STIRLING APPROXIMATION FORMULA

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ABSTRACT. This note contains an elementary and complete proof of the Stirling approximation formula \( n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \) of the factorial function.

1. INTRODUCTION

It is quite easy to get an approximation of the number \( n! \) which gives an information about its ratio of growth. Namely, let us consider the sequence \( S(n) = \ln(n!) \). Then \( S(n) = \sum_{k=1}^{n} \ln k \). So we see that \( \ln n \leq S(n) \leq n \ln n \), so the sequence \( S(n) \) grows rather slowly. Look at this picture:

I placed at this picture the plot of the function \( x \mapsto \ln x \). The area of the figure below the plot is equal to \( S(15) \). This observation, generalized to arbitrary \( n \) gives us the bound \( S(n) \leq \int_{1}^{n+1} \ln x \, dx \). Look now at the following picture:

We read from this picture that \( S(n) \geq \int_{1}^{n} \ln x \, dx \). So we have derived the following bounds:

\[
\int_{1}^{n} \ln x \, dx \leq S(n) \leq \int_{1}^{n+1} \ln x \, dx.
\]
All what we need now is the formula \( \int \ln x \, dx = x \ln x - x + C \). You can derive this formula using the integration by parts (\( \int \ln x \, dx = \int (x)' \ln x \, dx = \ldots \)). Let us note that \( x \ln x - x = x(\ln x - 1) = x(\ln x - \ln e) = x \ln (\frac{x}{e}) \).

Using this formula we get

\[
\int_1^n \ln x \, dx = [x \ln (\frac{x}{e})]_{x=1}^n = n \ln \frac{n}{e} - \ln 1 = \ln (\frac{n}{e})^n + \ln e = \ln n^n \quad \text{and}
\]

\[
\int_1^{n+1} \ln x \, dx = \ldots = \ln \frac{(n+1)^{n+1}}{e^{n+1}}
\]

Hence

\[
\ln \frac{n^n}{e^{n-1}} \leq S(n) \leq \ln \frac{(n+1)^{n+1}}{e^n}
\]

Finally, we observe that \( n! = \exp(S(n)) \) and we transform this formula into the form

\[
\frac{n^n}{e^{n-1}} \leq n! \leq \frac{(n+1)^{n+1}}{e^n}
\]

or

\[
e^{(\frac{n}{e})^n} \leq n! \leq (\frac{n}{e})^n (n+1)(1 + \frac{1}{n})^n < e(n+1)(\frac{n}{e})^n
\]

Therefore using a very elementary tools we derived the following formula

\[
n! = \alpha(n)(\frac{n}{e})^n
\]

where \( \alpha \) is some function such that \( e \leq \alpha(n) \leq (n+1)e \). This is a quite precise result and is sufficient for many applications.

2. More Precise Result: Stirling’s Formula

We are going to prove in this section the Stirling approximation formula:

\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n
\]

i.e. that

\[
\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = 1.
\]

Let

\[
a_n = \frac{n!}{\sqrt{2n} \left(\frac{n}{e}\right)^n}
\]

Our plan is following:

(1) First we show that \( \lim_{n \to \infty} a_n = C \) for some constant \( C \). This will imply that

\[
\lim_{n \to \infty} \frac{n!}{C \sqrt{2n} \left(\frac{n}{e}\right)^n} = 1
\]
(2) Next we derive Wallis formula which gives a precise asymptotic result involving \( n! \).

(3) Finally we put in Wallis formula the approximation \( C \sqrt{2\pi} n^\frac{1}{2} \) and this will give us the precise value of the constant \( C \).

2.1. **Part 1.** Let

\[ b_n = \ln a_n. \]

After easy transformations we get the following equality

\[ b_n - b_{n+1} = \frac{1}{2} (n + 1) \ln \frac{n + 1}{n} - 1 \]

We are going to use an expansion of the function \( \ln \) into the Taylor series at point \( 1 \). However the most obvious approach

\[ \ln \frac{n}{n + 1} = \ln \frac{1}{1 + \frac{1}{n}} = -\frac{1}{n} + \frac{1}{2} \frac{1}{n^2} - \ldots \]

gives us a series with alternating terms which are usually difficult to handle. So we try to be more ingenious. Observe that for \(|t| < 1\) we have

\[ \ln(1 + t) = t - \frac{1}{2} t^2 + \frac{1}{3} t^3 - \frac{1}{4} t^4 + \frac{1}{5} t^5 + \ldots \]

\[ -\ln(1 - t) = t + \frac{1}{2} t^2 + \frac{1}{3} t^3 + \frac{1}{4} t^4 + \frac{1}{5} t^5 + \ldots \]

hence

\[ \ln \frac{1 + t}{1 - t} = \ln(1 + t) - \ln(1 - t) = 2 \sum_{k=0}^{\infty} \frac{1}{2k + 1} t^{2k+1} \]

The only solution of the equation \( \frac{n+1}{n} = \frac{1+t}{1-t} \) is equal to \( t = \frac{1}{2n+1} \) so we get

\[ \ln \frac{n + 1}{n} = 2 \sum_{k=0}^{\infty} \frac{1}{2k + 1} \left( \frac{1}{2n + 1} \right)^{2k+1} \]

The first term of this series is equal to \( \frac{2}{2n+1} \), hence

\[ b_n - b_{n+1} = \frac{1}{2} (2n + 1) \ln \frac{n + 1}{n} - 1 = \sum_{k=1}^{\infty} \frac{1}{2k + 1} \left( \frac{1}{2n + 1} \right)^{2k} \]

so the sequence \( (b_n) \) is decreasing. Next we have

\[ b_n - b_{n+1} < \sum_{k=1}^{\infty} \left( \frac{1}{(2n + 1)^2} \right)^k = \frac{1}{(2n + 1)^2} \frac{1}{1 - \frac{1}{(2n+1)^2}} = \frac{1}{4} \frac{1}{n(n+1)} \]

Observe that (a telescoping sum)

\[ b_1 - b_n = (b_1 - b_2) + (b_2 - b_3) + \ldots (b_{n-1} - b_n) \]
therefore

\[ b_1 - b_n < \frac{1}{4} \sum_{m=1}^{n-1} \frac{1}{m(m+1)} < \frac{1}{4} \sum_{m=1}^{\infty} \frac{1}{m(m+1)} = \frac{1}{4} \]

hence

\[ b_n > b_1 - \frac{1}{4} = \frac{e}{\sqrt{2}} - \frac{1}{4} \approx 1.67212 \]

so \((b_n)_{n \geq 1}\) is bounded from below, hence is convergent to some constant \(D\). This implies that

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} e^{b_1} = e^{\lim_{n \to \infty} a_n} = e^D. \]

2.2. Part 2. We will prove in this part the Wallis product formula

(1)

\[ \prod_{n=1}^{\infty} \frac{2n}{2n-1} \frac{2n}{2n+1} = \frac{\pi}{2} \]

This formula can be easily derived immediately from the Euler formula

\[ \sin(x) = x \prod_{n=1}^{\infty} \left( 1 - \left( \frac{x}{\pi n} \right)^2 \right) \]

but for completeness of our arguments we shall give its elementary proof.

Let us start from the interval \( \int \sin^n x \, dx \). Integrating by parts we get

\[
\int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx = - \int \sin^{n-1} x (\cos x)' \, dx =
\]

\[ = - \sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \cos^2 x \, dx =
\]

\[ = - \sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x (1 - \sin^2 x) \, dx = \ldots \]

and after easy calculus we get

(2) \[
\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.
\]

Hence

\[ \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx. \]

Notice that \( \int_0^{\pi/2} 1 \, dx = \pi/2 \) and \( \int_0^{\pi/2} \sin x \, dx = 1 \). Hence we are able to calculate the integral \( \int_0^{\pi/2} \sin^n x \, dx \) for arbitrary \( n \). After a while we get

\[ S_n = \int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k-1}{2k}, \]

\[ C_n = \int_0^{\pi/2} \sin^{2n+1} x \, dx = \prod_{k=1}^{n} \frac{2k}{2k+1}. \]
Hence, finally we get
\[
\frac{\pi}{2} = S_n \prod_{k=1}^{n} \frac{2k}{2k - 1} = S_n \prod_{k=1}^{n} \frac{2k}{2k + 1} \left( \prod_{k=1}^{n} \frac{2k}{2k - 1} \right)^{-1} = \frac{S_n}{C_n} \prod_{k=1}^{n} \frac{2k}{2k - 1} \frac{2k}{2k + 1}.
\]

Therefore the Wallis formula will be proved if we show that \( \lim_{n \to \infty} \frac{S_n}{C_n} = 1 \). Fortunately this step is easy. Namely for \( x \in (0, \frac{\pi}{2}) \) we have
\[
0 < \sin^{2n+2} x < \sin^{2n+1} x < \sin^{2n} x
\]
hence
\[
0 < S_{n+1} < C_n < S_n
\]
so
\[
1 > \frac{S_n}{S_n} > \frac{S_n}{C_n}, \quad \frac{S_{n+1}}{S_n} = \frac{n}{n + 1}.
\]
Hence the Wallis formula is proved.

2.3. Part 3. The Wallis formula may be written in a more compact way as
\[
(3) \quad \lim_{n \to \infty} \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} = \frac{\pi}{2}
\]
In Part 1 we proved that \( n! \sim C\sqrt{2n}(\frac{e}{\pi})^n \), for some constant \( C \). If we put this approximation into the formula then we get
\[
\frac{\pi}{2} = \lim_{n \to \infty} \frac{2^{4n}C^4(2n)^2(\frac{e}{\pi})^{4n}}{C^24n(\frac{e}{\pi})^2(2n+1)} = C^2 \lim_{n \to \infty} \frac{2^{4n}4n^2n^{4n}}{4n(2n+1)(2n)^{2n}} = \lim_{n \to \infty} C^2 \frac{n^2}{n(2n+1)} = \frac{C^2}{2}
\]
Therefore \( C = \sqrt{\pi} \) and the Strirling formula is proved.

3. Much more precise results

The Strirling approximation formula can be extended to the following inequality
\[
\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n+1}}
\]
A more precise version of the Stirling formula is given by

\[ n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + O\left(\frac{1}{n^4}\right)\right) \]

4. Remarks

You should do yourself the following:

(1) complete derivation of Equation 3 from Equation 1
(2) complete derivation of Equation 2

Final remark: this document may be used without any limitations.