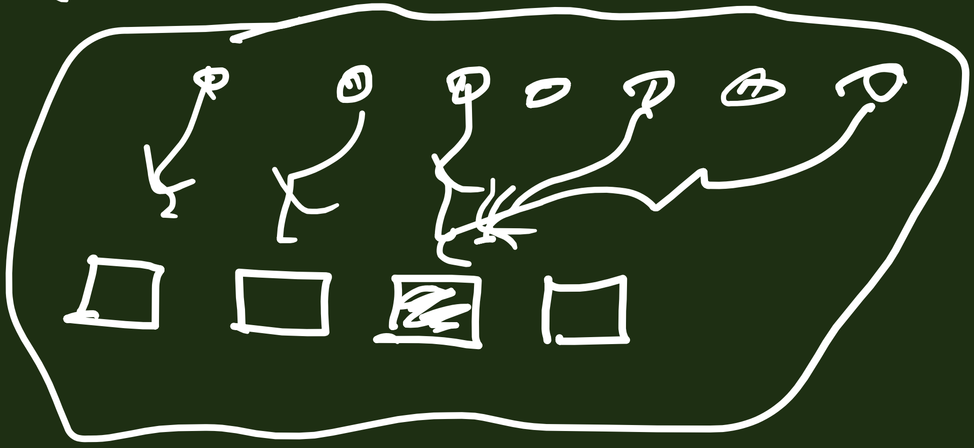
zadanie: $n \in \mathbb{N}$; istnieje $K \in \mathbb{N}^+$ t. że

$$n \cdot K = \sum_{i=0}^{L} \varepsilon_i \cdot 10^i \qquad \varepsilon_i \in \{0,1\}$$

FAKT (± zasada Dirichletta dla ∞)

$$\left[f : \mathbb{N} \longrightarrow \{1, \ldots, n\}\right] \Rightarrow \exists i \in \{1 .. n\}$$

$$|f^{-1}[\{i\}]| = \aleph_0$$



$$f(k) = 10^k \bmod n$$

$$f : \mathbb{N} \longrightarrow \{0, \dots, n-1\}$$

Jest $a \in \{0, \dots, n-1\}$ $\quad |f^{-1}(a)| = \infty$

$$k_1 < k_2 < \dots < k_n \qquad f(k_i) = a$$

$$(*) \begin{cases} 10^{k_1} = \alpha_1 \cdot n + a \\ 10^{k_2} = \alpha_2 \cdot n + a \\ \vdots \\ 10^{k_n} = \alpha_n \cdot n + a \end{cases}$$

$$n(\alpha_1 + \dots + \alpha_n) + n \cdot a = n(\alpha_1 + \dots + \alpha_n + a)$$

$$= 10^{k_1} + 10^{k_2} + \dots + 10^{k_n}$$

Liczby porządkowe: $\mathrm{ord}(\alpha) \equiv \mathrm{tran}(\alpha) \wedge$

$$\wedge (\forall \beta, \gamma \in \alpha)\,\underbrace{(\beta \in \gamma \vee \beta = \gamma \vee \gamma \in \beta)}_{(\beta < \gamma \,\vee\, \beta = \gamma \,\vee\, \gamma < \beta)}$$

Klasyfikacja:

$$\begin{cases} \mathrm{succ}(\alpha) \equiv (\exists \beta)(\alpha = \beta + 1) \\ \lim(\alpha) \equiv (\alpha \neq 0) \wedge \neg\,\mathrm{succ}(\alpha) \end{cases} \qquad \beta + 1 = \beta \cup \{\beta\}$$

Tw (o rekursji pozaskończonej)

Zał. że $(\forall x)(\exists ! y)\varphi(x,y)$. Niech $\alpha \in ON$.
Istnieje dokł. jedna funkcja f t. że

1) $dom(f) = \alpha$

2) $(\forall \beta \in \alpha)\ \varphi(f \upharpoonright \beta, f(\beta))$

$\beta = 0$: $\varphi(f\upharpoonright_0, f(0)) \equiv \varphi(\emptyset, f(0))$

$\beta = 1$ $\varphi(f\upharpoonright_1, f(1)) \equiv \varphi(\{(0, f(0))\}, f(1))$

$1 = \{0\}$

$\beta = 5$ $\varphi(f\upharpoonright\{0,1,2,3,4\}, f(5))$ $\beta = \omega$ $\varphi(f\upharpoonright_\omega, f(\omega))$.

Uwaga: $f, g : \alpha \longrightarrow$

$$(\forall \beta \in \alpha)\ \varphi(f\restriction_{\beta}, f(\beta)) \wedge \varphi(g\restriction_{\beta}, g(\beta))$$

Zał. że $f \neq g$. Jest $\beta \in \alpha$: $f(\beta) \neq g(\beta)$

$$Z = \{\beta \in \alpha : f(\beta) \neq g(\beta)\} \neq \emptyset \quad \subseteq \alpha$$

$$\beta_0 = \min(Z)$$

$$f\restriction_{\beta_0} = g\restriction_{\beta_0} \qquad \exists! y\, \varphi(f|\beta_0, y)$$

$$f(\beta_0) = g(\beta_0)$$ sprz.

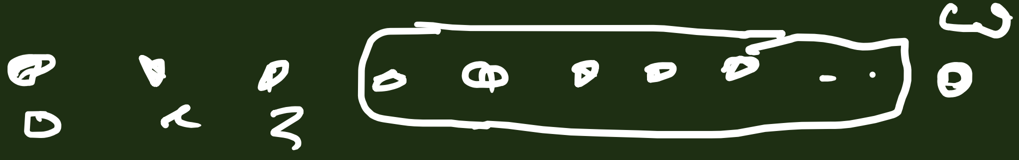
Ⓟ Dodawanie:

def.
$$\begin{cases} \alpha+0=\alpha \\ \alpha+(\beta+1)=(\alpha+\beta)+1 \\ \alpha+\lambda=\sup\{\alpha+\xi : \xi<\lambda\} \quad : \lim(\lambda) \end{cases}$$

Ⓩ popr. tej def $\Leftarrow$ tw. o rekursji porządk.

$$3+\omega = \sup\{3+n : n\in\omega\} = \omega$$

$3+\omega \neq \omega+3$



$\omega+3 > \omega$

$$\text{def}\begin{cases}\alpha\cdot 0 = 0\\ \alpha\cdot(\beta+1) = \alpha\cdot\beta+\alpha\\ \alpha\cdot\lambda = \bigcup\limits_{\xi<\lambda}(\alpha\cdot\xi)\end{cases}\qquad \operatorname{Lim}(\lambda)$$

$$2\cdot\omega = \bigcup_{n<\omega}(2\cdot n) = \omega$$

$$\omega\cdot 2 = \omega\cdot(1+1) = \omega\cdot 1+\omega$$

$$= \omega\cdot(0+1)+\omega = (\omega\cdot 0+\omega)+\omega$$

$$= \omega+\omega.$$

$$\omega\cdot 3 = (\omega+\omega)+\omega$$

def

$$\begin{cases} \alpha^0 = 1 \\ \alpha^{\beta+1} = \alpha^\beta \cdot \alpha \\ \alpha^\lambda = \bigcup\limits_{\xi<\lambda} \alpha^\xi \end{cases}$$

$$\omega^1 = \omega^{0+1} = \omega^0 \cdot \omega$$
$$= 1 \cdot \omega = \omega$$

$$2^\omega = \bigcup_{n<\omega} 2^n = \omega$$

$$\omega^2 = \omega^{1+1} = \omega^1 \cdot \omega = \omega \cdot \omega$$

$$\omega^3 = (\omega \cdot \omega) \cdot \omega$$