $-0.2\eta_{\alpha,m}$. We can check when the left side is greater than the right side

$$-\frac{\eta_{\alpha,m}}{2\pi - \eta_{\alpha,m}} \geqslant -0.2\eta_{\alpha,m} \iff \eta_{\alpha,m} \leqslant 2\pi - 5$$

So for $\eta_{\alpha,m} \leqslant 2\pi - 5$ we have

$$\int_{2\pi-\eta_{\alpha,m}}^{2\pi} \frac{\sin t}{t} dt \geqslant -\frac{\eta_{\alpha,m}^2}{10}$$

Now we pick $\eta_{\alpha,m}$ so that

$$-0.1\eta_{\alpha,m}^2 \cdot \frac{4\alpha m}{\pi(\alpha-1)^2} = -0.1.$$

That gives us

$$\eta_{\alpha,m} = \sqrt{\frac{\pi(\alpha-1)^2}{4\alpha m}}.$$

Plugging it all to our formula for expected magnitude of noise yields

$$E|Z| \geqslant \int_0^{\eta_{\alpha,m}} \frac{4 \cdot a \cdot m \cdot \cos t \cdot (\alpha-1)^{2m}}{\pi \cdot (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt - 0.1.$$

We are now interested in the lower bound for this integral. One can see that

$$\int_0^{\eta_{\alpha,m}} \frac{4 \cdot a \cdot m \cdot \cos t \cdot (\alpha-1)^{2m}}{\pi \cdot (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt \geqslant \int_0^{\eta_{\alpha,m}} \frac{4 \cdot a \cdot m \cdot \cos(\eta_{\alpha,m}) \cdot (\alpha-1)^{2m}}{\pi \cdot (\alpha^2 - 2\alpha\cos t + 1)^{m+1}} dt.$$

This inequality is just plugging the smallest cosine possible on this interval. Furthermore, we have

$$\int_0^{\eta_{\alpha,m}} \frac{4\alpha m \cdot \cos(\eta_{\alpha,m}) \cdot (\alpha-1)^{2m}}{\pi\left(\alpha^2 - 2\alpha\cos t + 1\right)^{m+1}} dt \geqslant \int_0^{\eta_{\alpha,m}} \frac{4\alpha m \cdot \left(1-\frac{\eta_{\alpha,m}^2}{2}\right) \cdot (\alpha-1)^{2m}}{\pi\left(\alpha^2 - 2\alpha\cdot\left(1-\frac{t^2}{2}\right) + 1\right)^{m+1}} dt.$$

This bound comes from the fact that $\cos t \geqslant 1 - \frac{t^2}{2}$. Let us call the integrand function $g(t)$. This function has a following anti-derivative $G(t)$:

$$G(t) = \frac{4(\alpha-1)^{2m-2}\alpha m t \left(1+\frac{\alpha t^2}{(\alpha-1)^2}\right)^m \left(1-\frac{\eta_{\alpha,m}^2}{2}\right) {}_2F_1\left(\frac{1}{2}, 1+m, \frac{3}{2}, -\frac{\alpha\cdot t^2}{(\alpha-1)^2}\right)}{\left(\alpha^2+\alpha(t^2-2)+1\right)^m \cdot \pi},$$

where the ${}_2F_1(a,b,c,z)$ denotes ordinary hypergeometric function (see [38]) . One can easily see, that $G(0) = 0$. That leaves us with

$$E|Z| \geqslant G(\eta_{\alpha,m}) - 0.1.$$

Function $G(\eta_{\alpha,m})$ is quite complicated, but we can greatly simplify it. Let us begin with taking some of the $G(\eta_{\alpha,m})$ factors

$$\begin{aligned}\frac{(\alpha-1)^{2m-2}\cdot\left(1+\frac{\alpha\cdot\eta_{\alpha,m}^2}{(\alpha-1)^2}\right)^m}{\left(\alpha^2+\alpha\cdot(\eta_{\alpha,m}^2-2)+1\right)^m} &= \frac{(\alpha-1)^{-2}\cdot\left(1+\frac{\alpha\cdot\eta_{\alpha,m}^2}{(\alpha-1)^2}\right)^m}{\left(\frac{\alpha^2}{(\alpha-1)^2}+\frac{\alpha}{(\alpha-1)^2}\cdot(\eta_{\alpha,m}^2-2)+\frac{1}{(\alpha-1)^2}\right)^m} = \\ &= \frac{(\alpha-1)^{-2}\cdot\left(1+\frac{\alpha\cdot\eta_{\alpha,m}^2}{(\alpha-1)^2}\right)^m}{\left(1+\frac{\alpha\cdot\eta_{\alpha,m}^2}{(\alpha-1)^2}\right)^m} = (\alpha-1)^{-2}.\end{aligned}$$

Furthermore, we can expand ${}_2F_1(a,b,c,z)$ into Taylor series around 0 in a following way:

$${}_2F_1\left(\frac{1}{2}, 1+m, \frac{3}{2}, -\frac{\alpha\cdot t^2}{(\alpha-1)^2}\right) = 1 - \frac{(m+1)\alpha t^2}{3(\alpha-1)^2} + O(t^4) \geqslant 1 - \frac{\alpha\cdot(m+1)\cdot\eta_{\alpha,m}^2}{3\cdot(\alpha-1)^2}.$$

Using these we obtain

$$G(\eta_{\alpha,m}) \geqslant \frac{4(\alpha-1)^{-2}\cdot\alpha\cdot m\cdot\eta_{\alpha,m}\cdot\left(1-\frac{\eta_{\alpha,m}^2}{2}\right)\cdot\left(1-\frac{\alpha\cdot(m+1)\cdot\eta_{\alpha,m}^2}{3\cdot(\alpha-1)^2}\right)}{\pi}$$

We can further simplify this by recalling that $\alpha = e^{-\frac{\epsilon}{\log 2n+2}}$ and $m = \gamma n$ and observing that $\left(1-\frac{\eta_{\alpha,m}^2}{2}\right)\cdot\left(1-\frac{\alpha\cdot(m+1)\cdot\eta_{\alpha,m}^2}{3\cdot(\alpha-1)^2}\right)$ is increasing with n, therefore we can calculate this for the smallest n that we want to consider. Let us call this value c_n^*. See that, for

example, for $n \geqslant 2^7$ we have $c_n^* \geqslant 1.43$. This leaves us with

$$G(\eta_{\alpha,m}) \geqslant \frac{4(\alpha-1)^{-2} \cdot \alpha \cdot m \cdot \eta_{\alpha,m} \cdot c_n^*}{\pi} =$$
$$= \frac{4c_n^* \cdot (\alpha-1)^{-2} \cdot \alpha \cdot m \cdot \sqrt{\frac{\pi(\alpha-1)^2}{4\alpha m}}}{\pi} =$$
$$= \frac{2c_n^* \cdot \sqrt{\alpha \cdot m}}{\sqrt{\pi} \cdot (\alpha-1)} \geqslant \frac{2c_n^* \cdot \sqrt{m}}{\sqrt{\pi} \cdot (\alpha-1)} = \frac{2c_n^* \cdot \sqrt{\gamma n}\sqrt{\pi} \cdot (e^{\frac{\epsilon}{\log_2(n)+2}} - 1)}{\geqslant}$$
$$\geqslant \frac{2c_n^* \cdot \sqrt{\gamma n}}{\sqrt{\pi} \cdot (e^{\frac{\epsilon}{\log_2(n)}} - 1)} \geqslant \frac{\xi \log_2(n) \cdot 2c_n^* \cdot \sqrt{\gamma n}}{\epsilon\sqrt{\pi}},$$

where ξ is such that $e^{\xi \cdot x} \geqslant (1+x)$ for $x = (\frac{1}{2\log_2(n)+2})$. For example, in case we hav $\epsilon = 0.5$ and $n \geqslant 2^7$ it suffice to take $\xi = 0.99$. In the end we have

$$G(\eta_{\alpha,m}) \geqslant c_{n,\epsilon} \cdot \sqrt{\gamma} \cdot \frac{\log_2(n) \cdot \sqrt{n}}{\epsilon\sqrt{\pi}},$$

where $c_{n,\epsilon} = 2\xi c_n^*$ which is, for moderate n and ϵ, greater than 1.4. In the end we have

$$E|Z| \geqslant c_{n,\epsilon} \cdot \sqrt{\gamma} \cdot \frac{\log_2(n) \cdot \sqrt{n}}{\epsilon\sqrt{\pi}} - 0.1\ ,$$

which completes the proof of this lemma. □

E Proof of Theorem 4

Proof. Let us note that $\mathcal{S}$ is **not** connected if and only if there exists a subset of nodes from $\mathcal{S}$ with cardinality $1 \leq k \leq m/2$ such that there is no connection to any of remaining $m-k$ nodes. For a given subset of cardinality k probability that no edge connects it to other $m-k$ nodes is $(1-p)^{k(m-k)}$.

Let A_k be an event that there exists such a "cut-off" subset of cardinality k. Clearly, using union bound we get

$$\Pr[A_k] \leq (1-p)^{k(m-k)} \binom{m}{k}.$$

Probability that $\mathcal{S}$ is not connected is equivalent to the event $A_1 \cup \ldots \cup A_k$ for $k = 1, \ldots, m/2$. Again, using union bound

$$\Pr[A_1 \cup \ldots \cup A_k] \leq \leq \sum_{i=1}^{m/2} \Pr[A_i] \leq \sum_{i=1}^{m/2} (1-p)^{k(m-k)} \binom{m}{k}$$
$$\leq \sum_{i=1}^{m/2} (1-p)^{k\frac{m}{2}} \binom{m}{k}.$$

Since $\binom{m}{k} < m^k$ we get

$$\leq \sum_{i=1}^{m/2} \left((1-p)^{\frac{m}{2}} m\right)^k \leq \sum_{i=1}^{\infty} \left((1-p)^{\frac{m}{2}} m\right)^k = \frac{(1-p)^{m/2} m}{1-(1-p)^{m/2} m} = (\star).$$

Since the function $f(x) = \frac{a^x x}{1-a^x x}$ is decreasing for $x > -\frac{1}{\log(a)}$ (if $0 < a < 1$) and from the assumption that $m \geq n/2$ we have

$$(\star) \leq \frac{(1-p)^{n/4} \frac{n}{2}}{1-(1-p)^{n/2} \frac{n}{2}} \ .$$

Applying inequality $\exp(x) > 1 + x$ and substituting $p = \frac{8 \log n}{n}$ we obtain

$$\frac{\exp\left(-\frac{8\log(n)}{n}\right) \frac{n}{2}}{1-1/2} = \exp\left(-\log(n^2)\right) n = \frac{1}{n}.$$

□