Lim sup, lim inf : $\Omega = \mathbb{N}$

intuicja

$(\exists^\infty n)\varphi(n) \equiv$ "$\{n : \varphi(n)\}$ – nieskończ."

$(\forall^\infty n)\varphi(n) \equiv$ "poza skończoną liczbą wyjątków $\varphi(n)$ jest prawdziwe"

definicja

$$(\exists^\infty n)\varphi(n) \overset{def}{\equiv} (\forall N\in\mathbb{N})(\exists n>N)\varphi(n)$$

$$(\forall^\infty n)\varphi(n) \equiv (\exists N)(\forall n>N)\varphi(n).$$

(P) $\lim a_n = g \equiv (\forall \varepsilon > 0)(\exists N)(\forall n > N)\, |a_n - g| < \varepsilon$

$$\equiv (\forall \varepsilon > 0)(\forall^{\infty} n)\,(|a_n - g| < \varepsilon)$$

g jest p. skupienia ciągu (a_n) $\equiv$

$$(\forall \varepsilon > 0)(\forall N)(\exists n > N)\,(|a_n - g| < \varepsilon)$$

$$\equiv (\forall \varepsilon > 0)(\exists^{\infty} n)\,(|a_n - g| < \varepsilon)$$

∞ wiele

ZADANIE:

$$(\forall^{\infty} n)\,\varphi(n) \longrightarrow (\exists^{\infty} n)\,\varphi(n)$$

FAKT: $g = \lim_n a_n \longrightarrow$ g jest p. skupienia ciągu (a_n)

$$x \in \bigcup_N \bigcap_{n>N} A_n \equiv (\exists N)(\forall n>N)(x \in A_n)$$
$$\equiv (\forall^\infty n)(x \in A_n)$$

$$x \in \bigcap_N \bigcup_{n>N} A_n \equiv (\forall N)(\exists n>N)(x \in A_n)$$
$$\equiv (\exists^\infty n)(x \in A_n)$$

$$(\forall n)\varphi(n) \to (\forall^\infty n)\varphi(n) \to (\exists^\infty n)\varphi(n) \to (\exists n)\varphi(n)$$

$$x \in \bigcap_n A_n \subseteq \liminf A_n \subseteq \limsup A_n \subseteq \bigcup A_n$$

$\phi, [0,1], \phi, [0,1], \phi, [0,1], \ldots$

FAKT:

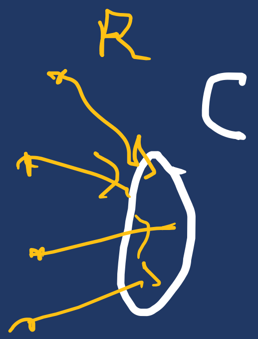
$$\boxed{\begin{array}{l} f: X \longrightarrow Y; \ A, B \text{ - zbiory} \\ f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B] \end{array}}$$

<u>D-d.</u>

$$x \in R^{-1}[C] \equiv x \in (R^{-1})[C]$$

$$\equiv (\exists y \in C)\big((y, x) \in R^{-1}\big)$$

$$\equiv (\exists y \in C)\big((x, y) \in R\big)$$



$$\underline{x \in f^{-1}[C]} \equiv (\exists y \in C)\big((x, y) \in f\big) \equiv \underline{f(x) \in C}$$

$$x \in f^{-1}[A \cap B] \equiv f(x) \in A \cap B \equiv f(x) \in A \wedge f(x) \in B$$

$$\equiv x \in f^{-1}[A] \wedge x \in f^{-1}[B] \equiv x \in f^{-1}[A] \cap f^{-1}[B] \ \blacksquare$$

UWAGA: mamy $f: X \longrightarrow Y$.

Dla $C \subseteq Y$ określamy

$$\Phi(C) = f^{-1}[C] \quad (\subseteq \mathrm{dom}(f) = X)$$

- $\Phi: P(Y) \longrightarrow P(X)$
- $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$
- $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$

$$\left.\begin{aligned} \Phi\Big(\bigcup_{i \in I} A_i\Big) &= \bigcup_{i \in I} \Phi(A_i) \\ \Phi\Big(\bigcap_{i} A_i\Big) &= \bigcap_{i \in I} \Phi(A_i) \end{aligned}\right\}$$

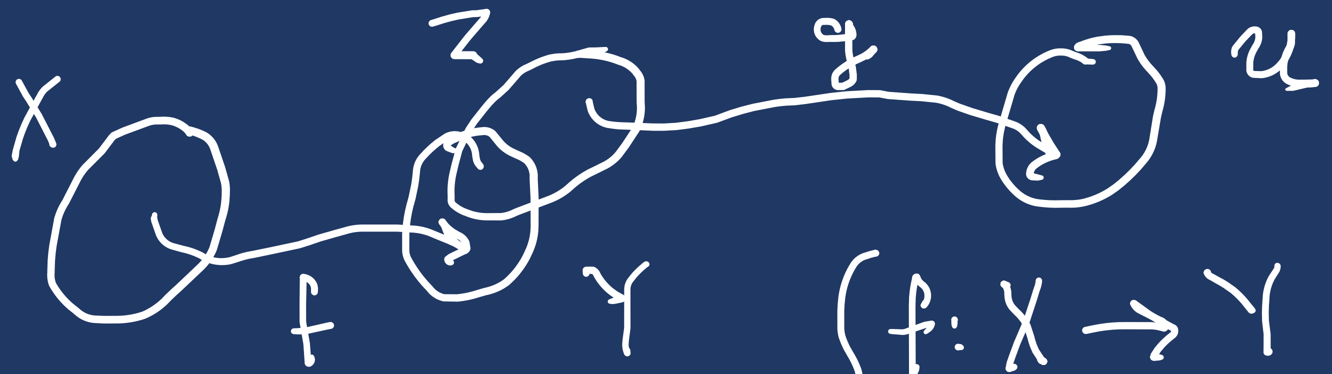
zadanie

- zadanie:

$\Phi(A^c) = ?$

$A \subseteq Y$

UWAGA:



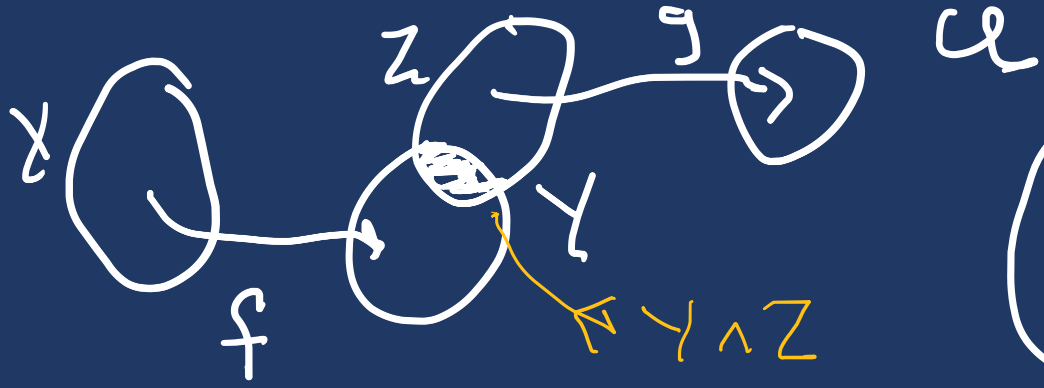
$$\begin{cases} f: X \to Y \\ g: Z \to U \end{cases}$$

- $g \circ f$ – funkcja

$$\left.\begin{array}{l} (x, z_1) \in g \circ f \\ (x, z_2) \in g \circ f \end{array}\right\} \equiv \begin{array}{l} (\exists y_1)\big((x, y_1) \in f \wedge (y_1, z_1) \in g\big) \wedge \\ (\exists y_2)\big((x, y_2) \in f \wedge (y_2, z_2) \in g\big) \end{array}$$

$$\Rightarrow (\exists y)\big((x, y) \in f \wedge (y, z_1) \in g \wedge (y, z_2) \in g\big)$$

$$\Rightarrow z_1 = z_2 \quad \blacksquare$$



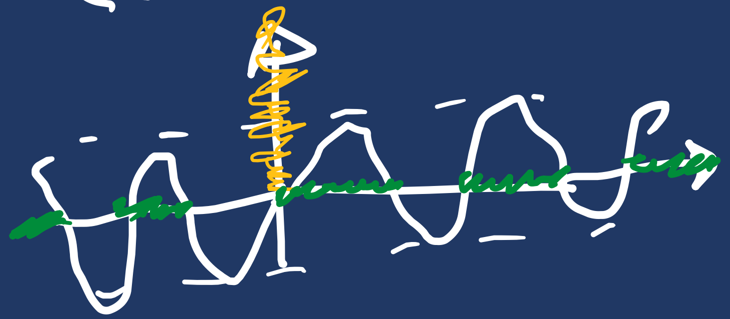
$$\boxed{\operatorname{dom}(g\circ f) = f^{-1}[\operatorname{dom}(g)]}$$

P. $f(x) = \sin(x)$; $f : \mathbb{R} \to \mathbb{R}$

$g(x) = \sqrt{x}$; $g : [0, \infty) \to \mathbb{R}$

$(g\circ f)(x) = \sqrt{\sin(x)}$

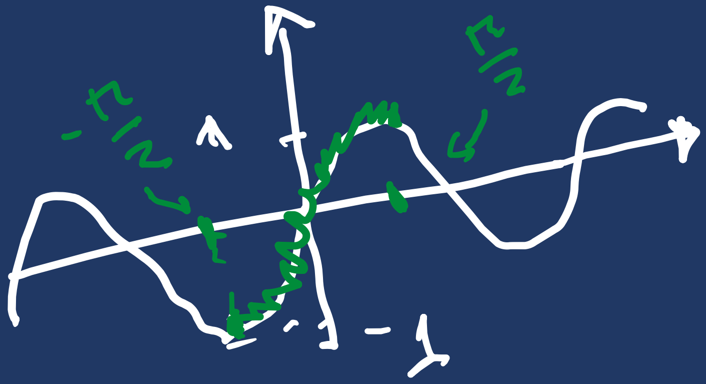
$\operatorname{dom}(g) = [0, \infty)$



$\sin^{-1}[[0,\infty)] = \sin^{-1}([0,1])$

$= \bigcup_{k\in\mathbb{Z}} [2k\pi, (2k+1)\pi]$

Uwaga:

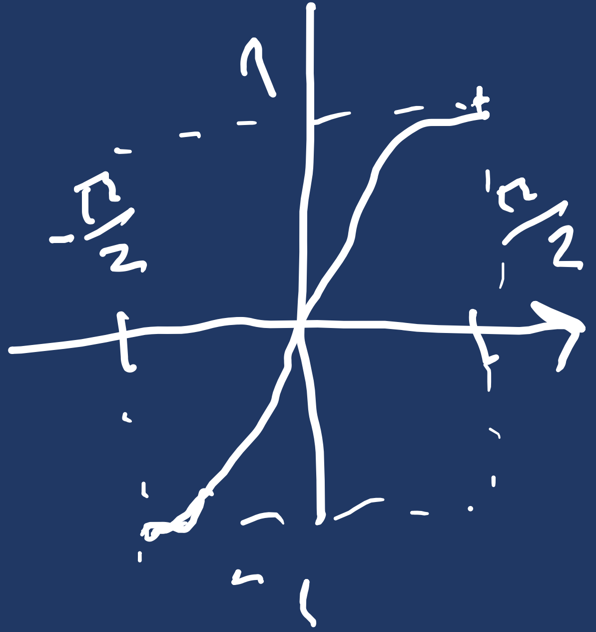


$$\sin\restriction\left[-\frac{\pi}{2},\frac{\pi}{2}\right] = \underline{\sin}(x)$$

$$f: X \to Y, \quad A$$

$$f\restriction A = \{(x,f(x)) : x\in A\}$$

$$= f\cap (A\times Y)$$



$$\underline{\sin} : \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \xrightarrow[\text{na}]{1-1} [-1,1]$$

$$\underline{\sin}^{-1} : [-1,1] \longrightarrow \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

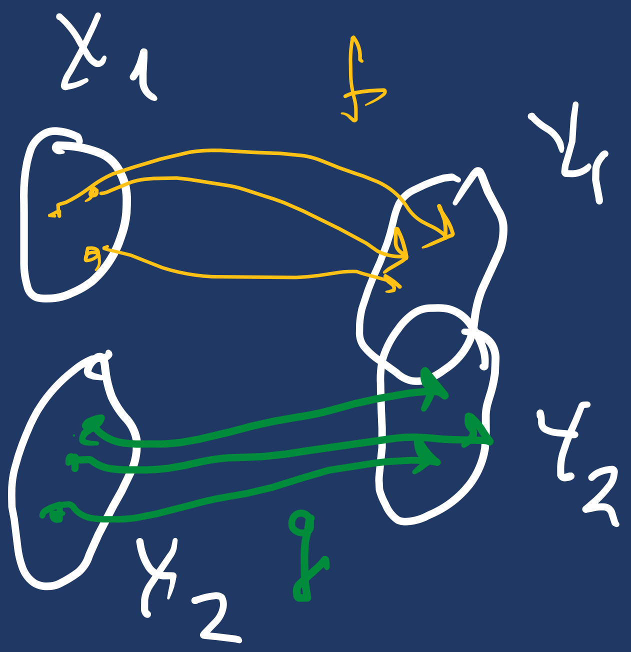
arcsin

$\sin^{-1}(x) \leftarrow$ wartość funkcji

$\sin^{-1}[A]$ = przeciwobraz A przez sin.

ZLEPIANIE Funkcji

$$f : X_1 \longrightarrow Y_1$$
$$g : X_2 \longrightarrow Y_2$$



- $X_1 \cap X_2 = \emptyset$

$$f \cup g : X_1 \cup X_2 \longrightarrow Y_1 \cup Y_2$$

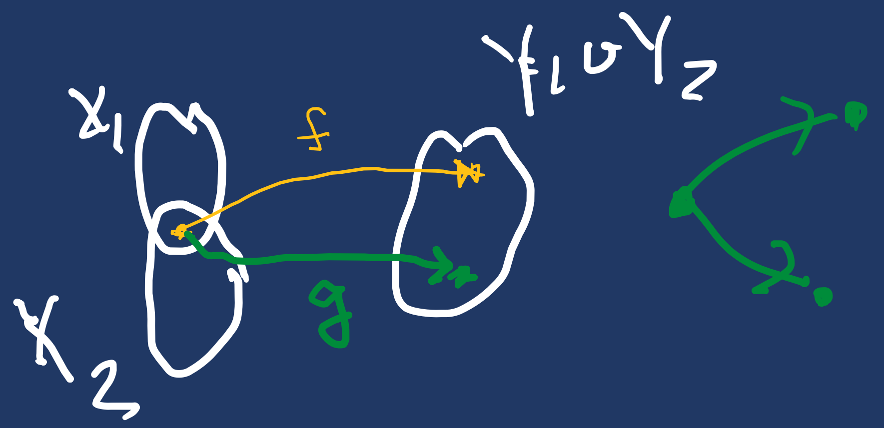
- $f \cup g$ jest funkcją

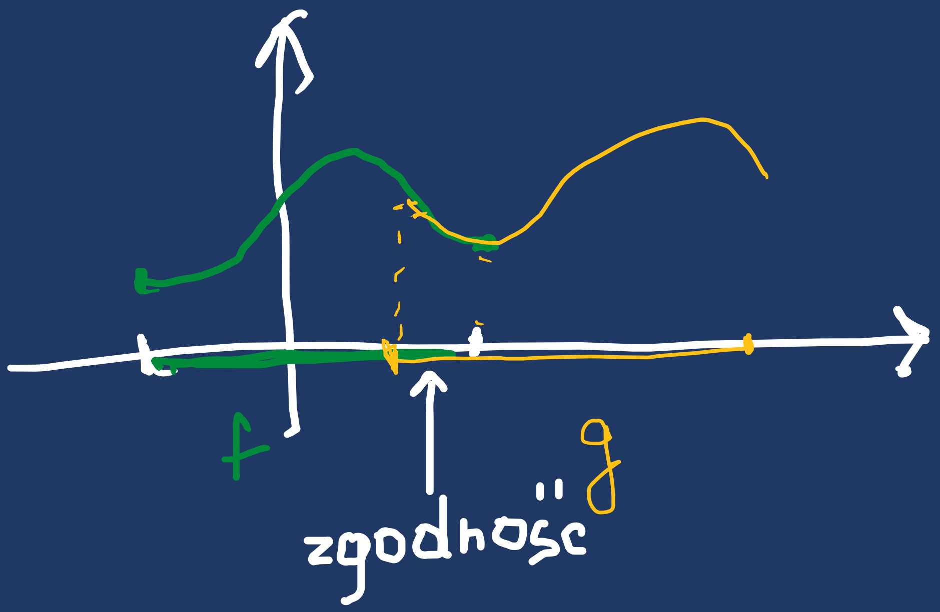
$$\equiv$$

$$(\forall x \in X_1 \cap X_2)\,(f(x) = g(x))$$

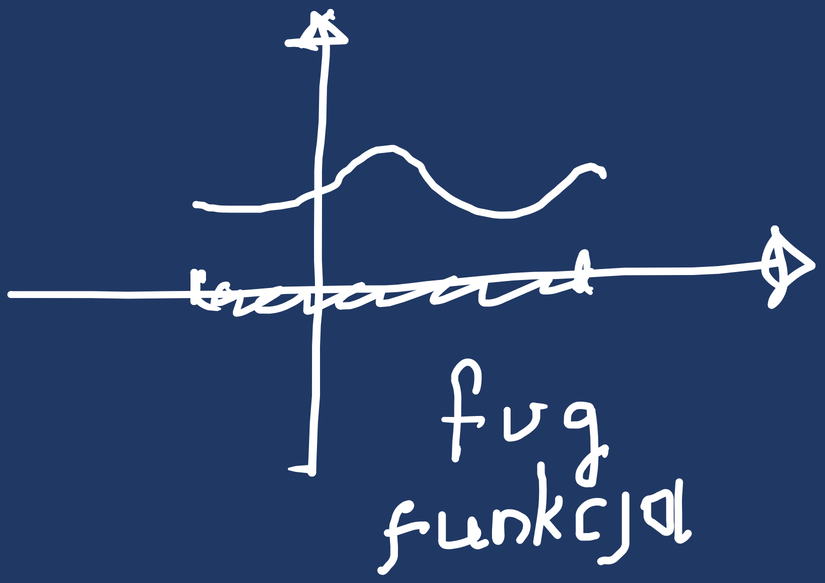
$$\equiv$$

$$f \restriction (X_1 \cap X_2) = g \restriction (X_1 \cap X_2)$$





$\Longrightarrow$



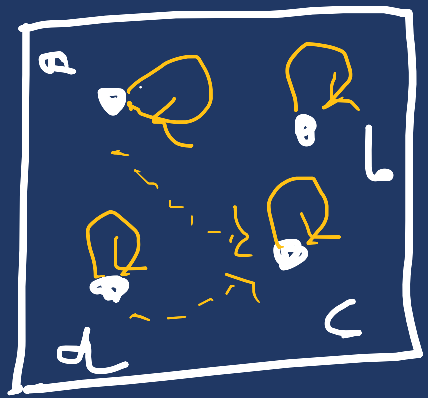
RELACYJNE ZOO

Def. $R \subseteq X \times X$

1) R jest <u>ZWROTNA</u> na $X \equiv (\forall x \in X)\big((x,x) \in R\big)$
2) R jest <u>SYMETRYCZNA</u> $\equiv (\forall x,y)\big((x,y) \in R \rightarrow (y,x) \in R\big)$
3) R jest <u>PRZECHODNIA</u> $\equiv (\forall x,y,z)\Big(\big((x,y) \in R \wedge (y,z) \in R\big) \rightarrow (x,z) \in R\Big)$
4) R jest <u>SŁABO-ANTYSYMETRYCZNA</u> $\equiv (\forall x,y)\Big(\big((x,y) \in R \wedge (y,x) \in R\big) \rightarrow x = y\Big)$

• ZWROTNOŚĆ

$$(\forall x \in X)((x,x) \in R)$$



$X = \{a, b, c, d\}$

$$\{(x,x) : x \in X\} \subseteq R$$

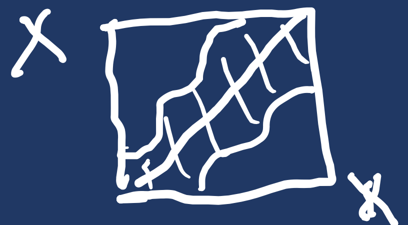
$$id_X \subseteq R$$

P na $X = \mathbb{R}$: $\leq$ - zwrotna, $=$ - zwrotna

$<$ - nie jest zwrotna

P $X = \mathbb{N}^+$: $x P y \equiv$ "$x | y$" $\leftarrow$ zwrotna

P $|X| = n$ $X = \{x_1 .. x_n\}$ $|X \times X| = n \cdot n = n^2$



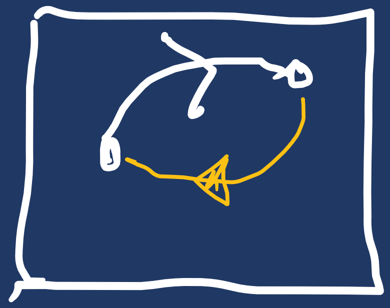
$R \subseteq X \times X$ $|(X \times X) \setminus id_X| = n^2 - n$

zwr.

liczba rel. zwr. na X : $2^{n(n-1)}$

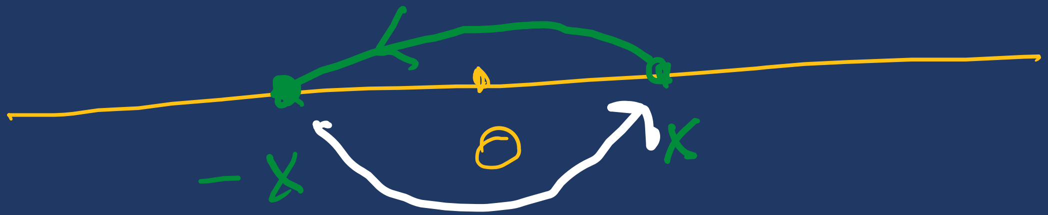
- SYMETRIA

$$(\forall x,y)\big((x,y)\in R \longrightarrow (y,x)\in R\big)$$



- $<$ (na $\mathbb{R}$) ← nie jest symetryczna

- $\mathbb{R}$: $\{(x,-x) : x \in \mathbb{R}\}$ ← jest symetryczna



$-(-x) = x$

- $\mathcal{G} = (G, \cdot)$

$$I = \{(g, g^{-1}) : g \in G\}$$

$$(g^{-1})^{-1} = g$$

symetryczna

R - sym. na X

$X = \{1, 2, 3, \dots n\}$

$(x, y) \in R$

$\Downarrow$

$(y, x) \in R$

$\equiv$

$(x, y) \in R^{-1}$

$(x, y) \in R \longrightarrow (x, y) \in R^{-1}$

$R \subseteq R^{-1}$

$R = R^{-1}$ = symetria R

$(x, y) \in R^{-1}$

$\equiv$

$(y, x) \in R$

sym.

$(x, y) \in R$