ADVANCED TOPICS IN ALGEBRA

LECTURE 4

(lecture and problems to solve)

2020/21

QUOTIENT SPACES

Let us recall the notion of a quotient space V/W, where $W < V$. Geometrically elements of V/W are all shifts of W in V, namely copies of W parallel to W. Formally, let $\cong$ be the following equivalence relation of elements of V:

$$\boldsymbol{v} \cong \boldsymbol{w} \text{ iff } \boldsymbol{v} - \boldsymbol{w} \in W.$$

The set of all vectors $\boldsymbol{w}$ such that $\boldsymbol{v} \cong \boldsymbol{w}$ is called the *coset* or *congruence class* determined by $\boldsymbol{v}$. It will be denoted by $[\boldsymbol{v}]_W$ or simply by $[\boldsymbol{v}]$ if W has been fixed.

Theorem 24. $[\boldsymbol{u}]_W = [\boldsymbol{v}]_W$ if and only if $\boldsymbol{u} \cong \boldsymbol{v}$.

Proof. Assume that $[\boldsymbol{u}]_W = [\boldsymbol{v}]_W$. Note that $\boldsymbol{u} \in [\boldsymbol{u}]_W$ (because $\boldsymbol{u} - \boldsymbol{u} = \mathbf{0} \in W$). Thus $\boldsymbol{u} \in [\boldsymbol{v}]_W$ which means that $\boldsymbol{u} \cong \boldsymbol{v}$.

Assume now that $\boldsymbol{u} \cong \boldsymbol{v}$ and that $\boldsymbol{w} \in [\boldsymbol{v}]_W$. Then $\boldsymbol{v} \cong \boldsymbol{w}$, which means $\boldsymbol{w}-\boldsymbol{v} \in W$. But by our assumption we know that $\boldsymbol{v}-\boldsymbol{u} \in W$. Hence, because the sum of vectors from W is in W,

$$\boldsymbol{w} - \boldsymbol{u} = (\boldsymbol{w} - \boldsymbol{v}) + (\boldsymbol{v} - \boldsymbol{u}) \in W,$$

which means that $\boldsymbol{w} \cong \boldsymbol{u}$ and hence $\boldsymbol{w} \in [\boldsymbol{u}]_W$. Thus we proved

$$[\boldsymbol{v}]_W \subseteq [\boldsymbol{u}]_W.$$

In the same way prove

$$[\boldsymbol{u}]_W \subseteq [\boldsymbol{v}]_W.$$

Finally,

$$[\boldsymbol{u}]_W = [\boldsymbol{v}]_W.$$

□

Problem 1. Prove that $[\mathbf{0}]_W = W$.

Problem 2. Prove that that $[\boldsymbol{u}]_W = [\boldsymbol{v}]_W$ or the sets $[\boldsymbol{u}]_W, [\boldsymbol{v}]_W$ are disjoint.

Let us define a sum of congruence classes

$$[\boldsymbol{u}]_W + [\boldsymbol{v}]_W := [\boldsymbol{u} + \boldsymbol{v}]_W.$$

Theorem 25. The sum of congruence classes is well defined, it means that if $[\boldsymbol{u}]_W = [\boldsymbol{u'}]_W$ (which is equivalent to $\boldsymbol{u} \cong \boldsymbol{u'}$) and $[\boldsymbol{v}]_W = [\boldsymbol{v'}]_W$ (which is equivalent to $\boldsymbol{v} \cong \boldsymbol{v'}$), then

$$[\boldsymbol{u} + \boldsymbol{v}]_W = [\boldsymbol{u'} + \boldsymbol{v'}]_W.$$

Proof. Assume $\boldsymbol{u} \cong \boldsymbol{u'}$ and $[\boldsymbol{v}]_W = [\boldsymbol{v'}]_W$. Then

$$(\boldsymbol{u}+\boldsymbol{v}) - (\boldsymbol{u'}+\boldsymbol{v'}) = (\boldsymbol{u}-\boldsymbol{u'}) + (\boldsymbol{v}-\boldsymbol{v'}) \in W.$$

Hence

$$\boldsymbol{u}+\boldsymbol{v} \cong \boldsymbol{u'}+\boldsymbol{v'}$$

and by Theorem 24

$$[\boldsymbol{u}+\boldsymbol{v}]_W = [\boldsymbol{u'}+\boldsymbol{v'}]_W.$$

□

Now let us define how to multiply a scalar by a congruence class.

$$\alpha[\boldsymbol{u}]_W := [\alpha\boldsymbol{u}]_W.$$

Problem 3. Prove that the multiplication of a scalar by a congruence class is well defined, it means that if if $[\boldsymbol{u}]_W = [\boldsymbol{u'}]_W$ then if $[\alpha\boldsymbol{u}]_W = [\alpha\boldsymbol{u'}]_W$.

Problem 4. Prove that the space $V/W = \{[\boldsymbol{v}]_W : \boldsymbol{v} \in V\}$ is a linear space with the addition and the outer multiplication defined above.

Problem 5. Say W is a one-dimensional subspace of $\mathbb{R}^2$. Prove that it means it is a line going through the point $(0,0)$. Say, the equation of W is $y = ax$. Prove that elements of $\mathbb{R}^2/W$ are all lines given by equations $y = ax + b$, $b \in \mathbb{R}$.

Theorem 26. Let $f : V \to V$ be a linear mapping. Let W be an invariant (with respect to f) subspace of V. Let

$$f^{(W)}([\boldsymbol{v}]_W) := [f(\boldsymbol{v})]_W.$$

Then $f^{(W)}$ is well defined and that it is a linear mapping from V/W into V/W.

Proof. Let $[\boldsymbol{v}]_W = [\boldsymbol{u}]_W$. Then

$$f(\boldsymbol{v}) - f(\boldsymbol{u}) = f(\boldsymbol{v}-\boldsymbol{u}) \in W$$

because $\boldsymbol{v}-\boldsymbol{u} \in W$ and W is invariant. Thus

$$f(\boldsymbol{v}) \cong f(\boldsymbol{u})$$

and by Theorem 24

$$[f(\boldsymbol{v})]_W = [f(\boldsymbol{v})]_W.$$

□

Theorem 27. Let $f : V \to V$ be a linear mapping and let W be a subspace of V invariant with respect to f. Let $\phi(x)$ be an annihilator of $\boldsymbol{v} \in V$. Then $\phi(f^{(W)})([\boldsymbol{v}]_W) = [\boldsymbol{0}] = W$.

Proof. We have

$$\phi(f^{(W)})([\boldsymbol{v}]_W) = [f(\boldsymbol{v})]_W = [\boldsymbol{0}]_W = W.$$

□

Problem 6. It follows from Theorem 26 that the annihilator of $[\boldsymbol{v}]_W$ divides $\phi(x)$. Must it be equal to $\phi(x)$?

Problem 7. Let a finite-dimensional linear space V be a direct sum of linear spaces V_1 and V_2. Show that there exists a one to one linear mapping from V/V_1 onto V_2 (in other words V/V_1 and V_2 are isomorphic).

Theorem 28. Let $W < V$ be linear spaces. If $\dim(V) < \infty$, then

$$\dim(W) + \dim(V/W) = \dim(V).$$

Let $f : V \to V/W$ be defined by the formula

$$f(\boldsymbol{v}) = [\boldsymbol{v}]_W.$$

By the very properties of congruence classes f is a linear mapping. Let us notice that

$$\mathrm{Ker}(f) := \{\boldsymbol{v} : f(\boldsymbol{v})) = \mathbf{0}_{V/W}(= W))\} = W$$

and

$$\mathrm{Im}(f) = V/W.$$

Now it is enough to use the (well known) equality

$$\dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f)) = \dim(V).$$

$\square$

Problem 8. Derive the theorem from Problem 7 from Theorem 28.

Problem 9. Derive Theorem 28 from the theorem from Problem 7.