Wsp. dwumianowe: III

Tw dwumianowy

$$(1+x)^{\alpha} = \sum_{k\geq 0} \binom{\alpha}{k} x^{k} \quad ; \ |x|<1, \ \alpha \in \mathbb{R}.$$

Uwaga: $\alpha \in \mathbb{N}$, $k>\alpha$: $\binom{\alpha}{k} = 0$

Uwaga na temat dowodu

$$f(x) = (1+x)^{\alpha} \qquad f^{(k)}(x) = \alpha^{\underline{k}} (1+x)^{\alpha-k}$$

wzór Taylora:

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^{k} + R_n(x) =$$

$$= \sum_{k=0}^{n-1} \binom{\alpha}{k} x^{k} + R_n(x).$$

$$R_n(x) = \frac{f^{(n)}(0+\theta_n x)}{n!} x^n \qquad 0 \le \theta_n < 1.$$

dla wst. $|x|<1$

$$= \binom{\alpha}{n}(1+\theta_n x)^{\alpha-n} \cdot x^n.$$

Q: ¿ $R_n(x) \xrightarrow[n\to\infty]{} 0$?

$0<x<1$ $\quad \left|\binom{\alpha}{n}(1+\theta_n x)^{\alpha-n}\right| = \left|(1+\theta_n x)^{\alpha} \frac{\binom{\alpha}{n}}{(1+\theta_n x)^n}\right|$

$$\le 2^{\alpha}\binom{\alpha}{n} = C \cdot \binom{\alpha}{n}$$

$$\binom{\alpha}{n} = \frac{\alpha^{\underline{n}}}{n!} = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{1\cdot 2\cdot \ldots \cdot n} =$$

$$\prod_{k=0}^{n-1} \frac{\alpha-k}{k+1} = \prod_{k=0}^{n-1} \frac{\alpha+1-(k+1)}{k+1} =$$

$$= \prod_{k=0}^{n-1} \left(\frac{\alpha+1}{k+1} - 1 \right)$$

$$\left|\binom{\alpha}{n}\right| = \prod_{k=0}^{n-1} \left|1 - \frac{\alpha+1}{k+1}\right| = \prod_{k\leq\alpha} \left|1 - \frac{\alpha+1}{k+1}\right| \cdot$$

$n \gg \alpha$

$$\cdot \prod_{\alpha<k}^{n-1} \left|1 - \frac{\alpha+1}{k+1}\right| = D \cdot \prod_{\alpha<k}^{n-1} \left(1 - \frac{\alpha+1}{k+1}\right)$$

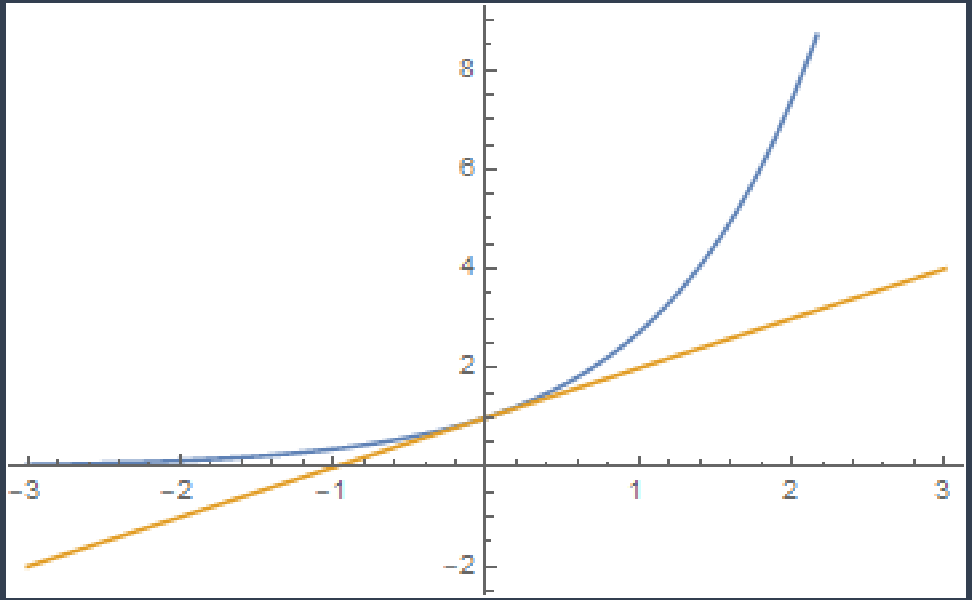
Użyjemy $\boxed{1 - x \leq e^{-x}} \equiv 1 + x \leq e^{x}$

...

P1. $x + 1 < 0$

P2. $x + 1 \geq 0$

...



(p) $\frac{1}{(1+x)^2} = (1+x)^{-2} = \sum_{k\geq 0} \binom{-2}{k} x^k \quad ; \quad |x|<1$

$$x^{\underline{k+1}} = x\cdot(x-1)\cdot\ldots\cdot(x-(k-1))(x-k) =$$

$$= (-1)^{k+1}(k-x)((k-x)-1)((k-x)-2)\cdot\ldots\cdot((k-x)-k)$$

$$= (-1)^{k+1}(k-x)^{\overline{k+1}}$$

$$x^{\underline{k}} = (-1)^k (k-x-1)^{\overline{k}}$$

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!} = \frac{(-1)^k (k-x-1)^{\underline{k}}}{k!}$$

górnoz negacja

$$\binom{x}{k} = (-1)^k \binom{k-x-1}{k}$$

(P)

$$(1-x)^{-2} = \sum_{k\geq 0} \binom{-2}{k} (-x)^k = \sum_{k\geq 0} (-1)^k \binom{k+2-1}{k} (-1)^k x^k$$

$$= \sum_{k\geq 0} \binom{k+1}{k} x^k = \sum_{k\geq 0} \binom{k+1}{1} x^k = \sum_{k\geq 0} (k+1) x^k$$

$|x| < 1$

$$x^{\underline{k}} \cdot \left(x - \frac{1}{2}\right)^{\underline{k}} = x\left(x - \frac{1}{2}\right)(x-1)\left(x - \frac{3}{2}\right) \cdot \ldots \cdot (x-k+1)\left(x-k+\frac{1}{2}\right)$$

$$= \frac{1}{2^{2k}}(2x)(2x-1)(2x-2)\cdot \ldots \cdot (2x-2k+1)$$

$$= \frac{1}{2^{2k}}(2x)^{\underline{2k}}$$

$$x^{\underline{k}}\left(x - \frac{1}{2}\right)^{\underline{k}} = \frac{1}{4^k}(2x)^{\underline{2k}}$$

wzór połówkowy

dzielimy obie strony przez $(k!)^2$

$$\frac{x^{\underline{k}}}{k!} \cdot \frac{\left(x - \frac{1}{2}\right)^{\underline{k}}}{k!} = \frac{1}{4^k} \frac{(2x)^{\underline{2k}}}{(2k)!} \cdot \frac{(2k)!}{k! \cdot k!}$$

$$\binom{x}{k}\binom{x-\frac{1}{2}}{k} = \frac{1}{4^k}\binom{2x}{2k}\binom{2k}{k}$$

$$\binom{x}{k}\binom{x-\frac{1}{2}}{k} = \frac{1}{4^k}\binom{2x}{2k}\binom{2k}{k}$$

$$\begin{cases} x \leftarrow n \\ k \leftarrow n \end{cases} \qquad \binom{n}{n}\binom{n-\frac{1}{2}}{n} = \frac{1}{4^n}\binom{2n}{2n}\binom{2n}{n}$$

$$\frac{1}{4^n}\binom{2n}{n} = \binom{n-\frac{1}{2}}{n}$$

$$(-1)^n$$

(P)

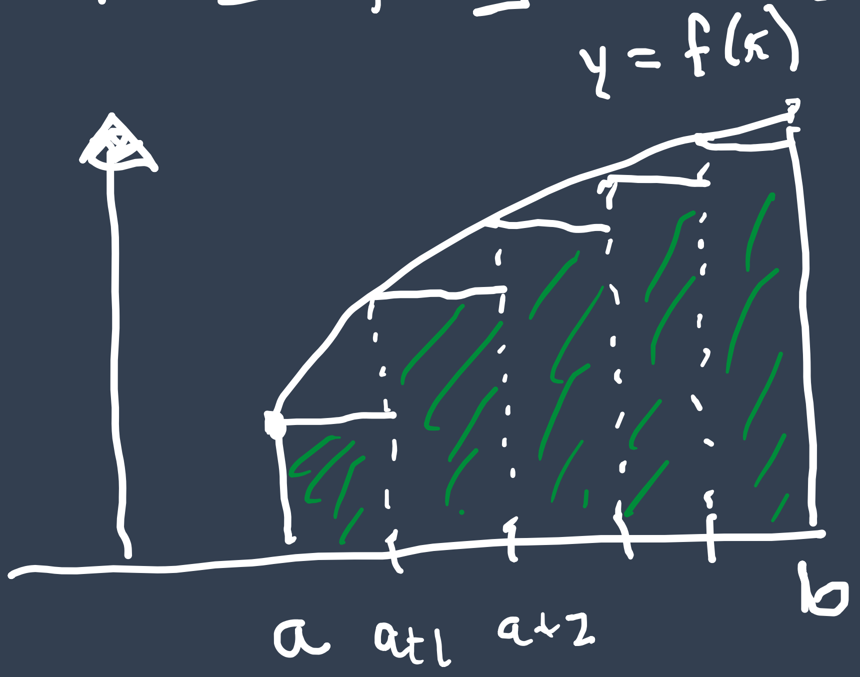
$$\sum_n \binom{2n}{n}x^n = \sum_n \binom{n-\frac{1}{2}}{n}4^n x^n = \sum_n \binom{n-(n-\frac{1}{2})-1}{n}(4x)^n$$

$$= \sum_n \binom{-\frac{1}{2}}{n}(-4x)^n = (1-4x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-4x}}.$$

Ograniczenia dolne i górne na $\sum_{k=a}^{b} a_k$.

- $a, b \in \mathbb{N}$; $a < b$;

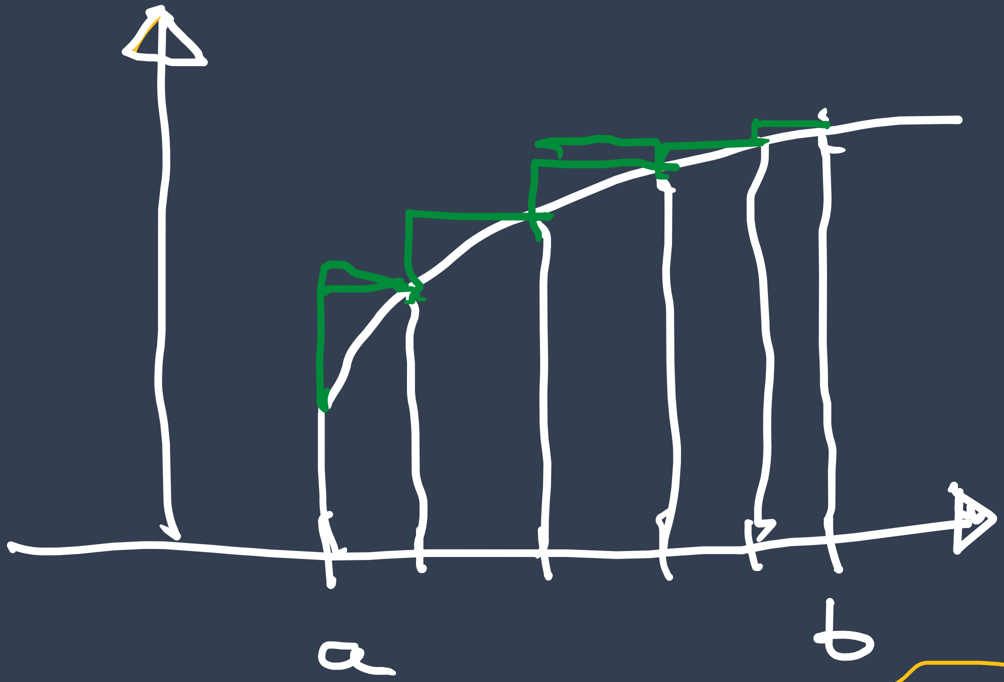
$f: [a, b] \longrightarrow \mathbb{R}$ niemalejąca



$a \leq x < y \leq b \rightarrow f(x) \leq f(y)$

$$\sum_{k=a}^{b-1} f(k) \leq \int_a^b f(x)\,dx$$

$$\sum_{k=a}^{b} f(k) \leq \int_a^b f(x)\,dx + f(b)$$



$$\sum_{k=a+1}^{b} f(k) \geq \int_a^b f(x)\,dx$$

$$\sum_{k=a}^{b} f(k) \geq \int_a^b f\,dx + f(a)$$

$$f(a) + \int_a^b f(x)\,dx \leq \sum_{k=a}^{b} f(k) \leq \int_a^b f(x)\,dx + f(b)$$

$f: [a,b] \to \mathbb{R}$ ненамаляваща

(P)

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

$$\ln(n!) = \sum_{k=1}^{n} \ln(k)$$

$$\int_1^n \ln(x)\,dx = \int_1^n (x)' \ln(x)\,dx = x\ln(x)\Big|_1^n - \int_1^n x \frac{1}{x}\,dx$$

$$= n\ln(n) - (n-1) =$$

$$= n(\ln(n) - 1) + 1 = n(\ln(n) - \ln(e)) + 1$$

$$= n\ln\left(\frac{n}{e}\right) + 1 = \ln\left(\left(\frac{n}{e}\right)^n\right) + 1$$

$$e^{\ln u} = u$$

$$\ln\left(\left(\frac{n}{e}\right)^n\right) + 1 \leq \ln(n!) \leq \ln\left(\left(\frac{n}{e}\right)^n\right) + 1 + \ln(n)$$

$$\left(\frac{n}{e}\right)^n \cdot e \leq n! \leq \left(\frac{n}{e}\right)^n \cdot e \cdot n$$

$$e\left(\frac{n}{e}\right)^n < n! < e\cdot n\left(\frac{n}{e}\right)^n$$

Wzór Stirlinga: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$!!!

ZADANIE:

$$\binom{2n}{n} \approx \frac{1}{\sqrt{\pi n}} 4^n$$

$$\binom{n}{\lfloor n/2 \rfloor} \sim \sqrt{\frac{2}{\pi n}}\, 2^n$$

asymptotyczna równoważność

- $f : [a,b] \to \mathbb{R}$ nierosnąca

$x \leq y \to f(x) \geq f(y)$

$$\int_a^b f\,dx + f(b) \leq \sum_{k=a}^{b} f(k) \leq \int_a^b f\,dx + f(a)$$

ZADANIE ↗

DEF: n-ta liczba harmoniczna:

!!!

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

FAKT:

$$\sum_{n\geq 1} \frac{1}{n} = +\infty$$

$$\lim_{n} H_n = +\infty$$

Oszacowania H_n: $\sum_{k=1}^{n} \frac{1}{k}$

$$f(x) = \frac{1}{x}$$

$$\int_1^n f(x)\,dx = \int_1^n \frac{1}{x}\,dx = \left[\ln(x)\right]_1^n = \ln n - \ln 1$$

$$= \ln(n)$$

$$\ln(n) + \frac{1}{n} \leq H_n \leq \ln(n) + 1$$

Knuth + ...
"Mat. dyskr"

$$0 < \frac{1}{n} \leq H_n - \ln(n) \leq 1$$

FAKT: $\lim_{n \to \infty} (H_n - \ln n) = \gamma \simeq 0.5772\ldots$

stała Eulera-Mascheroniego

$$H_n = \ln(n) + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + O\left(\frac{1}{n^5}\right)$$

$$H_n = \ln(n) + \gamma + O\left(\frac{1}{n}\right)$$