More intuitive analysis of bit-reversal scheduling RBO*

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This document is replacement of mki-seminar_current.pdf from https://sites.google.com/site/rbo-protocol/rbo-files. It contains more intuitive presentation of the analysis of the the RBO (bit-reversal scheduling protocol) [1], [2], [3], [5], [4]. It will be updated to improve its readability.

First we present the bounds on the tuning costs (i.e. external-energy costs) of the RBO receiver in the case of reliable and unreliable transmission channel.

Then we present and analyze the algorithm NSI that efficiently computes the next wake-up time-slot for the RBO receiver.

Introduction

Broadcast scheduling: There is a single server (*broadcaster*) that broadcasts a set of messages and a dynamic set of clients (*receivers*). Each receiver wants to receive some specified subset of the messages. Each message has a *key* and we assume that the keys are from some linearly ordered universe.

The broadcaster is unaware of the receivers and what are the keys requested by the receivers. He simply broadcasts the set periodically in so called *broadcast cycles*. The broadcast cycle is divided into *time-slots* and in each time-slot a single message is transmitted.

On the other hand the receiver is initially unaware of the contents of broadcast cycle. He simply wants to receive all the messages with the keys in some specified interval, say $[\kappa', \kappa'']$. The receiver may start at arbitrary time-slot s during the broadcast cycle and we want it to receive all the messages with the keys in $[\kappa', \kappa'']$ transmitted since the time-slot s. Since there is some cost (usually – energetic) of message reception, we also want to minimize the number of time-slots when receiver to listens to anything else. We call the number of such time-slots *extra energy* or *tuning cost* of the receiver.

We propose a broadcast scheduling method based on properties of bit-reversal permutation. We assume that the length of broadcast cycle is $n = 2^k$, for some positive integer k. (Otherwise, we may duplicate some messages to increase their number to the power of two, since we do not require that the keys are distinct.)

Preliminaries

Notation:

- $\mathrm{bin}_k(x)$ denotes k-bit binary representation of $x \bmod 2^k$

  (e.g. $\mathrm{bin}_4(5) = (0101)$)

∗. This document has been written using the GNU $\mathrm{T_EX_{MACS}}$ text editor (see www.texmacs.org).

- for $x \geq 0$, $\operatorname{bin}(x) = \operatorname{bin}_l(x)$, where $l = \lceil \lg_2(x+1) \rceil$
  (e.g. $\operatorname{bin}(5) = (101)$, $\operatorname{bin}(0) = ()$)
- for a binary representation α, $(\alpha)_2$ denotes the integer represented by α
  (e.g. $(0101)_2 = 5$)
- for a sequence α, $\operatorname{rev} \alpha$ denotes the reversal of the sequence α
- $\alpha\beta$ denotes concatenation of sequences α and β
- α^i denotes concatenation of i copies of the sequence α.
- $\operatorname{rev}_k(t)$ is the integer represented by reversal of k-bit representation of $t \bmod 2^k$, i.e.:
  $\operatorname{rev}_k(t) = (\operatorname{rev} \operatorname{bin}_k(t))_2$
- $\mathbb{Z}$ denotes the set of all integer numbers
- $[[a, b]] = [a, b] \cap \mathbb{Z}$
- for $S \subseteq \mathbb{Z}$, $\operatorname{rev}_k S = \{\operatorname{rev}_k(x) \mid x \in S\}$
  (i.e. image of S under rev_k)
- For a finite set S, the number of elements of S is denoted by $|S|$
- For a random variable v, $\mathrm{EX}[v]$ denotes the expected value of v.

RBO operation

The length of the *broadcast cycle* is 2^k. Let n denote 2^k.

The sequence $\kappa_0, \ldots, \kappa_{n-1}$ is the sorted sequence of the keys (i.e. $\kappa_0 \leq \ldots \leq \kappa_{n-1}$).

The *broadcast cycle* is the sequence: $\kappa_{\operatorname{rev}_k(0)}, \ldots, \kappa_{\operatorname{rev}_k(n-1)}$.

Let us define κ_{-1} and κ_n as follows: $\kappa_{-1} = -\infty$ and $\kappa_n = +\infty$.

The receiver wishes to receive the keys from the interval $[\kappa', \kappa'']$.

We assume that $\kappa' \leq \kappa''$.

Let $r' = \min\{r : r \in [[0, n]] \wedge \kappa' \leq \kappa_r\}$ and $r'' = \max\{r : r \in [[-1, n-1]] \wedge \kappa_r \leq \kappa''\}$.

Note that:

- $0 \leq r' \leq r'' + 1 \leq n$, and
- $\{\kappa_i : r' \leq i \leq r''\} = [\kappa', \kappa''] \cap \{\kappa_0, \ldots, \kappa_{n-1}\}$, and
- $r' = r'' + 1$ implies $[\kappa', \kappa''] \cap \{\kappa_0, \ldots, \kappa_{n-1}\} = \emptyset$.

By s we denote the time-slot when the receiver starts. (W.l.o.g. we assume that $s \geq 0$.)

RBO Broadcaster in time-slot t broadcasts $\kappa_{\operatorname{rev}_k(t)}$.

RBO Receiver requesting for the keys from $[\kappa', \kappa'']$ starting in time-slot s:

- starts in time slot s with $\mathrm{lb} = 0$, $\mathrm{ub} = n - 1$
- In time slot $t \geq s$:
  - **if** $\mathrm{lb} \leq \mathrm{rev}_k(t) \leq \mathrm{ub}$ **then** the receiver listens
  - **if** the receiver successfully receives the key $\kappa = \kappa_{\mathrm{rev}_k(t)}$

    then

    - **if** $\kappa < \kappa'$ **then** $\mathrm{lb} \leftarrow \mathrm{rev}_k(t) + 1$
    - **if** $\kappa'' < \kappa$ **then** $\mathrm{ub} \leftarrow \mathrm{rev}_k(t) - 1$
    - **if** $\kappa' \leq \kappa \leq \kappa''$ **then** the receiver reports κ
  - **if** $\mathrm{ub} < \mathrm{lb}$ **then** the receiver reports absence of the keys from $[\kappa', \kappa'']$ in the broadcast cycle and stops

Let lb_t denote the value of lb just before time-slot t. (For $t \leq s$, we have $\mathrm{lb}_t = 0$.)

Let ub_t denote the value of ub just before time-slot t. (For $t \leq s$, we have $\mathrm{ub}_t = 2^k - 1$.)

The *left-side energy used in the set of time-slots* Y (denoted: $\mathrm{le}\, Y$) is the size of the set $\{t \in Y : \mathrm{lb}_t \leq \mathrm{rev}_k(t) \leq r' - 1\}$.

The *right-side energy used in the set of time-slots* Y (denoted: $\mathrm{re}\, Y$) is the size of the set $\{t \in Y : r'' + 1 \leq \mathrm{rev}_k(t) \leq \mathrm{ub}_t\}$.

Example:

Recall that s is the starting time-slot of the RBO receiver.

If s is such that:

$$\mathrm{bin}_k(\mathrm{rev}_k(s)) = (000111011010011)$$

then, we present $\mathrm{bin}_k(\mathrm{rev}_k(s))$ as the following concatenation:

$$(000)(111)(011)(01)(0)(011).$$

Note that the sequence $\mathrm{bin}_k(\mathrm{rev}_k(s)), \mathrm{bin}_k(\mathrm{rev}_k(s+1)), \ldots$ looks as follows:

- $(000)(111)(011)(01)(0)(011)$,
- $(100)(111)(011)(01)(0)(011)$,
- $(010)(111)(011)(01)(0)(011)$,
- ...
- $(111)(111)(011)(01)(0)(011)$,

- (000)(000)(111)(01)(0)(011),
- (100)(000)(111)(01)(0)(011),
- ...
- (111)(111)(111)(01)(0)(011),
- (000)(000)(000)(11)(0)(011),
- ...
- (111)(111)(111)(11)(0)(011),
- (000)(000)(000)(00)(1)(011),
- ...
- (111)(111)(111)(11)(1)(011),
- (000)(000)(000)(00)(0)(111),
- ...
- (111)(111)(111)(11)(1)(111),
- (000)(000)(000)(00)(0)(000),
- ...
- (111)(111)(111)(11)(1)(111),
- (000)(000)(000)(00)(0)(000),
- ...
- (111)(111)(111)(11)(1)(111),
- ...

Definitions:

k is fixed positive integer such that the length of the broadcast cycle is $n = 2^k$.

Thus, the broadcast cycle is the sequence: $\kappa_{\mathrm{rev}_k(0)}, \ldots, \kappa_{\mathrm{rev}_k(2^k-1)}$.

s is the time-slot when the RBO receiver starts.

Let the non-negative integer: last and the sequence $l_0, \ldots, l_{\mathrm{last}}$ be defined as follows:

If $\mathrm{bin}_k(\mathrm{rev}_k(s)) = (0)^k$ then last $= 0$ and $l_0 = k$, else

if $\mathrm{bin}_k(\mathrm{rev}_k(s)) = (0)^{k'}(1)^{k-k'}$, for some k', $0 \le k' < k$, then last $= 1$, and $l_0 = k'$, and $l_1 = k$, else

last and $l_0, \ldots, l_{\mathrm{last}}$ are defined so that the following holds:

$$\mathrm{bin}_k(\mathrm{rev}_k(s)) = (0)^{l_0}(1)^{l_1 - l_0}((0)(1)^{l_2 - l_1 - 1})\ldots((0)(1)^{l_{\mathrm{last}} - l_{\mathrm{last}-1} - 1}).$$

Note that we have always $l_{\mathrm{last}} = k$.

For $i > \text{last}$, let $l_i = k$.

Let $\gamma_i = (0)(1)^{l_{i+1} - l_i - 1}$.

Let $\gamma_i' = (1)^{l_{i+1} - l_i}$.

Old definitions of β_i and α_i:

$\beta_i = \gamma_i' \gamma_{i+1} \ldots \gamma_{\text{last}-1}$

$(0)\alpha_i = \gamma_{i+1} \ldots \gamma_{\text{last}-1}$

Note that: $\text{rev}_k(s) = ((0)^{l_0} \gamma_0' \gamma_1 \ldots \gamma_{\text{last}-1})_2$

<table>
<tr><td colspan="16">$\leftarrow$ $\text{bin}_k(\text{rev}_k(s))$ $\rightarrow$</td></tr>
<tr><td colspan="3"></td><td colspan="13">$\leftarrow$ β_0 $\rightarrow$</td></tr>
<tr><td colspan="7"></td><td colspan="9">$\leftarrow$ α_0 $\rightarrow$</td></tr>
<tr><td>0</td><td>...</td><td>0</td><td>1</td><td>...</td><td>1</td><td>0</td><td>1</td><td>...</td><td>1</td><td>0</td><td>1</td><td>...</td><td>1</td><td>0</td><td>α_2</td></tr>
<tr><td colspan="3">$\leftarrow$ l_0 $\rightarrow$</td><td colspan="3">$\leftarrow$ γ_0' $\rightarrow$</td><td colspan="4">$\leftarrow$ γ_1 $\rightarrow$</td><td colspan="4">$\leftarrow$ γ_2 $\rightarrow$</td><td colspan="2"></td></tr>
<tr><td colspan="6">$\leftarrow$ l_1 $\rightarrow$</td><td colspan="10"></td></tr>
<tr><td colspan="10">$\leftarrow$ l_2 $\rightarrow$</td><td colspan="6"></td></tr>
<tr><td colspan="14">$\leftarrow$ l_3 $\rightarrow$</td><td colspan="2"></td></tr>
<tr><td colspan="16">$\leftarrow$ k $\rightarrow$</td></tr>
</table>

Let $t_0 = s$.

For $i \geq 0$, let $t_{i+1} = t_i + 2^{l_i}$.

Note that: $\text{rev}_k(t_i) = ((0)^{l_i} \gamma_i' \gamma_{i+1} \ldots \gamma_{\text{last}-1})_2$ (If $i = \text{last} - 1$, then $\gamma_{i+1} \ldots \gamma_{\text{last}-1}$ is empty sequence, and, if $i \geq \text{last}$, then $\gamma_i' \gamma_{i+1} \ldots \gamma_{\text{last}-1}$ is empty sequence.)

Division of the time-slots $s, s+1, \ldots$ into blocks Y_i and sub-blocks $Y_{i,j}$:

For $i \geq 0$, let $Y_i = [[t_i, t_{i+1} - 1]]$.

For $i \geq 0$, for $j \in [[0, l_i]]$, let $Y_{i,j} = [[t_i + \lfloor 2^{j-1} \rfloor, t_i + 2^j - 1]]$.

Note that $Y_{i,0} = \{t_i\}$ and, for $j \in [[0, l_i - 1]]$, $Y_{i,j+1} \subseteq Y_i \setminus \left(\bigcup_{j'=0}^{j} Y_{i,j'} \right)$.

Sets of keys' indexes broadcast during blocks Y_i and sub-blocks $Y_{i,j}$:

Let $X_i = \text{rev}_k Y_i$.

Let $X_{i,j} = \text{rev}_k Y_{i,j}$.

Subset X_i contains the elements of the following sequence of indexes:

- $(\text{rev}(\text{bin}_{l_i}(0)) \gamma_i' \gamma_{i+1} \ldots \gamma_{\text{last}-1})_2$
- ...
- $(\text{rev}(\text{bin}_{l_i}(2^{l_i} - 1)) \gamma_i' \gamma_{i+1} \ldots \gamma_{\text{last}-1})_2$

Note that: For $i = \text{last} - 1$, $\gamma_{i+1} \ldots \gamma_{\text{last}-1}$ is an empty sequence and, for $i = \text{last}$, $\gamma_i' \gamma_{i+1} \ldots \gamma_{\text{last}-1}$ is an empty sequence. If $l_0 = 0$, then $\text{rev}(\text{bin}_{l_0}(0)) = \text{rev}(\text{bin}_{l_0}(2^{l_0} - 1))$ is an empty sequence.

We have $X_{i,0} = \{(0)^{l_i}\gamma_i'\,\gamma_{i+1}\ldots\gamma_{\text{last}-1}\}$.

For $1 \leq j \leq l_i$, subset $X_{i,j}$ contains the elements of the following sequence of indexes:

- $(\text{rev}(\text{bin}_{j-1}(0))\,1\,(0)^{l_i-j}\gamma_i'\,\gamma_{i+1}\ldots\gamma_{\text{last}-1})_2$
- ...
- $(\text{rev}(\text{bin}_{j-1}(2^{j-1}-1))\,1\,(0)^{l_i-j}\gamma_i'\,\gamma_{i+1}\ldots\gamma_{\text{last}-1})_2$

Note that: For $j = 1$, $\text{rev}(\text{bin}_{j-1}(0)) = \text{rev}(\text{bin}_{j-1}(2^{j-1}-1))$ is an empty sequence.

Note that: If $X_{i,j}$ contains more than one element, then $j \geq 2$ and the minimal distance between the elements of $X_{i,j}$ is 2^{k-j+1}.

Infinite extensions of $\boldsymbol{X_i}$ and of $\boldsymbol{\bigcup_{j'=0}^{j} X_{i,j'}}$:

For $i \geq 0$, let $\mathbb{X}_i = \{x \cdot 2^{k-l_i} + (\gamma_i'\gamma_{i+1}\ldots\gamma_{\text{last}-1})_2 : x \in \mathbb{Z}\}$ and,

for $j \in [[0, l_i]]$, let $\mathbb{X}_{i,j} = \{x \cdot 2^{k-j} + ((0)^{l_i-j}\,\gamma_i'\gamma_{i+1}\ldots\gamma_{\text{last}-1})_2 : x \in \mathbb{Z}\}$.

Note that: $X_i = \mathbb{X}_i \cap [[0, 2^k - 1]]$ and $\bigcup_{j'=0}^{j} X_{i,j'} = \mathbb{X}_{i,j} \cap [[0, 2^k - 1]]$.

Definitions of $\boldsymbol{p_{i,j}'}$, $\boldsymbol{p_i'}$, $\boldsymbol{x_i'}$ and $\boldsymbol{p_{i,j}''}$, $\boldsymbol{p_i''}$, $\boldsymbol{x_i''}$:

- $p_{i,j}' = \max\{x : x \in \mathbb{X}_{i,j} \wedge x < r'\}$
- $x_{i,j}' = \lfloor p_{i,j}'/2^{k-j} \rfloor$ (note that: $x_{i,j}' \geq -1$, since $r' \geq 0$)
- $p_i' = \max\{x : x \in \mathbb{X}_i \wedge x < r'\}$
- $x_i' = \lfloor p_i'/2^{k-l_i} \rfloor$ (note that: $x_i' \geq -1$, since $r' \geq 0$)
- $p_{i,j}'' = \min\{x : x \in \mathbb{X}_{i,j} \wedge r'' < x\}$
- $x_{i,j}'' = \lfloor p_{i,j}''/2^{k-j} \rfloor$ (note that: $x_{i,j}'' \leq 2^j$, since $r'' \leq 2^k - 1$)
- $p_i'' = \min\{x : x \in \mathbb{X}_i \wedge r'' < x\}$
- $x_i'' = \lfloor p_i''/2^{k-l_i} \rfloor$ (note that: $x_i'' \leq 2^{l_i}$, since $r'' \leq 2^k - 1$)

Note that:

- $p_i' = 2^{k-l_i} \cdot x_i' + (\gamma_i'\gamma_{i+1}\ldots\gamma_{\text{last}-1})_2$
- $p_{i,j}' = 2^{k-j} \cdot x_{i,j}' + ((0)^{l_i-j}\gamma_i'\gamma_{i+1}\ldots\gamma_{\text{last}-1})_2$
- $p_i'' = 2^{k-l_i} \cdot x_i'' + (\gamma_i'\gamma_{i+1}\ldots\gamma_{\text{last}-1})_2$
- $p_{i,j}'' = 2^{k-j} \cdot x_{i,j}'' + ((0)^{l_i-j}\gamma_i'\gamma_{i+1}\ldots\gamma_{\text{last}-1})_2$

ENERGY IN RELIABLE NETWORK

In *reliable* network the receiver successfully receives the key in each time-slot it listens.

LEFT-SIDE ENERGY:

Let lb_t denote the value of lb just before time-slot t. (For $t \le s$, we have $\mathrm{lb}_t = 0$.)

Note that:

- if $t < t'$ then $\mathrm{lb}_t \le \mathrm{lb}_{t'}$, and
- RBO receiver uses one unit of left-side energy in time-slot t (i.e. $\mathrm{le}\{t\}{=}1$) if and only if $\mathrm{rev}_k(t) \in [[\mathrm{lb}_t, r' - 1]]$ and $\mathrm{lb}_{t+1} = \mathrm{rev}_k(t) + 1$.
- $\mathrm{le}\,[[t, t+m]] \le |\,[[\mathrm{lb}_t, r' - 1]] \cap \mathrm{rev}_k\,[[t, t+m]]\,|$.
- The left-side energy used in Y_{i+1} is not greater than the size of $[[p'_i + 1, r' - 1]] \cap \mathbb{X}_{i+1} = [[p'_i + 1, p'_{i+1}]] \cap \mathbb{X}_{i+1}$. (I.e. $\mathrm{le}\,Y_{i+1} \le |\,[[p'_i + 1, p'_{i+1}]] \cap \mathbb{X}_{i+1}\,|$.)
- The left-side energy used in $Y_{i,j+1}$ is not greater than the size of $[[p'_{i,j} + 1, r' - 1]] \cap \mathbb{X}_{i,j+1} = [[p'_{i,j} + 1, p'_{i,j+1}]] \cap \mathbb{X}_{i,j+1}$. (I.e. $\mathrm{le}\,Y_{i,j+1} \le |\,[[p'_{i,j} + 1, p'_{i,j+1}]] \cap \mathbb{X}_{i,j+1}\,|$.)

Bounds on r':

- $p'_i + 1 \le r' \le p'_i + 2^{k-l_i}$
- $p'_{i,j} + 1 \le r' \le p'_{i,j} + 2^{k-j}$

Bounds on lb:

After the time-slots $\bigcup_{i'=0}^{i} Y_{i'}$ the value of lb is in $[[p'_i + 1, r']]$.

After the time-slots $\bigcup_{i'=0}^{i-1} Y_{i'} \cup \bigcup_{j'=0}^{j} Y_{i,j'}$ the value of lb is in $[[p'_{i,j} + 1, r']]$.

(In other words: if $t \ge \max\, Y_i + 1$, then $\mathrm{lb}_t \in [[p'_i + 1,\ r']]$ and if $t \ge \max\, Y_{i,j} + 1$, then $\mathrm{lb}_t \in [[p'_{i,j} + 1, r']]$.)

Left-energy bound for $Y_{i,0}$:

The left-side energy used during the time-slots $Y_{i,0}$ is at most 1, since $|Y_{i,0}| = |X_{i,0}| = 1$.

Left-energy bound for $Y_{i,j+1}$:

For $0 \le j \le l_i - 1$, the left-side energy used during the time-slots $Y_{i,j+1}$ is at most 1, since:

- $[[p'_{i,j} + 1, r' - 1]] \cap \mathbb{X}_{i,j+1} \subseteq [[p'_{i,j} + 1, p'_{i,j} + 2^{k-j} - 1]] \cap \mathbb{X}_{i,j+1}$, since $r' \le p'_{i,j} + 2^{k-j}$, and
- $[[p'_{i,j} + 1, p'_{i,j} + 2^{k-j} - 1]] \cap \mathbb{X}_{i,j+1} \subseteq \{p'_{i,j} + 2^{k-j-1}\}$.

Left-side energy bound for Y_0:

Thus, *the left-side energy used during the time-slots Y_0 is at most $1 + l_0$.*

Lemma 1. (Lemma L0.) le $Y_0 \leq 1 + l_0$.

Left-side energy bound for $\bigcup_{j'=0}^{l_i} Y_{i+1,j'}$, where $i+1 \leq \text{last} - 1$:

Remark: If $i+1 = \text{last} - 1$, then $\gamma_{i+2} \ldots \gamma_{\text{last}-1}$ is an empty sequence.

For $0 \leq i \leq \text{last} - 1$*, the left-side energy used during the time-slots* $\bigcup_{j'=0}^{l_i} Y_{i+1,j'}$ *is at most* 1, since:

- $[[\text{lb}_{\min Y_{i+1}}, r' - 1]] \cap \left(\bigcup_{j'=0}^{l_i} X_{i+1,j'}\right) \subseteq [[p_i' + 1, p_i' + 2^{k-l_i} - 1]] \cap \mathbb{X}_{i+1,l_i}$, and

- $p_i' + 1 = 2^{k-l_i} \cdot x_i' + (\gamma_i' \gamma_{i+1} \ldots \gamma_{\text{last}-1})_2 + 1 > 2^{k-l_i} \cdot x_i' + ((0)^{l_{i+1}-l_i} \gamma_{i+1}' \gamma_{i+2} \ldots \gamma_{\text{last}-1})_2$, since $(\gamma_i' \gamma_{i+1})_2 > ((0)^{l_{i+1}-l_i} \gamma_{i+1}')_2$, and

- $p_i' + 2^{k-l_i} - 1 = 2^{k-l_i} \cdot (x_i' + 1) + (\gamma_i' \gamma_{i+1} \ldots \gamma_{\text{last}-1})_2 - 1 < 2^{k-l_i} \cdot (x_i' + 2) + ((0)^{l_{i+1}-l_i} \gamma_{i+1}' \gamma_{i+2} \ldots \gamma_{\text{last}-1})_2$, and, hence,

- $[[p_i' + 1, p_i' + 2^{k-l_i} - 1]] \cap \mathbb{X}_{i+1,l_i} \subseteq \{2^{k-l_i} \cdot (x_i' + 1) + ((0)^{l_{i+1}-l_i} \gamma_{i+1}' \gamma_{i+2} \ldots \gamma_{\text{last}-1})_2\}$.

Left-side energy bound for Y_{i+1}, where $i+1 \leq \text{last} - 1$:

For each j, $j > l_i$, the left-side energy used in $Y_{i+1,j}$ is at most 1.

If the left-side energy used during the time-slots $\bigcup_{j'=0}^{l_{i+1}-1} Y_{i+1,j'}$ is $l_{i+1} - 1$, then

- the left-side energy used during the time-slots $\bigcup_{j'=0}^{l_i} Y_{i+1,j'}$ is 1 and

- $p_{i+1,l_i}' = 2^{k-l_i} \cdot (x_i' + 1) + ((0)^{l_{i+1}-l_i} \gamma_{i+1}' \gamma_{i+2} \ldots \gamma_{\text{last}-1})_2$, and,

- for each j, $l_i + 1 \leq j \leq l_{i+1} - 1$, the left-side energy used during the time-slots $Y_{i+1,j}$ is one and $p_{i+1,j}' = 2^{k-l_i} \cdot (x_i' + 1) + ((1)^{j-l_i} (0)^{l_{i+1}-j} \gamma_{i+1}' \gamma_{i+2} \ldots \gamma_{\text{last}-1})_2$.

  Note that, in this case:

- $p_{i+1,l_{i+1}-1}' = 2^{k-l_i} \cdot (x_i' + 1) + ((1)^{l_{i+1}-1-l_i} 0 \gamma_{i+1}' \gamma_{i+2} \ldots \gamma_{\text{last}-1})_2$, and

- $r' \leq p_i' + 2^{k-l_i} = 2^{k-l_i} \cdot (x_i' + 1) + (\gamma_i' \gamma_{i+1} \ldots \gamma_{\text{last}-1})_2 < 2^{k-l_i} \cdot (x_i' + 1) + ((1)^{l_{i+1}-l_i} \gamma_{i+1}' \gamma_{i+2} \ldots \gamma_{\text{last}-1})_2$, and, hence,

- $[[p_{i+1,l_{i+1}-1}' + 1, r' - 1]] \cap \mathbb{X}_{i+1,l_{i+1}} = \emptyset$, thus, the left-side energy used in time-slots $Y_{i+1,l_{i+1}}$ is zero.

Thus, *the left-side energy used in* Y_{i+1} *is at most* $l_{i+1} - l_i$.

Left-side energy bound for $\bigcup_{j'=0}^{l_i} Y_{i+1,j'}$, where $i+1 = \text{last}$:

Remark: If $i+1 = \text{last}$, then $\gamma_{i+1} \ldots \gamma_{\text{last}-1}$ is an empty sequence.

For $i+1 = \text{last}$*, the left-side energy used during the time-slots* $\bigcup_{j'=0}^{l_i} Y_{i+1,j'}$ *is at most* 1, since:

- $[[\text{lb}_{\min Y_{i+1}}, r' - 1]] \cap \left(\bigcup_{j'=0}^{l_i} X_{i+1,j'}\right) \subseteq [[p_i' + 1, p_i' + 2^{k-l_i} - 1]] \cap \mathbb{X}_{i+1,l_i}$, and

- $p_i' + 1 = 2^{k-l_i} \cdot x_i' + (\gamma_i')_2 + 1 > 2^{k-l_i} \cdot x_i' + ((0)^{l_{i+1}-l_i})_2$, since $(\gamma_i')_2 > ((0)^{l_{i+1}-l_i})_2$, and

- $p_i' + 2^{k-l_i} - 1 = 2^{k-l_i} \cdot (x_i' + 1) + (\gamma_i')_2 - 1 < 2^{k-l_i} \cdot (x_i' + 2) + ((0)^{l_{i+1}-l_i})_2$, and, hence,

- $[[p'_i + 1, p'_i + 2^{k-l_i} - 1]] \cap \mathbb{X}_{i+1,l_i} \subseteq \{2^{k-l_i} \cdot (x'_i + 1) + ((0)^{l_{i+1}-l_i})_2\}$.

Left-side energy bound for Y_{i+1}, where $i+1 = \text{last}$:

For each j, $j > l_i$, the left-side energy used in $Y_{i+1,j}$ is at most 1.

If the left-side energy used during the time-slots $\bigcup_{j'=0}^{l_{i+1}-1} Y_{i+1,j'}$ is $l_{i+1} - 1$, then the left-side energy used during the time-slots $\bigcup_{j'=0}^{l_i} Y_{i+1,j'}$ is 1 and $p'_{i+1,l_i} = 2^{k-l_i} \cdot (x'_i + 1) + ((0)^{l_{i+1}-l_i})_2$, and, for each j, $l_i + 1 \leq j \leq l_{i+1} - 1$, the left-side energy used during the time-slots $Y_{i+1,j}$ is one and $p'_{i+1,j} = 2^{k-l_i} \cdot (x'_i + 1) + ((1)^{j-l_i}(0)^{l_{i+1}-j})_2$. Note that, in this case:

- $p'_{i+1,l_{i+1}-1} = 2^{k-l_i} \cdot (x'_i + 1) + ((1)^{l_{i+1}-1-l_i}0)_2$, and
- $r' \leq p'_i + 2^{k-l_i} = 2^{k-l_i} \cdot (x'_i + 1) + (\gamma'_i)_2 = 2^{k-l_i} \cdot (x'_i + 1) + ((1)^{l_{i+1}-l_i})_2$, and, hence,
- $[[p'_{i+1,l_{i+1}-1} + 1, r' - 1]] \cap \mathbb{X}_{i+1,l_{i+1}} = \emptyset$, thus, the left-side energy used in time-slots $Y_{i+1,l_{i+1}}$ is zero.

Thus, the *left-side energy used in Y_{i+1} is at most $l_{i+1} - l_i$.*

Lemma 2. (Lemma L1.) $\forall_{0 \leq i \leq \text{last}-1} \, \text{le}\, Y_{i+1} \leq l_{i+1} - l_i$.

Total bound for the left-side energy:

Left-side energy is at most $1 + l_0 + \sum_{i=0}^{\text{last}-1} (l_{i+1} - l_i) = 1 + l_{\text{last}} = 1 + k$.

Lemma 3. (Lemma L.) $\text{le}\,\{t : t \in \mathbb{Z} \wedge t \geq s\} \leq k + 1$.

RIGHT-SIDE ENERGY

Let ub_t denote the value of ub just before time-slot t. (For $t \leq s$, we have $\text{ub}_t = 2^k - 1$.)

Note that:

- if $t < t'$ then $\text{ub}_t \geq \text{ub}_{t'}$, and
- RBO receiver uses one unit of right-side energy in time-slot t (i.e. $\text{re}\{t\} = 1$) if and only if $\text{rev}_k(t) \in [[r'' + 1, \text{ub}_t]]$ and $\text{ub}_{t+1} = \text{rev}_k(t) - 1$.
- $\text{re}\,[[t, t+m]] \leq |\, [[r'' + 1, \text{ub}_t]] \cap \text{rev}_k\, [[t, t+m]] \,|$.
- The right-side energy used in Y_{i+1} is not greater than the size of $[[r'' + 1, p''_i - 1]] \cap \mathbb{X}_{i+1} = [[p''_{i+1}, p''_i - 1]] \cap \mathbb{X}_{i+1}$. (I.e. $\text{re}\, Y_{i+1} \leq |\, [[p''_{i+1}, p''_i - 1]] \cap \mathbb{X}_{i+1} \,|$.)
- The right-side energy used in $Y_{i,j+1}$ is not greater than the size of $[[r'' + 1, p''_{i,j} - 1]] \cap \mathbb{X}_{i,j+1} = [[p''_{i,j+1}, p''_{i,j} - 1]] \cap \mathbb{X}_{i,j+1}$. (I.e. $\text{re}\, Y_{i,j+1} \leq |\, [[p''_{i,j+1}, p''_{i,j} - 1]] \cap \mathbb{X}_{i,j+1} \,|$.)
- For $l \in [[0, l_{i+1}]]$, $\text{re} \bigcup_{j'=0}^{l} Y_{i+1,j'} \leq |\, [[p''_{i+1,l}, p''_i - 1]] \cap \mathbb{X}_{i+1,l} \,|$

Bounds on r'':

- $p''_i - 2^{k-l_i} \leq r'' \leq p''_i - 1$
- $p''_{i,j} - 2^{k-j} \leq r'' \leq p''_{i,j} - 1$

Bounds on ub:

After the time-slots $\bigcup_{i'=0}^{i} Y_{i'}$ the value of ub is in $[[r'', p''_i - 1]]$.

After the time-slots $\bigcup_{i'=0}^{i-1} Y_{i'} \cup \bigcup_{j'=0}^{j} Y_{i,j'}$ the value of ub is in $[[r'', p''_{i,j} - 1]]$.

(In other words: if $t \geq \max Y_i + 1$, then $\mathrm{ub}_t \in [[r'', p''_i - 1]]$ and if $t \geq \max Y_{i,j} + 1$, then $\mathrm{lb}_t \in [[r'', p''_{i,j} - 1]]$.)

Right-energy bound for $Y_{i,0}$:

The right-side energy used during the time-slots $Y_{i,0}$ is at most 1, since $|Y_{i,0}| = |X_{i,0}| = 1$.

Right-energy bound for $Y_{i,j+1}$:

For $0 \leq j \leq l_i - 1$, the right-side energy used during the time-slots $Y_{i,j+1}$ is at most 1, since:

- $[[r'' + 1, \mathrm{ub}_{1+\max Y_{i,j}}]] \cap X_{i,j+1} \subseteq [[p''_{i,j} - 2^{k-j} + 1, p''_{i,j} - 1]] \cap \mathbb{X}_{i,j+1}$, and
- $[[p''_{i,j} - 2^{k-j} + 1, p''_{i,j} - 1]] \cap \mathbb{X}_{i,j+1} \subseteq \{p''_{i,j} - 2^{k-j-1}\}$.

Right-side energy bound for Y_0:

Thus, *the right-side energy used during the time-slots Y_0 is at most $1 + l_0$.*

Lemma 4. (Lemma R0.) $\mathrm{re}\, Y_0 \leq 1 + l_0$.

Right-side energy bound for Y_{i+1}, where $i+1 \leq \mathrm{last}$, in the case $2^{k-l_i} \cdot x''_i + \big((0)^{l_{i+1}-l_i} \gamma'_{i+1} \gamma_{i+2} \ldots \gamma_{\mathrm{last}-1}\big)_2 \leq r''$:

Remark: If $i+1 = \mathrm{last}$, then $\gamma'_{i+1} \gamma_{i+2} \ldots \gamma_{\mathrm{last}-1} = \gamma_{i+1} \ldots \gamma_{\mathrm{last}-1}$ is an empty sequence.

We have:

- $2^{k-l_i} \cdot x''_i + ((0)^{l_{i+1}-l_i} \gamma'_{i+1} \gamma_{i+2} \ldots \gamma_{\mathrm{last}-1})_2 \leq r''$, and
- $p''_i = 2^{k-l_i} \cdot x''_i + (\gamma'_i \gamma_{i+1} \ldots \gamma_{\mathrm{last}-1})_2 < 2^{k-l_i} \cdot (x''_i + 1) + ((0)^{l_{i+1}-l_i} \gamma'_{i+1} \gamma_{i+2} \ldots \gamma_{\mathrm{last}-1})_2$.

Thus, $[[r'' + 1, p''_i - 1]] \cap \mathbb{X}_{i,l_i} = \emptyset$.

Since, for $j+1 \in [[l_i + 1, l_{i+1}]]$, the right-side energy used in each $Y_{i+1,j+1}$ is at most one, *the total energy in Y_{i+1} is at most $l_{i+1} - l_i$ in this case.*

Right-side energy bound for Y_{i+1}, where $i+1 = \mathrm{last}$, in the case $r'' < 2^{k-l_i} \cdot x''_i + \big((0)^{l_{i+1}-l_i} \gamma'_{i+1} \gamma_{i+2} \ldots \gamma_{\mathrm{last}-1}\big)_2$:

Remark: If $i+1 = \mathrm{last}$, then $\gamma'_{i+1} \gamma_{i+2} \ldots \gamma_{\mathrm{last}-1} = \gamma_{i+1} \ldots \gamma_{\mathrm{last}-1}$ is an empty sequence.

We have:

- $p''_i - 2^{k-l_i} \le r'' < 2^{k-l_i} \cdot x''_i + ((0)^{l_{i+1}-l_i})_2$, and,
- since $\gamma'_i = (1)^{l_{i+1}-l_i}$, $p''_i - 2^{k-l_i} = 2^{k-l_i} \cdot (x''_i - 1) + ((1)^{l_{i+1}-l_i})_2 = 2^{k-l_i} \cdot x''_i + ((0)^{l_{i+1}-l_i})_2 - 1$.

Thus, $r'' = p''_i - 2^{k-l_i} = 2^{k-l_i} \cdot (x''_i - 1) + ((1)^{l_{i+1}-l_i})_2$ and $r'' + 1 = 2^{k-l_i} \cdot x''_i + ((0)^{l_{i+1}-l_i})_2$.

Recall that $p''_i - 1 = 2^{k-l_i} \cdot x''_i + ((1)^{l_{i+1}-l_i-1}(0))_2$ and $Y_{i+1} = Y_{\text{last}} = [[0, 2^k - 1]]$.

Let $t' = \min\{t \in Y_{\text{last}} : \text{rev}_k(t) \in [[r''+1, p''_i - 1]]\}$. Note that $\text{rev}_k(t') = r'' + 1$ and, hence, *the right-side energy used in* Y_{last} *is at most one in this case.* (After t' the value of ub is r''.)

Right-side energy bound for Y_{i+1}, where $i+1 \le \text{last} - 1$, in the case $r'' < 2^{k-l_i} \cdot x''_i + \left((0)^{l_{i+1}-l_i}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\text{last}-1}\right)_2$:

We have:

- $p''_i = 2^{k-l_i} \cdot x''_i + (\gamma'_i\gamma_{i+1}\ldots\gamma_{\text{last}-1})_2$
- $p''_i - 2^{k-l_i} \le r'' < 2^{k-l_i} \cdot x''_i + ((0)^{l_{i+1}-l_i}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\text{last}-1})_2$.

Thus:

- $p''_i - 1 < 2^{k-l_i} \cdot (x''_i + 1) + ((0)^{l_{i+1}-l_i}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\text{last}-1})_2$, and
- $2^{k-l_i} \cdot (x''_i - 1) + ((0)^{l_{i+1}-l_i}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\text{last}-1})_2 < r'' + 1$.

This implies:

- $[[r''+1, p''_i + 1]] \cap \mathbb{X}_{i+1,l_i} = \{2^{k-l_i} \cdot x''_i + ((0)^{l_{i+1}-l_i}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\text{last}-1})_2\}$ and,
- $p''_{i+1,l_i} = 2^{k-l_i} \cdot x''_i + ((0)^{l_{i+1}-l_i}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\text{last}-1})_2$.

Thus the right-side energy used in $\bigcup_{j'=0}^{l_i} Y_{i+1,j'}$ is at most one.

Since $2^{k-l_i} \cdot (x''_i - 1) + (\gamma'_i\gamma_{i+1}\ldots\gamma_{\text{last}-1})_2 = p''_i - 2^{k-l_i} \le r''$, we have $2^{k-l_i} \cdot (x''_i - 1) + ((1)^{l_{i+1}-l_i-1}(0)\gamma'_{i+1}\ldots\gamma_{\text{last}-1})_2 < r'' + 1$ and, hence, $[[r''+1, p''_{i+1,l_i}]] \cap \mathbb{X}_{i+1} \subseteq \{2^{k-l_i} \cdot (x''_i - 1) + ((1)^{l_{i+1}-l_i}\gamma'_{i+1}\ldots\gamma_{\text{last}-1})_2\}$. Thus, the right-side energy used in $Y_{i+1} \setminus \left(\bigcup_{j'=0}^{l_i} Y_{i+1,j'}\right)$ is at most one.

Hence, *the total right-side energy used in* Y_{i+1} *is at most two in this case.*

The relation between the right-side energy used in Y_{i+1} and p''_{i+1}:

It follows that

Lemma 5. (Lemma R1.) *For* $0 \le i \le \text{last} - 1$, *either the right-side energy used in* Y_{i+1} *is at most* $l_{i+1} - l_i$ *(i.e.* $\text{re}\, Y_{i+1} \le l_{i+1} - l_i$*) or:*

- $i + 1 < \text{last}$, *and*
- $l_{i+1} - l_i = 1$, *and*
- *the right-side energy used in* Y_{i+1} *is two (i.e.* $\text{re}\, Y_{i+1} = 2$*), and*

- $\mathrm{ub}_{1+\max Y_{i+1}}+1=p''_{i+1}=2^{k-l_i}\cdot(x''_i-1)+((1)^{l_{i+1}-l_i}\gamma'_{i+1}\ldots\gamma_{\mathrm{last}-1})_2$.

Compensations for the cases, when the right-side energy used in Y_{i+1} is two and $l_{i+1}-l_i=1$:

In the following, we show that all the cases (except the last one) when the right-side energy used in Y_{i+1} is greater than $l_{i+1}-l_i$ must be compensated by using less than $l_{i+2+c}-l_{i+1+c}$ of right-side energy in Y_{i+2+c}, for some $c\geq 0$.

Bound on the right-side energy used in Y_{i+1} and the values of x''_{i+1} and $\mathrm{ub}_{1+\max Y_{i+1}}$, in the case when the energy used in Y_{i+1} is at least $l_{i+1}-l_i$ and $x''_{i+1}\geq\big(\mathrm{bin}(x''_i)(0)^{l_{i+1}-l_i-1}(1)\big)_2$:

We show that, *in this case, the right-side energy is exactly* $l_{i+1}-l_i$ *and* $x''_{i+1}=(\mathrm{bin}(x''_i)(0)^{l_{i+1}-l_i-1}(1))_2$ *and* $\mathrm{ub}_{1+\max Y_{i+1}}=p''_{i+1}-1$.

Lemma 6. (Lemma S.) *If* $\mathrm{re}\,Y_{i+1}\geq l_{i+1}-l_i$ *and* $x''_{i+1}\geq(\mathrm{bin}(x''_i)(0)^{l_{i+1}-l_i-1}(1))_2$ *then*

- $\mathrm{re}\,Y_{i+1}=l_{i+1}-l_i$ *and*
- $x''_{i+1}=(\mathrm{bin}(x''_i)(0)^{l_{i+1}-l_i-1}(1))_2$ *and*
- $\mathrm{ub}_{1+\max Y_{i+1}}=p''_{i+1}-1$.

Proof:

Since $p''_{i+1}\geq(\mathrm{bin}(x''_i)(0)^{l_{i+1}-l_i-1}(1)\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\mathrm{last}-1})_2$ and $p''_{i+1}-2^{k-l_{i+1}}\leq r''$, we have $(\mathrm{bin}(x''_i)(0)^{l_{i+1}-l_i}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\mathrm{last}-1})_2\leq r''$.

We also have

$r''<p''_i=(\mathrm{bin}(x''_i)(1)^{l_{i+1}-l_i}\gamma_{i+1}\gamma_{i+2}\ldots\gamma_{\mathrm{last}-1})_2<(\mathrm{bin}(x''_i+1)(0)^{l_{i+1}-l_i}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\mathrm{last}-1})_2$.

Hence,

- $p''_{i+1,l_i}=(\mathrm{bin}(x''_i+1)(0)^{l_{i+1}-l_i}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\mathrm{last}-1})_2$ and
- $[[p''_{i+1,l_i},p''_i-1]]\cap\mathbb{X}_{i+1,l_i}=\emptyset$ and
- the right-side energy used in $\bigcup_{j'=0}^{l_i}Y_{j'}$ is zero and
- $p''_{i+1,l_i}\geq p''_i\geq\mathrm{ub}_{1+\max Y_i}+1=\mathrm{ub}_{1+\max Y_{i+1,l_i}}+1$.

Since the right-side energy used in Y_{i+1} is at least $l_{i+1}-l_i$ and the right-side energy used in each $Y_{i+1,j}$ is at most one, the right-side energy used in each $Y_{i+1,j+1}$, for each $j+1\in[[l_i+1,l_{i+1}]]$, must be exactly one.

Thus, for each $j+1\in[[l_i+1,l_{i+1}]]$, we must have $\mathrm{ub}_{i+1,j+1}+1=p''_{i+1,j+1}<p''_{i+1,j}$.

Note that, $p''_{i+1,j+1}\in\{p''_{i+1,j},p''_{i+1,j}-2^{k-j-1}\}$. Thus, for each $j+1\in[[l_i+1,l_{i+1}]]$, we must have $p''_{i+1,j+1}=p''_{i+1,j}-2^{k-j-1}$.

Since $p''_{i+1,l_i}=(\mathrm{bin}(x''_i+1)(0)^{l_{i+1}-l_i}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\mathrm{last}-1})_2$, it follows by induction that, for each $j+1\in[[l_i+1,l_{i+1}]]$, $p''_{i+1,j+1}=(\mathrm{bin}(x''_i)(0)^{j-l_i}(1)(0)^{l_{i+1}-j-1}\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\mathrm{last}-1})_2$.

Thus, for $j+1=l_{i+1}$, we have $p''_{i+1,j+1}=(\mathrm{bin}(x''_i)(0)^{l_{i+1}-l_i-1}(1)\gamma'_{i+1}\gamma_{i+2}\ldots\gamma_{\mathrm{last}-1})_2=p''_{i+1}$.

Compensation Lemma:

Lemma 7. (Lemma C.) *If, for some* $i \in [[0, \text{last} - 2]]$,

- $x''_{i+1} = (\text{bin}(x''_i - 1)(1))_2$, *and,*
- *for some* d, *where* $i + 2 + d \leq \text{last}$, *for each* $c \in [[0, d]]$, *the right-side energy used in* Y_{i+2+c} *is at least* $l_{i+2+c} - l_{i+1+c}$,

then, for each $c \in [[0, d]]$, *we have*

- $l_{i+2+c} - l_{i+1+c} \leq 2$, *and*
- *right-side energy used in* Y_{i+2+c} *is exactly* $l_{i+2+c} - l_{i+1+c}$, *and*
- $x''_{i+2+c} = (\text{bin}(x''_{i+1})\gamma_{i+1} \ldots \gamma_{i+1+c})_2$, *and*
- $p''_{i+2+c} = \text{ub}_{1+\max Y_{i+2+c}} + 1$.

Proof:

Note that

- $p''_i = (\text{bin}(x''_i)\gamma'_i\gamma_{i+1} \ldots \gamma_{\text{last}-1})_2$, and, hence,
- $r'' \geq p''_i - 2^{k-l_i} = (\text{bin}(x''_i - 1)\gamma'_i\gamma_{i+1} \ldots \gamma_{\text{last}-1})_2$.

Since $r'' \geq (\text{bin}(x''_i - 1)\gamma'_i\gamma_{i+1} \ldots \gamma_{\text{last}-1})_2$, we also have, for arbitrary $c' \in [[0, \text{last} - 1 - i]]$,

$$p''_{i+1+c} > r'' \geq (\text{bin}(x''_i - 1)\gamma_i\gamma_{i+1} \ldots \gamma_{i+c'}\gamma'_{i+c'+1}\gamma_{i+c'+2} \ldots \gamma_{\text{last}-1})_2.$$

(Note that if $c' = \text{last} - 2 - i$, then $\gamma_{i+c'+2} \ldots \gamma_{\text{last}-1}$ is an empty sequence, and if $c' = \text{last} - 1 - i$, then $\gamma'_{i+c'+1}\gamma_{i+c'+2} \ldots \gamma_{\text{last}-1}$ is an empty sequence.)

Hence, we also have

Proposition 8. (Proposition CP.) $x''_{i+c'+1} \geq (\text{bin}(x''_i - 1)\gamma_i\gamma_{i+1} \ldots \gamma'_{i+c'})_2$.

We proof the Lemma 7 (Lemma C) is by induction on c. However, we start by noting that, for $c = -1$, we have $x''_{i+2+c} = (\text{bin}(x''_{i+1}))_2$.

The induction step for $c \in [[0, d]]$:

By inductive assumption we have $x''_{i+1+c} = (\text{bin}(x''_{i+1})\gamma_{i+1} \ldots \gamma_{i+c})_2$.

Case $\boldsymbol{l_{i+2+c} - l_{i+1+c} = 1}$:

Then $\gamma_{i+1+c} = (0)$ and

$p''_{i+1+c} = (\text{bin}(x''_{i+1})\gamma_{i+1} \ldots \gamma_{i+c}(1)\gamma_{i+c+2} \ldots \gamma_{\text{last}-1})_2$, since $\gamma'_{i+1+c} = (1)$.

Let $p = (\mathrm{bin}(x''_{i+1})\gamma_{i+1}...\gamma_{i+c}(0)\gamma'_{i+c+2}...\gamma_{\mathrm{last}-1})_2$.

Note that $p \in \mathbb{X}_{i+2+c}$ and

the minimal element greater than p in $\mathbb{X}_{i+2+c}$ is $(\mathrm{bin}(x''_{i+1})\gamma_{i+1}...\gamma_{i+c}(1)\gamma'_{i+c+2}...\gamma_{\mathrm{last}-1})_2$,

which is greater than $p''_{i+1+c} - 1$.

In other words: $[[p, p''_{i+1+c} - 1]] \cap \mathbb{X}_{i+2+c} = \{p\}$.

We have $p''_{i+2+c} \geq p$, since, by Proposition 8 (Proposition CP),

$x''_{i+2+c} \geq (\mathrm{bin}(x''_{i+1})\gamma_{i+1}...\gamma_{i+c}\gamma_{i+1+c})_2 = (\mathrm{bin}(x''_{i+1})\gamma_{i+1}...\gamma_{i+c}(0))_2$.

Since the right-side energy used during Y_{i+2+c} is at least one, we must have

$\mathrm{ub}_{1+\max Y_{i+2+c}} + 1 = p''_{i+2+c} = p = (\mathrm{bin}(x''_{i+1})\gamma_{i+1}...\gamma_{i+c}(0)\gamma'_{i+c+2}...\gamma_{\mathrm{last}-1})_2$.

Case $l_{i+2+c} - l_{i+1+c} = 2$:

Then $\gamma_{i+1+c} = (01) = (0)^{l_{i+2+c} - l_{i+1+c} - 1}(1)$.

The right-side energy during Y_{i+2+c} is at least $l_{i+2+c} - l_{i+1+c}$ and, by Proposition 8 (Proposition CP),

$$x''_{i+2} \geq (\mathrm{bin}(x''_{i+1})\gamma_{i+1}...\gamma_{i+c}\gamma_{i+1+c})_2 = (\mathrm{bin}(x''_{i+1+c})\gamma_{i+1+c})_2.$$

Hence, by Lemma 6 (Lemma S),

- the right-side energy during Y_{i+2+c} is exactly $l_{i+2+c} - l_{i+1+c}$, and
- $\mathrm{ub}_{1+\max Y_{i+2+c}} + 1 = p''_{i+2+c}$, and
- $x''_{i+2+c} = (\mathrm{bin}(x''_{i+1})\gamma_{i+1}...\gamma_{i+1+c})_2$.

Case $l_{i+2+c} - l_{i+1+c} \geq 2$:

Let $q = l_{i+2+c} - l_{i+1+c} - 1$.

We have $q > 1$ and $\gamma_{i+1+c} = (0)(1)^q$.

The right-side energy used in Y_{i+2+c} is at least $l_{i+2+c} - l_{i+1+c}$ and, by Proposition 8 (Proposition CP),

$$x''_{i+2+c} \geq (\mathrm{bin}(x''_{i+1})\gamma_{i+1}...\gamma_{i+1+c})_2 \geq (\mathrm{bin}(x''_{i+1+c})(0)^q(1))_2.$$

Hence, by Lemma 6 (Lemma S), we have $x''_{i+2+c} = (\mathrm{bin}(x''_{i+1+c})(0)^q(1))_2$.

However, by Proposition 8 (Proposition CP),

$$\begin{aligned} x''_{i+2+c} &\geq (\mathrm{bin}(x''_{i+1})\gamma_{i+1}...\gamma_{i+1+c})_2 \\ &= (\mathrm{bin}(x''_{i+1+c})(0)(1)^q)_2 \\ &> (\mathrm{bin}(x''_{i+1+c})(0)^q(1))_2, \end{aligned}$$

where the last inequality follows from: $q > 1$.

Thus we have contradiction and the case $l_{i+2+c} - l_{i+1+c} \geq 2$ is impossible.

Total right-side energy:

Let i be such that the right-side energy used in Y_{i+1} is greater than $l_{i+1} - l_i$.

Then $l_{i+1} - l_i = 1$ and the right-side energy used in Y_{i+1} is two, and $\mathrm{ub}_{1+\max Y_{i+1}} + 1 = p''_{i+1} = 2^{k-l_i} \cdot (x''_i - 1) + ((1)^{l_{i+1}-l_i} \gamma'_{i+1} \ldots \gamma_{\mathrm{last}-1})_2$.

Thus, $x''_{i+1} = (\mathrm{bin}(x''_i - 1)(1))_2$.

Let d be the maximal integer value such that, $i + 2 + d \leq \mathrm{last}$ and, for each $c \in [[0, d]]$, the right-side energy used in Y_{i+2+c} is at least $l_{i+2+c} - l_{i+1+c}$. (Note that $d \geq -1$.)

Then, by Lemma 7 (Lemma C), for each $c \in [[0, d]]$, the right-side energy used in Y_{i+2+c} is exactly $l_{i+2+c} - l_{i+1+c}$.

Let i' be the minimal integer such that $i < i' < \mathrm{last}$ and the right-side energy used in $Y_{i'+1}$ is greater than $l_{i'+1} - l_{i'}$.

It follows that there must be some integer i'', such that $i + 2 + d < i'' + 1 < i' + 1$ and the right-side energy in $Y_{i''+1}$ is at most $l_{i''+1} - l_{i''} - 1$.

Thus, *the total right-side energy is at most:*

$(1 + l_0) + \left(\sum_{i=0}^{\mathrm{last}-1} l_{i+1} - l_i\right) + 1 \leq l_{\mathrm{last}} + 2 = k + 2.$

Lemma 9. (Lemma R.) $\mathrm{re}\,\{t : t \in \mathbb{Z} \wedge t \geq s\} \leq k + 2.$

EXTERNAL ENERGY IN RELIABLE NETWORK:

By Lemma 3 (Lemma L) and Lemma 9 (Lemma R), we have the bound $2k + 3$ on the total external energy:

Theorem 10. (Theorem E.) $\mathrm{le}\,\{t : t \in \mathbb{Z} \wedge t \geq s\} + \mathrm{re}\,\{t : t \in \mathbb{Z} \wedge t \geq s\} \leq 2k + 3.$

ENERGY IN UNRELIABLE NETWORK

In *unreliable* network, in each time-slot when the receiver listens, the receiver successfully receives the key with some probability p, where $0 < p < 1$.

For each $t \in \mathbb{Z}$, let $\mathrm{success}_{p,t}$ be a random variable such that $\mathrm{success}_{p,t} = 1$ with probability p and $\mathrm{success}_{p,t} = 0$ with probability $1 - p$. Let q denote $1 - p$.

The receiver successfully receives a key in time-slot t if and only if the receiver listens in time-slot t and $\mathrm{success}_{p,t} = 1$.

Let $\mathrm{lb}_{p,t}$ and $\mathrm{ub}_{p,t}$ denote the values of variables lb and ub, respectively, just before the time slot t.

Note that, for each t, $\mathrm{lb}_{p,t}$ and $\mathrm{ub}_{p,t}$ are random variables.

For each $t \geq s$, we have:

- if $\mathrm{rev}_k(t) \in [[\mathrm{lb}_{p,t}, r' - 1]]$, then
  - if $\mathrm{success}_{p,t} = 1$, then $\mathrm{lb}_{p,t+1} = \mathrm{rev}_k(t) + 1$ else $\mathrm{lb}_{p,t+1} = \mathrm{lb}_{p,t}$,

  and
- if $\mathrm{rev}_k(t) \in [[r'' + 1, \mathrm{ub}_{p,t}]]$ then
  - if $\mathrm{success}_{p,t} = 1$, then $\mathrm{ub}_{p,t+1} = \mathrm{rev}_k(t) + 1$ else $\mathrm{ub}_{p,t+1} = \mathrm{ub}_{p,t}$.

In *reliable* network we had $\mathrm{lb}_{\max Y_{i,j}+1} - 1 \geq p'_{i,j}$ and $\mathrm{ub}_{\max Y_{i,j}+1} + 1 \leq p''_{i,j}$.

In *unreliable* network we can show the corresponding bounds on the expected values of $\mathrm{lb}_{p,\max Y_{i,j}+1} - 1$ and $\mathrm{ub}_{p,\max Y_{i,j}+1} + 1$:

Lemma 11. (Lemma U1) *For $i \in [[0, \mathrm{last}]]$, for $j \in [[0, l_i]]$, we have*

- $\mathrm{EX}[\, \mathrm{lb}_{p,\max Y_{i,j}+1} - 1\,] \geq p'_{i,j} - \frac{q}{p} \cdot 2^{k-j}$ *and*
- $\mathrm{EX}[\, \mathrm{ub}_{p,\max Y_{i,j}+1} + 1\,] \leq p''_{i,j} + \frac{q}{p} \cdot 2^{k-j}$,

where $q = p - 1$.

Proof.

We have $\mathrm{lb}_{p,\max Y_{i,j}+1} - 1 = \max(\{\mathrm{lb}_{p,\min Y_i} - 1\} \cup A)$, where

$A = \{\mathrm{rev}_k(t) : t \in \bigcup_{j'=0}^{j} Y_{i,j'} \wedge \mathrm{success}_{p,t} = 1\} \cap [[0, p'_{i,j}]]$.

Since $\bigcup_{j'=0}^{j} X_{i,j'} = \mathbb{X}_{i,j} \cap [[0, 2^k - 1]]$ and $\mathbb{X}_{i,j} = \{p'_{i,j} + 2^{k-j} - d \cdot 2^{k-j} : d \in \mathbb{Z}\}$, we have

$$\begin{aligned}\mathrm{rev}_k\left(\bigcup_{j'=0}^{j} Y_{i,j'}\right) &= \bigcup_{j'=0}^{j} X_{i,j'} \\ &= \{p'_{i,j} + 2^{k-j} - d \cdot 2^{k-j} : d \in \mathbb{Z}\} \cap [[0, 2^k - 1]].\end{aligned}$$

Recall that we also have: $p'_{i,j} < r' \leq p'_{i,j} + 2^{k-j}$.

For $t \in \bigcup_{j'=0}^{j} Y_{i,j'}$, let $d_t = (p'_{i,j} + 2^{k-j} - \mathrm{rev}_k(t))/2^{k-j}$.

Thus, $\mathrm{rev}_k(t) = p'_{i,j} + 2^{k-j} - d_t \cdot 2^{k-j}$.

Note that

$$\begin{aligned}\left\{\mathrm{rev}_k(t) : t \in \bigcup_{j'=0}^{j} Y_{i,j'} \wedge d_t \geq 1\right\} &= \left\{\mathrm{rev}_k(t) : t \in \bigcup_{j'=0}^{j} Y_{i,j'} \wedge \mathrm{rev}_k(t) < r'\right\} \\ &= \left(\bigcup_{j'=0}^{j} X_{i,j'}\right) \cap [[0, r'-1]].\end{aligned}$$

Let $g = \min \{d : d \in \mathbb{Z} \wedge p'_{i,j} + 2^{k-j} - d \cdot 2^{k-j} \leq -1\}$.

Since $p' + 2^{k-j} \geq 0$, we have $g \in \mathbb{Z}$, $g \geq 1$.

Let $d' = \min \{d : d \in \mathbb{Z} \wedge d \geq 1 \wedge p'_{i,j} + 2^{k-j} - d \cdot 2^{k-j} \leq \mathrm{lb}_{p,\max Y_{i,j}+1} - 1\}$.

We have

$$\begin{aligned}\mathrm{EX}[\mathrm{lb}_{p,\max Y_{i,j}+1} \quad - 1] &\geq p'_{i,j} + 2^{k-j} - \mathrm{EX}[d'] \cdot 2^{k-j} \\ &= p'_{i,j} - \mathrm{EX}[d'-1] \cdot 2^{k-j}.\end{aligned}$$

Since $\mathrm{lb}_{p,\max Y_{i,j}+1} - 1 \geq -1$, we have

$$d' = \min\left(\left\{d_t : t \in \bigcup_{j'=0}^{j} Y_{i,j'} \wedge d_t \geq 1 \wedge \mathrm{success}_{p,t} = 1\right\} \cup \{g\}\right).$$

Since $p'_{i,j} + 2^{k-j} - (g-1) \cdot 2^{k-j} \geq -1 + 2^{k-j} \geq 0$, we have

$$[[1, g-1]] \subseteq \left\{d_t \mid t \in \bigcup_{j'=0}^{j} Y_{i,j'} \wedge d_t \geq 1\right\}.$$

Remark: For $0 < p < 1$, we have $\sum_{i=1}^{+\infty} (1-p)^{i-1} = \sum_{i=0}^{+\infty} (1-p)^i = \frac{1}{1-(1-p)} = \frac{1}{p}$.

Thus:

$$\begin{aligned}1 &= \sum_{i=1}^{+\infty} (1-p)^{i-1} \cdot p \\ &= \sum_{i \in [[1,g-1]]} (1-p)^{i-1} \cdot p + \sum_{i=g}^{+\infty} (1-p)^{i-1} \cdot p \\ &= \sum_{i \in [[1,g-1]]} (1-p)^{i-1} \cdot p + (1-p)^{g-1} \cdot \sum_{i=1}^{+\infty} (1-p)^{i-1} \cdot p \\ &= \sum_{i \in [[1,g-1]]} (1-p)^{i-1} \cdot p + (1-p)^{g-1} \cdot 1.\end{aligned}$$

For $i \in [[1, g-1]]$, the probability that $d' = i$ is $(1-p)^{i-1} \cdot p$.

If $d' \notin [[1, g-1]]$, then $d' = g$ and this happens with probability $(1-p)^{g-1}$.

Hence, we have

$$\begin{array}{rcl} \mathrm{EX}[d'] &=& \sum_{i \in [[1, g-1]]} i \cdot (1-p)^{i-1} \cdot p + g \cdot (1-p)^{g-1} \\ &=& \sum_{i \in [[1, g-1]]} i \cdot (1-p)^{i-1} \cdot p + g \cdot \sum_{i=g}^{+\infty} (1-p)^{i-1} \cdot p \\ &\leq& \sum_{i \in [[1, g-1]]} i \cdot (1-p)^{i-1} \cdot p + \sum_{i=g}^{+\infty} i \cdot (1-p)^{i-1} \cdot p \\ &=& \sum_{i \geq 1} i \cdot (1-p)^{i-1} \cdot p \\ &=& \frac{1}{p}. \end{array}$$

(Last equality is by geometric distribution.)

Thus we have $\mathrm{EX}[d'-1] \leq \frac{1}{p} - 1$ and, hence,

$$\begin{array}{rcl} \mathrm{EX}[\mathrm{lb}_{p, \max Y_{i,j+1}} - 1] &\geq& p'_{i,j} - \mathrm{EX}[d'-1] \cdot 2^{k-j} \\ &\geq& p'_{i,j} - (\frac{1}{p} - 1) \cdot 2^{k-j} \\ &=& p'_{i,j} - \frac{q}{p} \cdot 2^{k-j}. \end{array}$$

For last equality, observe that $(\frac{1}{p} - 1) = \frac{q}{p}$, where $q = 1 - p$.

By analogy: $\mathrm{EX}[\ \mathrm{ub}_{p, \max Y_{i,j+1}} + 1] \leq p''_{i,j} + \frac{q}{p} \cdot 2^{k-j}$.

□

Let $\mathrm{le}_{p,t}$ and $\mathrm{re}_{p,t}$ be random variables defined as follows:

- if $\mathrm{rev}_k(t) \in [[\mathrm{lb}_{p,t}, r' - 1]]$ then $\mathrm{le}_{p,t} = 1$ else $\mathrm{le}_{p,t} = 0$, and
- if $\mathrm{rev}_k(t) \in [[r'' + 1, \mathrm{ub}_{p,t}]]$ then $\mathrm{rb}_{p,t} = 1$ else $\mathrm{rb}_{p,t} = 0$.

Note that, for each set of time-slots Y,

- the left-side energy used in Y is $\sum_{t \in Y} \mathrm{le}_{p,t}$, and
- the right-side energy used in Y is $\sum_{t \in Y} \mathrm{re}_{p,t}$.

In *reliable* network we had $\mathrm{le}\, Y_{i,j+1} \leq 1$ and $\mathrm{re}\, Y_{i,j+1} \leq 1$.

We show the corresponding bounds on the expected energy costs for *unreliable* network:

Lemma 12. (Lemma U2) *If* $j \in [[0, l_i - 1]]$, *then*

- $\mathrm{EX}\Big[\sum_{t \in Y_{i,j+1}} \mathrm{le}_{p,t}\Big] \leq \frac{1}{p} + 1$, *and*
- $\mathrm{EX}\Big[\sum_{t \in Y_{i,j+1}} \mathrm{re}_{p,t}\Big] \leq \frac{1}{p} + 1$.

Proof.

By Lemma 11 (Lemma U1), we have

$$\begin{aligned} \mathrm{EX}[\,\mathrm{lb}_{p,\max Y_{i,j}+1}-1] &\geq p'_{i,j}-\frac{1-p}{p}\cdot 2^{k-j} \\ &= p'_{i,j}+2^{k-j}-\frac{1}{p}\cdot 2^{k-j}. \end{aligned}$$

We also have $r'-1<p'_{i,j}+2^{k-j}$.

Thus $\mathrm{EX}[r'-1-(\mathrm{lb}_{p,\max Y_{i,j}+1}-1)]<\frac{1}{p}\cdot 2^{k-j}$.

Let $A=[[\mathrm{lb}_{p,\max Y_{i,j}+1}, r'-1]]$.

If the receiver listens in time slot $t\in Y_{i,j+1}$, then $\mathrm{rev}_k(t)\in X_{i,j+1}\cap A$.

Thus $\sum_{t\in Y_{i,j+1}} \mathrm{le}_{p,t}\leq |\,\mathrm{rev}_k(X_{i,j+1})\cap A\,|$.

Recall that the minimal distance between the elements of $X_{i,j+1}$ is $2^{k-(j+1)+1}=2^{k-j}$.

Let $B=\{u\in\mathbb{Z}:\ \min X_{i,j+1}+u\cdot 2^{k-j}\in A\}$.

Then $B=[[\,\lceil(\mathrm{lb}_{p,\max Y_{i,j}+1}-\min X_{i,j+1})/2^{k-j}\rceil, \lfloor((r'-1)-\min X_{i,j+1})/2^{k-j}\rfloor\,]]$.

Since $X_{i,j+1}\cap A\subseteq\{x:\ x=\min X_{i,j+1}+u\cdot 2^{k-j}\wedge u\in B\}$, we have:

$$\begin{aligned} |X_{i,j+1}\cap A| &\leq |B| \\ &= \lfloor((r'-1)-\min X_{i,j+1})/2^{k-j}\rfloor-\lceil(\mathrm{lb}_{p,\max Y_{i,j}+1}-\min X_{i,j+1})/2^{k-j}\rceil+1 \\ &\leq (r'-1-\mathrm{lb}_{p,\max Y_{i,j}+1})/2^{k-j}+1. \end{aligned}$$

Thus:

$$\begin{aligned} \mathrm{EX}\left[\sum_{t\in Y_{i,j+1}} \mathrm{le}_{p,t}\right] &\leq \mathrm{EX}[\,|\,\mathrm{rev}_k(X_{i,j+1})\cap A\,|\,] \\ &\leq \mathrm{EX}[r'-1-\mathrm{lb}_{p,\max Y_{i,j}+1}]/2^{k-j}+1 \\ &\leq 1/p+1 \end{aligned}$$

By analogy: $\mathrm{EX}\left[\sum_{t\in Y_{i,j+1}} \mathrm{re}_{p,t}\right]\leq 1/p+1$.

□

In *reliable* network we had $\mathrm{le}\bigcup_{j=0}^{l_i} Y_{i+1,j}\leq 1$ and $\mathrm{re}\bigcup_{j=0}^{l_i} Y_{i+1,j}\leq 1$.

We show the corresponding bounds on the expected energy costs for *unreliable* network:

Lemma 13. (Lemma U3) *If* $i<\mathrm{last}$ *then*

- $\mathrm{EX}\left[\sum_{t\in[[\min Y_{i+1},\max Y_{i+1,l_i}]]} \mathrm{le}_{p,t}\right]\leq 1/p+1$ *and*
- $\mathrm{EX}\left[\sum_{t\in[[\min Y_{i+1},\max Y_{i+1,l_i}]]} \mathrm{re}_{p,t}\right]\leq 1/p+1$.

Proof.

By Lemma 11 (Lemma U1) we have $\mathrm{EX}[\mathrm{lb}_{p,\max Y_{i,l_i}+1}-1]\geq p'_{i,l_i}+2^{k-l_i}-\frac{1}{p}\cdot 2^{k-l_i}$.

We also have $r'-1<p'_{i,l_i}+2^{k-l_i}$.

Thus $\mathrm{EX}[r'-1-\mathrm{lb}_{p,\max Y_{i,l_i}+1}+1]<\frac{1}{p}\cdot 2^{k-l_i}$.

Note that $\max Y_{i,l_i}+1=\max Y_i+1=\min Y_{i+1}$.

Let $A=[[\mathrm{lb}_{p,\min Y_{i+1}}, r'-1]]$.

If the receiver listens in time slot $t\in Y_{i,j+1}$, then $\mathrm{rev}_k(t)\in(\mathrm{rev}_k\,[[\min Y_{i+1},\max Y_{i+1,l_i}]])\cap A$.

We have $\mathrm{rev}_k\,[[\min Y_{i+1},\max Y_{i+1,l_i}]]\subseteq\bigcup_{j=0}^{l_i} X_{i+1,j}\subset\mathbb{X}_{i+1,l_i}$.

Recall that the minimal distance between elements of $\mathbb{X}_{i+1,l_i}$ is 2^{k-l_i} and

$\min X_{i+1}=\min X_{i+1,0}\in\mathbb{X}_{i+1,l_i}$.

Thus

$$\mathbb{X}_{i+1,l_i}\cap A \;=\; \{x\mid\exists_{u\in\mathbb{Z}}\, x=\min X_{i+1}+2^{k-l_i}\cdot u\}\cap A,$$

and

$$\begin{aligned}|\mathbb{X}_{i+1,l_i}\cap A| &\le \lfloor((r'-1)-\min X_{i+1})/2^{k-j}\rfloor-\lceil(\mathrm{lb}_{p,\min Y_{i+1}}-\min X_{i+1})/2^{k-j}\rceil+1\\ &\le (r'-1-\mathrm{lb}_{p,\min Y_{i+1}})/2^{k-j}+1.\end{aligned}$$

Thus $\mathrm{EX}[\,|\mathbb{X}_{i+1,l_i}\cap A|\,]\le 1/p+1$ and, hence $\mathrm{EX}\Big[\sum_{t\in[[\min Y_{i+1},\max Y_{i+1,l_i}]]}\mathrm{le}_{p,t}\Big]\le 1/p+1$.

By analogous reasoning, we have $\mathrm{EX}\Big[\sum_{t\in[[\min Y_{i+1},\max Y_{i+1,l_i}]]}\mathrm{re}_{p,t}\Big]\le 1/p+1$.

□

In *reliable* network we had $\mathrm{le}\,[[s,s+2^k-1]]\le k+1$ and $\mathrm{re}\,[[s,s+2^k-1]]\le k+2$.

We show the corresponding bounds on the expected energy costs for *unreliable* network:

Lemma 14. (Lemma U4)

- $\mathrm{EX}\Big[\sum_{t\in[[\min Y_0,\max Y_{\mathrm{last}}]]}\mathrm{le}_{p,t}\Big]\le(1/p+1)\cdot(2k+1)$, *and*
- $\mathrm{EX}\Big[\sum_{t\in[[\min Y_0,\max Y_{\mathrm{last}}]]}\mathrm{re}_{p,t}\Big]\le(1/p+1)\cdot(2k+1)$.

Proof.

We have $|Y_{0,0}|=1<(1/p+1)$ and, for $j\in[[1,l_0]]$, by Lemma 12 (Lemma U2), $\mathrm{EX}\Big[\sum_{t\in Y_{0,j}}\mathrm{le}_t\Big]\le(1/p+1)$.

Thus

$$\mathrm{EX}\Bigg[\sum_{t\in Y_0}\mathrm{le}_t\Bigg]\le(1/p+1)\cdot(l_0+1).$$

For $i\in[[0,\mathrm{last}-1]]$, by Lemma 13 (Lemma U3), $\mathrm{EX}\Big[\sum_{t\in[[\min Y_{i+1,0},\max Y_{i+1,l_i}]]}\mathrm{le}_{p,t}\Big]\le 1/p+1$.

For each $j\in[[l_i+1,l_{i+1}]]$, by Lemma 12 (Lemma U2), $\mathrm{EX}\Big[\sum_{t\in Y_{i+1,j}}\mathrm{le}_t\Big]\le(1/p+1)$.

Thus

$$\mathrm{EX}\Bigg[\sum_{t\in Y_{i+1}}\mathrm{le}_t\Bigg]\le(1/p+1)\cdot(l_{i+1}-l_i+1).$$

It follows that

$$\begin{aligned}
\mathrm{EX}\left[\sum_{t\in[[\min Y_0,\max Y_{\mathrm{last}}]]} \mathrm{le}_{p,t}\right] &= \mathrm{EX}\left[\sum_{i\in[[0,\mathrm{last}]]}\sum_{t\in Y_i} \mathrm{le}_t\right] \\
&\le (1/p+1)\cdot\left(l_0+1+\sum_{i=1}^{\mathrm{last}}(l_i-l_{i-1}+1)\right) \\
&= (1/p+1)\cdot(\mathrm{last}+1+l_{\mathrm{last}}) \\
&= (1/p+1)\cdot(2\cdot k+1).
\end{aligned}$$

By analogy: $\mathrm{EX}\left[\sum_{t\in[[\min Y_0,\max Y_{\mathrm{last}}]]}\mathrm{re}_t\right]\le(1/p+1)\cdot(2k+1)$.

□

In *reliable* network we had $\mathrm{le}\,\{t\in\mathbb{Z}:\, t\ge s+2^k\}=0$ and $\mathrm{re}\,\{t\in\mathbb{Z}:\, t\ge s+2^k\}=0$.

We show the corresponding bounds on the expected energy costs for *unreliable* network:

Lemma 15. (Lemma U5)

$\mathrm{EX}[\ \sum_{t\ge s+n}(\mathrm{le}_{p,t}+\mathrm{re}_{p,t})\ ]\le 2q/p^2$, *where* $q=1-p$.

Proof.

Recall that $n=2^k$ is the length of broadcast cycle.

For integer $i\ge1$, each of the values $r'-\mathrm{lb}_{p,s+i\cdot n}$ and $\mathrm{ub}_{p,s+i\cdot n}-r''$

is a random variable.

For each $x\in[[0,n-1]]$, let the event $F_i(x)$ be:

"For each $t\in[[s,s+i\cdot n-1]]$ such that $\mathrm{rev}_k(t)=x$, $\mathrm{success}_{p,t}=0$."

We have $\mathrm{rev}_k(x)=\mathrm{rev}_k(x+n)$ and,

for each integer j, $\mathrm{rev}_k\,[[s+j\cdot n, s+(j+1)\cdot n-1]]=[[0,n-1]]$.

Thus, for each $x\in[[0,n-1]]$,

there are exactly i time-slots t in $[[s,s+i\cdot n-1]]$ such that $\mathrm{rev}_k(t)=x$ and

the probability of the event $F_i(x)$ is q^i.

By the definition, $\mathrm{ub}_{p,s+i\cdot n}$ is the maximal $u\in[[r'',n-1]]$ such that,

for each $x\in[[r''+1,u]]$, $F_i(x)$ is true.

Hence, the expected value of $\mathrm{ub}_{p,s+i\cdot n}+1-(r''+1)=\mathrm{ub}_{p,s+i\cdot n}-r''$ is not greater than

$\sum_{j\ge1} j\cdot(q^i)^{j-1}(1-q^i)=1/(1-q^i)$.

By analogy: the expected value of $r'-\mathrm{lb}_{p,s+i\cdot n}$ is not greater than $1/(1-q^i)$.

In time slots $t\in[[s+i\cdot n, s+(i+1)\cdot n-1]]$,

$\mathrm{le}_{p,t}=1$ (respectively, $\mathrm{re}_{p,t}=1$) implies that

$\mathrm{rev}_k(t)\in[[\mathrm{lb}_{p,s+i\cdot n}, r'-1]]$ (respectively, $\mathrm{rev}_k(t)\in[[r''+1,\mathrm{ub}_{p,s+i\cdot n}]]$).

Thus, for $i\ge1$,we have

EX[$\sum_{t=[[s+i\cdot n, s+(i+1)\cdot n-1)]]} (\mathrm{le}_{p,t} + \mathrm{re}_{p,t})$] $\leq 2 \cdot (1/(1-q^i)-1)$.

Finally, note that

$$\begin{aligned}\sum_{j=1}^{\infty}(1/(1-q^j)-1) &= \sum_{j=1}^{\infty}(q^j/(1-q^j)) \\ &\leq \frac{1}{1-q}\sum_{j=1}^{\infty} q^j \\ &= q/(1-q)^2 \\ &= q/p^2.\end{aligned}$$

□

Corollary 16. (Corollary U6)

$\mathrm{EX}\big[\sum_{t\geq s} \mathrm{le}_{p,t} + \mathrm{re}_{p,t}\big] \leq (1/p+1)\cdot(4k+2) + 2(1-p)/p^2$.

Proof. From Lemmas 14 and 15 (Lemmas U4 and U5) □

Implementation

RBO receiver has to switch on the radio receiver in each time-slot t such that $\mathrm{rev}_k(t)$ – the index of transmitted key – is between the values of variables lb and ub.

Suppose that, just after the time slot t, the value of lb is r_1 and the value of ub is r_2.

Then the next the next time slot, when the RBO receiver has to listen is

the minimal $t' > t$ such that $\mathrm{rev}_k(t') \in [[r_1, r_2]]$.

Next wake-up time-slot

Definition of $\mathbf{nsi}_{\boldsymbol{k}}(\boldsymbol{t}, \boldsymbol{r_1}, \boldsymbol{r_2})$:

$$\mathrm{nsi}_k(t, r_1, r_2) = \min \{\, t' \mid t' > t \wedge \mathrm{rev}_k(t') \in [[r_1, r_2]] \,\}.$$

Computation of $\mathbf{nsi}_{\boldsymbol{k}}(\boldsymbol{t}, \boldsymbol{r_1}, \boldsymbol{r_2})$:

1. $t'' \leftarrow t + 1$
2. $l \leftarrow 0$
3. **repeat**
   a) $t' \leftarrow t''$
   b) **while** $l < k \wedge t' \bmod 2^{l+1} = 0$
      do $l \leftarrow l + 1$
   c) $x_1 \leftarrow \mathrm{rev}_k(t')$
   d) $t'' \leftarrow t' + 2^l$
   e) $x_2 \leftarrow \mathrm{rev}_k(t' + 2^l - 1)$
4. **until** $r_1 \leq x_2$ **and** $r_2 \geq x_1$ **and**
   $\lceil (r_1 - x_1)/2^{k-l} \rceil \leq \lfloor (r_2 - x_1)/2^{k-l} \rfloor$
5. $c \leftarrow 2^{k-1}$
6. **while** $x_1 < r_1 \vee x_1 > r_2$ **do**
   a) **if** $x_1 < r_1$
      then $x_1 \leftarrow x_1 + c$
      else $x_1 \leftarrow x_1 - c$
   b) $c \leftarrow c/2$
7. **return** $2^k \cdot \lfloor t'/2^k \rfloor + \mathrm{rev}_k(x_1)$

Example: Figure 1 illustrates the computation of $\mathrm{nsi}_k(t, r_1, r_2)$, for $k = 5$, $t = 5$, $r_1 = 7$ and $r_2 = 9$.

The black square dots are the graph of $x = \mathrm{rev}_k(t)$, where t is vertical dimension increasing downwards (representing time-slot), and x is horizontal dimension increasing rightwards (representing the index of transmitted key).

Let us define the blocks of time-slots $Y_0, Y_1, \ldots$, and the sets of indexes $X_0, X_1, \ldots$, under assumption that $s = t_0 = t + 1 = 6$. (Note that all of them are *bellow* the time-slot $t = 5$.)

In our example: $Y_0 = \{6, 7\}$, $Y_1 = \{8, 9, \ldots, 15\}$, $Y_2 = \{16, 17, \ldots, 31\}$, $Y_3 = \{32, 33, \ldots, 64\}$.

The lines represent Binary Search Trees on the subsets X_i, where the jth level of each such tree is the subset $X_{i,j}$. To see the correctness of these trees, recall that, for each $j \in [[1, l_i]]$, we have:

- $\left|\bigcup_{j'=0}^{j-1} X_{i,j'}\right| = |X_{i,j}|$, and
- $2^{k-(j-1)}$ is the minimal distance between distinct elements of $\bigcup_{j'=0}^{j-1} X_{i,j'}$, and
- $2^{k-(j-1)}$ is the minimal distance between distinct elements of $X_{i,j}$, and
- 2^{k-j} is the minimal distance between distinct elements of $\bigcup_{j'=0}^{j} X_{i,j'}$, and
- $\min X_{i,j} = \left(\bigcup_{j'=0}^{j-1} X_{i,j'}\right) + 2^{k-j}$.

Let i' be the minimal i, such that X_i intersects the interval $[r_1, r_2]$.

When the **repeat** loop finishes, we have: $x_1 = \min X_{i'}$, $x2 = \max X_{i'}$, $t' = \min Y_{i'}$, $t'' = \max Y_{i'}$, and $l = l_{i'}$.

In our example: $i' = 2$, $\min X_{i'} = 1$, $\max X_{i'} = 32$, $\min Y_{i'} = 16$, $\max Y_{i'} = 31$, and $l_{i'} = 4$.

In the **while** loop starting in line 6, we do binary search on the binary search tree on $X_{i'}$ until we enter the interval $[r_1, r_2]$ for the first time. When the loop finishes, the value of x_1 is in $[r_1, r_2]$ and $y = 2^k \cdot \lfloor t'/2^k \rfloor + \mathrm{rev}_k(x_1)$ is the minimal time-slot in $Y_{i'}$ such that $\mathrm{rev}_k(y) = x_1$.

In our example the final value of x_1 is 9 and the value returned by the algorithm is 18.

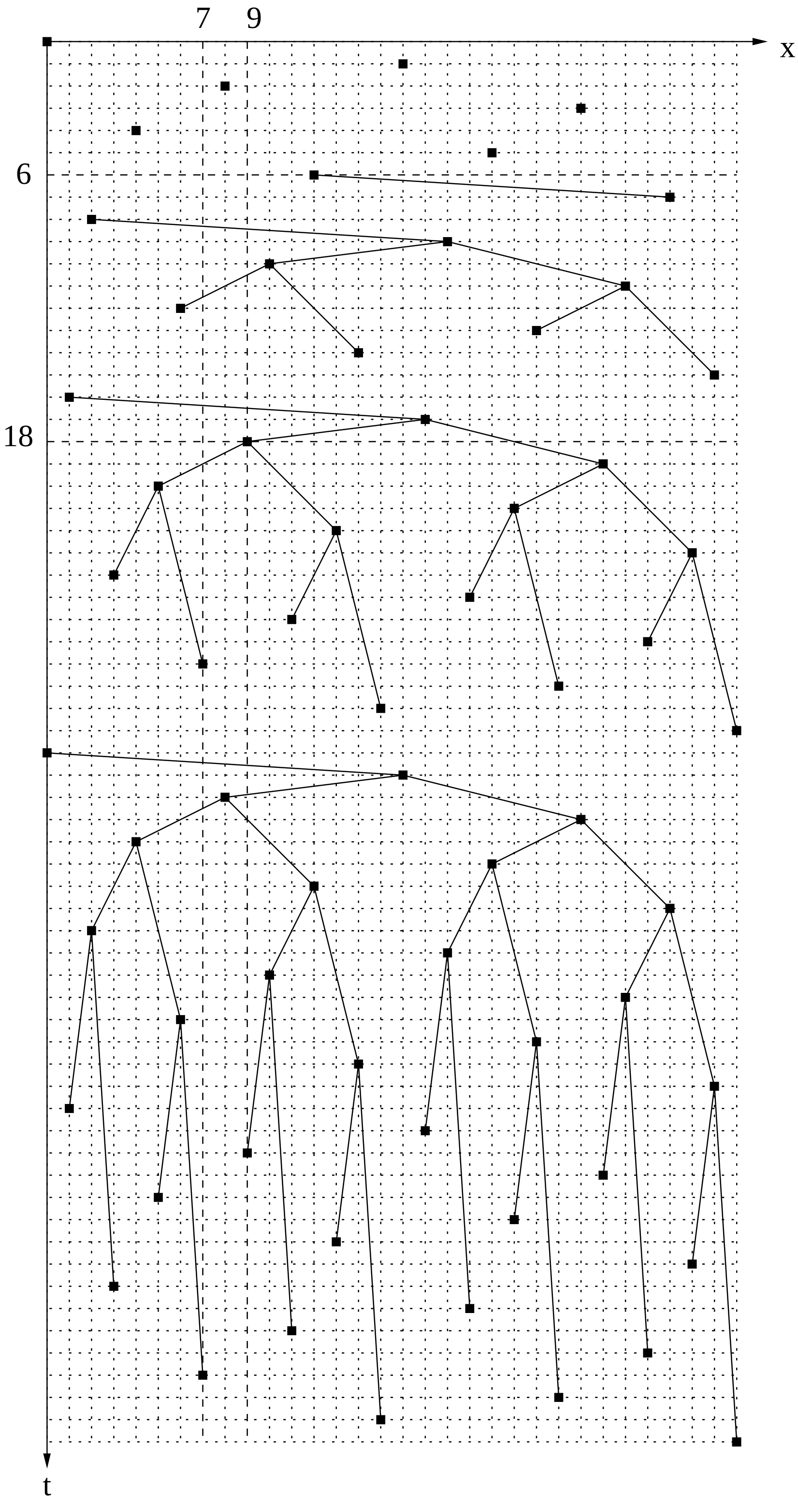

Figure 1. Computation of $\mathrm{nsi}_5(5, 7, 9)$

Correctness of the NSI algorithm:

For the analysis of the computation of $\mathrm{nsi}_k(t, r_1, r_2)$, we define:

- $t_0, t_1, \ldots,$
- $l_0, l_1, \ldots,$
- last,
- $Y_0, Y_1, \ldots,$
- $X_0, X_1, \ldots,$
- the subsets $X_{i,j}$,

under the assumption that $s = t_0 = t + 1$.

Lemma 17. (NSI1) *Let* $0 \leq r_1 \leq r_2 \leq n - 1$.
Then the "repeat" loop of line 3 finishes.
Let x_1' *and* x_2' *be the values of variables* x_1 *and* x_2*,*
respectively, just after the line 4.
Then $x_1' = \min X_{i'}$ *and*

$x_2' = \max X_{i'}$,

where $i' = \min \{i : i \geq 0 \wedge X_i \cap [[r_1, r_2]] \neq \emptyset\}$.

Proof. Let the iterations of the "repeat-until" loop be numbered starting from zero.
After the ith iteration, at line 4, we have

- $l = l_i$,
- $t' = t_i$,
- $x_1 = \min X_i = \mathrm{rev}_k(t_i)$,
- $t'' = t_{i+1}$,
- $x_2 = \mathrm{rev}_k(\max Y_i) = \max X_i$.

Thus, the condition

$$r_1 \leq x_2 \wedge r_2 \geq x_1 \wedge \lceil (r_1 - x_1)/2^{k-l} \rceil \leq \lfloor (r_2 - x_1)/2^{k-l} \rfloor$$

is equivalent to

$$r_1 \leq \max X_i \wedge r_2 \geq \min X_i \wedge \min \{j : \min X_i + 2^{k-l_i} \cdot j \geq r_1\} \leq \max \{j : \min X_i + 2^{k-l_i} \cdot j \leq r_2\},$$

which is equivalent to

$$X_i \cap [[r_1, r_2]] \neq \emptyset,$$

since $X_i = \{\min X_i + 2^{k-l_i} \cdot j : j \in \mathbb{Z}\} \cap [[0, n-1]]$.

We have $X_{\text{last}} = [[0, n-1]] \supseteq [[r_1, r_2]] \neq \emptyset$, thus the "repeat" loop is finite.

Hence, at line 5, we have $x_1 = \text{rev}_k(t_{i'}) = \min X_{i'}$, where $i' = \min \{i \geq 0 \,|\, X_i \cap [r_1, r_2] \neq \emptyset\}$. $\square$

Lemma 18. (NSI2) *Let $0 \leq r_1 \leq r_2 \leq n-1$ and let y' be the value returned in the line 7.*

We have and $y' \in Y_{i',j'}$,

where $i' = \min \{i \mid i \geq 0 \wedge X_i \cap [[r_1, r_2]] \neq \emptyset\}$ and

$j' = \min \{j \mid j \in [[0, l_{i'}]] \wedge X_{i',j} \cap [[r_1, r_2]] \neq \emptyset\}$ *and*

$X_{i',j'} \cap [[r_1, r_2]] = \{\text{rev}_k(y')\}$.

Thus $\text{nsi}_k(s', r_1, r_2) = y'$.

Proof.

Let $x_{1,0}$ (respectively, c_0) be the value of x_1 (respectively, c) just before the line 6.

Let $x_{1,j}$ (respectively, c_j) be the value of x_1 (respectively, c)

just after the jth iteration of the "while" loop of line 6.

We have $x_{1,0} = \min X_{i'} = \min X_{i',0}$, where, by Lemma 17 (Lemma NSI1),

$$i' = \min \{i \geq 0 \mid X_i \cap [[r_1, r_2]] \neq \emptyset\}.$$

Let j' be the number of iterations of the "while" loop.

For $0 \leq j \leq j'$, we have $c_j = 2^{k-1-j}$.

We have $\emptyset \neq X_{i'} \cap [[r_1, r_2]] \subseteq [[\min X_{i'}, \min X_{i'} + 2^k - 1]]$.

Thus $j' \leq l_{i'}$ and $x_{1,j'} \in [[r_1, r_2]]$.

For each $j \in [[0, j']]$, we have $x_{1,j} \in \mathbb{X}_{i',j}$.

If $j' = 0$, then $X_{i',j'} = \{\min X_{i'}\} = \{x_{1,0}\} \subseteq [[r_1, r_2]]$ and $\text{rev}_k(2^k \cdot \lfloor t'/2^k \rfloor + \text{rev}_k(x_{1,0})) = x_{1,0}$.

Let $j' \geq 1$.

We can show by induction that, for each $j \in [[0, j'-1]]$, we have x' such that

- $\mathbb{X}_{i',j} \ni x' < r_1 \leq r_2 < x' + 2^{k-j} \in \mathbb{X}_{i',j}$ and
- $x_{1,j} \in \{x', x' + 2^{k-j}\}$ and
- $\mathbb{X}_{i',j} \cap [x', x' + 2^{k-j}] = \{x', x' + 2^{k-j}\}$ and
- $x_{1,j+1} = x' + 2^{k-j-1}$.

Thus $x_{1,j'} - 2^{k-j'} < r_1$ and $r_2 < x_{1,j'} + 2^{k-j'}$.

Hence $\mathbb{X}_{i',j'} \cap [r_1, r_2] = \{x_{1,j'}\} \not\subseteq \mathbb{X}_{i',j'-1}$.

Since $(\mathbb{X}_{i',j'} \setminus \mathbb{X}_{i',j'-1}) \cap [[0, n-1]] = X_{i',j'}$ and $0 \le r_1 \le r_2 \le n-1$,

we have $x_{1,j'} \in X_{i',j'}$ and

(reverse of the returned value) $\mathrm{rev}_k(2^k \cdot \lfloor t'/2^k \rfloor + \mathrm{rev}_k(x_{1,j'})) = x_{1,j'}$. □

Complexity of the NSI algorithm:

Time complexity:

The number of iterations of the “repeat-until” loop is never greater than $k+1$.

Since l never decreases, the **total** number of iterations of the internal “while” loop (line 3 b) in **all** iterations of the “repeat-until” loop is never grater than $k+1$.

The total number of iterations of the binary search loop (starting at line 6) is never greater than k.

Complexity of the algorithm:

- **memory:** constant number of k-bit variables
- **time:** $O(k)$ elementary operations on k-bit integers

Example implementation in Java:

On Figure 2, we present the implementation in Java language used in the simulation of RBO on TinyOS (`https://github.com/mki1967/rbo-tinyos-java`).

It computes and returns the value: $\mathrm{nsi}_k(t, r_1, r_2) \bmod 2^k$.

(It uses also `revBits(k,t)` that computes $\mathrm{rev}_k(t)$.)

We replaced some operations such as e.g. divisions by the powers of two by bit-wise operations such as shifting and masking operations that should be more efficient on real processors.

We use the following bit-masks, related to the values k and l of the original algorithm:

- `twoToK` for $2^k = (1(0)^k)_2$,
- `modMaskK` for $2^k - 1 = ((1)^k)_2$,
- `twoToL` for $2^l = (1(0)^l)_2$,
- `stepLMinusOne` for $2^{k-l} - 1 = ((1)^{k-l})_2$,
- `stepDivMask` for $\mathrm{rev}_k(2^l - 1) = ((1)^l(0)^{k-l})_2$.

Variables: `t1`, `tNext`, `x1`, `x2`, and `s` correspond to the variables: t', t'', x_1, x_2 and c of the original algorithm, respectively.

```
public static int nextIn(int k, int t, int r1, int r2)
// we assume 0<=r1<=r2< 2^k
{
    int twoToK=(1<<k); // 2^k
    int modMaskK= twoToK-1; // 2^k-1
    int t1,x1,x2, stepDivMask;
    int twoToL=1;
    int stepLMinusOne=modMaskK;
    int tNext=((t+1)&modMaskK);
    do
        {
            t1=tNext;
            while(twoToL<twoToK && (t1&twoToL)==0)
                {
                    twoToL=twoToL<<1;
                    stepLMinusOne=stepLMinusOne>>1;
                }
            tNext=((t1+twoToL)&modMaskK);
            stepDivMask=((~stepLMinusOne) & modMaskK);
            x1=revBits(k,t1);
            x2= (x1 | stepDivMask );
        }while( r1>x2 || r2<x1 ||
                ((r1-x1+stepLMinusOne)&stepDivMask)>((r2-x1)&stepDivMask));
    int s= (twoToK>>1); // 2^(k-1)
    while(x1<r1 || x1>r2)
        {

            if(x1<r1) x1=x1+s;
            else x1=x1-s;
            s=s/2;
        }
    return revBits(k, x1);
}
```

Figure 2. Implementation in Java of computation of: $\mathrm{nsi}_k(t, r_1, r_2) \bmod 2^k$

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