ADVANCED TOPICS IN ALGEBRA

LECTURE 6

(lecture and problems to solve)

2020/21

JORDAN FORM OF A MATRIX OF A LINEAR MAPPING - THE FIELD OF REALS CASE

In the previous lecture we proved that in the case of a finite-dimensional linear space V over the field of complex numbers if $f : V \to V$ is a linear mapping then there exists a basis B of V such that the matrix F representing f in this basis has a diagonal-like form:

$$F = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & \dots & & \\ & & & D_r \end{bmatrix}. \tag{1}$$

for some r, where all blocks D_i have the form either

$$D_i = [t]$$

or

$$\begin{bmatrix} t & & & & & \\ 1 & t & & & & \\ & 1 & & & & \\ & & \dots & & & \\ & & & & t & \\ & & & & 1 & t \end{bmatrix}.$$

Now we assume that a finite dimensional space V is taken over the field of real numbers $\mathbb{R}$. Let $\dim(V) = n$. Let $f : V \to V$ be a linear mapping. Fixing any particular basis in V and taking the matrix representation F of f consider a linear mapping <u>from $\mathbb{R}^n$ to $\mathbb{R}^n$</u> having the matrix representation F in the standard basis. If we prove that we can change the basis using a transition matrix A to obtain a new representation $F' = AFA^{-1}$ in the new basis then we can do exactly the same for f, changing the old basis via means of A. Thus we can consider V as simply $\mathbb{R}^n$. Let f' be the so-called *extension* of f to the space $\mathbb{C}^n$. Namely, the basis

$$\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots, \boldsymbol{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

is also a basis for $\mathbb{C}^n$ and we can set $f'(\boldsymbol{e_i}) = f(\boldsymbol{e_i})$ ($f(\boldsymbol{e_1})$ being a real vector is also a complex vector). Thus f' has been defined as a mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$ and it has the same matrix representation as f and in the same basis.

We adopt the following notation for vectors in $\mathbb{C}^n$. If

$$\boldsymbol{v} = \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_{n-1} + ib_{n-1} \\ a_n + ib_n \end{bmatrix}$$

then

$$Re(\boldsymbol{v}) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} \text{ and } Im(\boldsymbol{v}) = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}.$$

Thus

$$\boldsymbol{v} = Re(\boldsymbol{v}) + iIm(\boldsymbol{v}).$$

Let

$$\overline{\boldsymbol{v}} := Re(\boldsymbol{v}) - iIm(\boldsymbol{v}).$$

Thus

$$\overline{\boldsymbol{v}} = \begin{bmatrix} a_1 - ib_1 \\ a_2 - ib_2 \\ \vdots \\ a_{n-1} - ib_{n-1} \\ a_n - ib_n \end{bmatrix}.$$

Problem 1. If vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ are linearly independent in $\mathbb{R}^k$, then, treated as complex vectors, they are also linearly independent in $\mathbb{C}^k$.

From Theorem we know that the space $\mathbb{R}^n$ (which is the domain of f) is a direct sum of cyclic spaces:

$$\mathbb{R}^n = V_1 \oplus \ldots \oplus V_m$$

whose annihilators are of the form $\phi_i(x) = (\psi_i(x))^{n_i}$, where $\psi_i(x)$ is a prime polynomial, i.e. it is either of the form

$$\psi_i(x) = x - t_i \tag{2}$$

or

$$\psi_i(x) = x^2 + b_i x + c_i \text{ and } b_i - 4c_i < 0. \tag{3}$$

Let $\boldsymbol{v_i}$ be its generator of the space V_i. Let V_i' be the space over the field $\mathbb{C}$ having the same cyclic basis as V_i. Let f' be the extension of f. If (2) holds the same polynomial $\phi_i(x) = (x - t_i)^{n_i}$ is the annihilator of V_i'. Then, as we know, we can change the basis of V_i' to obtain the matrix representation of f' on V in the form

$$D_i = [t]$$

or

$$\begin{bmatrix} t & & & & & \\ 1 & t & & & & \\ & 1 & & & & \\ & & \cdots & & & \\ & & & & t & \\ & & & & 1 & t \end{bmatrix}.$$

If (3) holds V' being cyclic is a direct sum of two subspaces $W_{i,1}$ and $W_{i,2}$ of the annihilators $\phi_{i,1}(x) = (x - \alpha_i)^{n_i}$ and $\phi_{i,1}(x) = (x - \beta_i)^{n_i}$, respectively, where, of course,

$$(x - \alpha_i)(x - \beta_i) = x^2 + bx + c.$$

Note that, because b, c are real numbers, from the fact that α_i is a root of the polynomial $x^2 + bx + c$ follows that $\overline{\alpha_i}$ is also a root of $x^2 + bx + c$. Thus $\beta_i = \overline{\alpha_i}$.

Problem 2. Show that if $\boldsymbol{w_i}$ is a generator of the space $W_{i,1}$ then $\overline{\boldsymbol{w_i}}$ is a generator of the space $W_{i,2}$.

Problem 3. Show that if $\boldsymbol{w_i}$ is a generator of of the space $W_{i,1}$ then the vector $\boldsymbol{w_i} + \overline{\boldsymbol{w_i}}$ is a generator of V_i' (Hence we can assume that $\boldsymbol{v_i} = \boldsymbol{w_i} + \overline{\boldsymbol{w_i}}$).

We shall now use Jordan bases of the spaces $W_{i,1}$ and $W_{i,2}$. Namely, for fixed i, let

$$\boldsymbol{p_1} = \boldsymbol{w_i},\ \boldsymbol{p_2} = (f' - \alpha_i \cdot \mathrm{id})(\boldsymbol{w_i}), \dots, \boldsymbol{p_{n_i}} = (f' - \alpha_i \cdot \mathrm{id})^{n_i - 1}(\boldsymbol{w_i})$$

and

$$\boldsymbol{r_1} = \overline{\boldsymbol{w_i}},\ \boldsymbol{r_2} = (f' - \overline{\alpha_i} \cdot \mathrm{id})(\overline{\boldsymbol{w_i}}), \dots, \boldsymbol{r_{n_i}} = (f' - \overline{\alpha_i} \cdot \mathrm{id})^{n_i - 1}(\overline{\boldsymbol{w_i}}).$$

We know from the reasoning concerning the complex case (in the previous theorem) that

$$f'(\boldsymbol{p_j}) = \alpha_i \boldsymbol{p_j} + \boldsymbol{p_{j+1}} \text{ for } j < n_i, \text{ and } f'(\boldsymbol{p_{n_i}}) = \alpha_i \boldsymbol{p_{n_i}}$$

and

$$f'(\boldsymbol{r_j}) = \overline{\alpha_i} \boldsymbol{r_j} + \boldsymbol{r_{j+1}} \text{ for } j < n_i, \text{ and } f'(\boldsymbol{r_{n_i}}) = \overline{\alpha_i} \boldsymbol{r_{n_i}}.$$

Problem 4. Prove that

$$f'(\boldsymbol{r_j}) = \overline{\alpha_i \boldsymbol{p_j}} + \overline{\boldsymbol{p_{j+1}}} \text{ for } j < n_i, \text{ and } f'(\boldsymbol{r_{n_i}}) = \overline{\alpha_i \boldsymbol{p_{n_i}}}.$$

Let $\alpha_i = a + ib$. Then $\overline{\alpha_i} = a - ib$. Thus we have

$$f'(\boldsymbol{p_j}) = (a + ib)\boldsymbol{p_j} + \boldsymbol{p_{j+1}} \text{ for } j < n_i, \text{ and } f'(\boldsymbol{p_{n_i}}) = (a + ib)\boldsymbol{p_{n_i}},$$

and

$$f'(\boldsymbol{r_j}) = (a - ib)\overline{\boldsymbol{p_j}} + \overline{\boldsymbol{p_{j+1}}} \text{ for } j < n_i, \text{ and } f'(\boldsymbol{r_{n_i}}) = (a - ib)\overline{\boldsymbol{p_{n_i}}}.$$

Let, for $1 \leqslant j \leqslant n_i$,

$$\boldsymbol{s_{2j-1}} = \boldsymbol{p_j} + \boldsymbol{r_j}$$

and

$$\boldsymbol{s_{2j}} = i \cdot (\boldsymbol{p_j} - \boldsymbol{r_j}).$$

Problem 5. Prove that the vectors $\boldsymbol{s_j}$ have real coefficients, i.e. $\boldsymbol{s_j} \in \mathbb{R}^n$.

Problem 6. Prove that the vectors $\boldsymbol{s_j}$, $j \leqslant 2n_i$ are linearly independent in the space $\mathbb{C}^n$.

By the fact stated in Problem 6 the vectors $\boldsymbol{s_j}$ form a basis for the space V_i. We have

$$f'(\boldsymbol{s_1}) = f'(\boldsymbol{p_1}) + f'(\boldsymbol{r_1}) = f'(\boldsymbol{p_1}) + \overline{f'(\boldsymbol{p_1})} =$$
$$(a+bi)\boldsymbol{p_1} + \boldsymbol{p_2} + (a-ib)\overline{\boldsymbol{p_1}} + \overline{\boldsymbol{p_2}} =$$
$$a(\boldsymbol{p_1} + \overline{\boldsymbol{p_1}}) + bi(\boldsymbol{p_1} - \overline{\boldsymbol{p_1}}) + (\boldsymbol{p_2} + \overline{\boldsymbol{p_2}}) =$$
$$a\boldsymbol{s_1} + b\boldsymbol{s_2} + \boldsymbol{s_3},$$

and

$$f'(\boldsymbol{s_2}) = f'(i(\boldsymbol{p_1} - \boldsymbol{r_1})) = if'(\boldsymbol{p_1}) - if'(\boldsymbol{r_1}) = if'(\boldsymbol{p_1}) - if'(\overline{\boldsymbol{p_1}}) =$$
$$if'(\boldsymbol{p_1}) - i\overline{f'(\boldsymbol{p_1})} = i((a+bi)\boldsymbol{p_1} + \boldsymbol{p_2}) - i\overline{(a+bi)\boldsymbol{p_1} + \boldsymbol{p_2}} =$$
$$i((a+bi)\boldsymbol{p_1} + \boldsymbol{p_2}) - i((a-bi)\overline{\boldsymbol{p_1}} + \overline{\boldsymbol{p_2}}) =$$
$$-b(\boldsymbol{p_1} + \overline{\boldsymbol{p_1}}) + a(i(\boldsymbol{p_1} - \overline{\boldsymbol{p_1}})) + i(\boldsymbol{p_2} - \overline{\boldsymbol{p_2}}) =$$
$$-b\boldsymbol{s_1} + a\boldsymbol{s_2} + \boldsymbol{s_4}.$$

Similarly, for $j < n_i$, we obtain:

$$f'(\boldsymbol{s_{2j-1}}) = a\boldsymbol{s_{2j-1}} + b\boldsymbol{s_{2j}} + \boldsymbol{s_{2j+1}}$$

and

$$f'(\boldsymbol{s_{2j}}) = -b\boldsymbol{s_{2j-1}} + a\boldsymbol{s_{2j}} + \boldsymbol{s_{2j+2}},$$

and

$$f'(\boldsymbol{s_{2n_i-1}}) = f'(\boldsymbol{p_{n_i}} + \overline{\boldsymbol{p_{n_i}}}) = \alpha_i\boldsymbol{p_{n_i}} + \overline{\alpha_i\boldsymbol{p_{n_i}}} = a(\boldsymbol{p_{n_i}} + \overline{\boldsymbol{p_{n_i}}}) + ib(\boldsymbol{p_{n_i}} - \overline{\boldsymbol{p_{n_i}}}) =$$
$$= a\boldsymbol{s_{2n_i-1}} + b\boldsymbol{s_{2n_i}},$$

and

$$f'(\boldsymbol{s_{2n_i}}) = f'(i(\boldsymbol{p_{n_i}} - \overline{\boldsymbol{p_{n_i}}})) = i\alpha_i\boldsymbol{p_{n_i}} - i\overline{\alpha_i\boldsymbol{p_{n_i}}}$$
$$= ai(\boldsymbol{p_{n_i}} - \overline{\boldsymbol{p_{n_i}}}) - b(\boldsymbol{p_{n_i}} + \overline{\boldsymbol{p_{n_i}}}) = -b\boldsymbol{s_{2n_i-1}} + a\boldsymbol{s_{2n_i-1}}.$$

Hence in the basis $\langle \boldsymbol{s_1}, \boldsymbol{s_2}, \ldots \boldsymbol{s_{2n_i-1}}, \boldsymbol{s_{2n_i}} \rangle$ the mapping f restricted to the space V' is represented by the matrix

$$F_i = \begin{bmatrix} A & & & & & \\ E & A & & & & \\ & E & & & & \\ & & \ldots & & & \\ & & & & A & \\ & & & & E & A \end{bmatrix}$$

where

$$A = \begin{bmatrix} a, -b \\ b,\ a \end{bmatrix} \text{ and } E = \begin{bmatrix} 1, 0 \\ 0, 1 \end{bmatrix}.$$

Thus we can state the following theorem.

Theorem. If V is a finite-dimensional linear space over the field of real naumbers $\mathbb{R}$ and $f : V \to V$ is a linear mapping, then there exists a basis of V such that in this basis f is represented by a matrix F

$$F = \begin{bmatrix} F_1 & & & \\ & F_2 & & \\ & & \cdots & \\ & & & F_m \end{bmatrix},$$

where the blocks F_i are of one of the following three forms:

$$[t],$$

$$\begin{bmatrix} t & & & & & \\ 1 & t & & & & \\ & 1 & & & & \\ & & \cdots & & & \\ & & & & t & \\ & & & & 1 & t \end{bmatrix},$$

$$\begin{bmatrix} A & & & & & \\ E & A & & & & \\ & E & & & & \\ & & \cdots & & & \\ & & & & A & \\ & & & & E & A \end{bmatrix},$$

where

$$A = \begin{bmatrix} a, -b \\ b, \ a \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1, 0 \\ 0, 1 \end{bmatrix}.$$