Liczby Fibonacciego

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} & n \geq 2 \end{cases}$$

n:	0	1	2	3	4	5	6	7	8	....
F_n:	0	1	1	2	3	5	8	13	21	....

FAKT:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

WNIOSEK

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

$n = 1$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$$

Ind :

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$$

Alg. szybkiego potęgowania

Idea algorytmu : dane a, n

wynik : a^n.

$n = x_0 + x_1 2 + x_2 \cdot 2^2 + \ldots + x_k 2^k$, $x_i \in \{0, 1\}$

$= (x_k x_{k-1} \ldots x_1 x_0)_{(2)}$ $k \leq \lg_2(n) + 1$

$a^{(1100110)_{(2)}}$

$= a^{0 + 2 \cdot 1 + 2^2 \cdot 1 + 2^3 \cdot 0 + 2^4 \cdot 0 + 2^5 \cdot 1 + 2^6 \cdot 1}$

$= a^0 \cdot a^2 \cdot a^{2^3} \cdot a^{2^5} \cdot a^{2^6}$

obliczam : $a \quad a^2 \quad a^{2^2} \quad a^{2^3} \quad a^{2^4} \quad a^{2^5} \quad a^{2^6}$

$a^{2^2} = (a^2)^2$

$\leq 2 \lg_2 n$ - operacji

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

- wartości własne M

$$0 = \det(M - \lambda \cdot I) = \det\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda(\lambda - 1) - 1$$

$$= \lambda^2 - \lambda - 1.$$

$$\Delta = 5$$

$$\omega_1 = \frac{1+\sqrt{5}}{2}$$

$$\omega_2 = \frac{1-\sqrt{5}}{2}$$

$(\omega_1 + \omega_2)$

są $f_1, f_2 \neq 0$

$M \cdot f_1 = \omega_1 f_1$

$M \cdot f_2 = \omega_2 f_2$

$$M = T^{-1} \cdot \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \cdot T$$

T – macierz przejścia $\{f_1, f_2\}$

a $\{e_1, e_2\}$

$\uparrow$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\uparrow$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$M \cdot M = (T^{-1} \cdot (\ \ ) \cdot T) \cdot (T^{-1} \cdot (\ \ ) \cdot T)$$

$$= T^{-1} (\ \ )^2 \cdot T$$

$$M^n = T^{-1} \cdot \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}^n \cdot T$$

$$M^n = T^{-1} \cdot \begin{pmatrix} \omega_1^n & 0 \\ 0 & \omega_2^n \end{pmatrix} \cdot T$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = M^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \qquad F_n = A\omega_1^n + B\omega_2^n$$

$$0 = F_0 = A + B \qquad B = -A$$

$$1 = F_1 = A\omega_1 + B\omega_2 = A\omega_1 - A\omega_2 =$$

$$= A(\omega_1 - \omega_2) = A\sqrt{5}$$

$$A = \frac{1}{\sqrt{5}}$$

$$F_n = \frac{1}{\sqrt{5}} \omega_1^n - \frac{1}{\sqrt{5}} \omega_2^n =$$

$$= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n + (-1)^{n+1} \left(\frac{\sqrt{5}-1}{2} \right)^n \right)$$

wzór Binet'a

$$F_n \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$$

$$\left(\frac{1+\sqrt{5}}{2} \right) = 1.618\ldots$$

$$\left(\frac{\sqrt{5}-1}{2} \right) = 0.618 \leftarrow \text{złota proporcja}$$

Tw. $\boxed{nwd(F_{n+1}, F_n) = 1}$

$nwd(a,b) = nwd(a-b, b)$, $a \geq b$.

- $nwd(F_1, F_0) = nwd(1, 0) = 1$
- $nwd(F_{n+2}, F_{n+1}) = nwd(F_{n+1} + F_n, F_{n+1})$

$$= nwd(F_n, F_{n+1}) = nwd(F_{n+1}, F_n) = 1$$

ind.

Zadanie: $\boxed{nwd(F_n, F_m) = F_{nwd(n,m)}}$

Tw. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \quad n \geq 1$

D-d. $n = 1$: OK

∘ $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} =$

$$= \begin{pmatrix} F_{n+1} + F_n & F_n + F_{n-1} \\ F_{n+1} & F_n \end{pmatrix} = \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}$$

OK

(Ind).

$$\det\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = F_{n+1}\cdot F_{n-1} - (F_n)^2$$

$$\|$$

$$\det\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n\right) = \left(\det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right)^n = (-1)^n$$

Wn (Cassini) $F_{n+1}\cdot F_{n-1} - (F_n)^2 = (-1)^n$

Fibonacci : ~ 1202

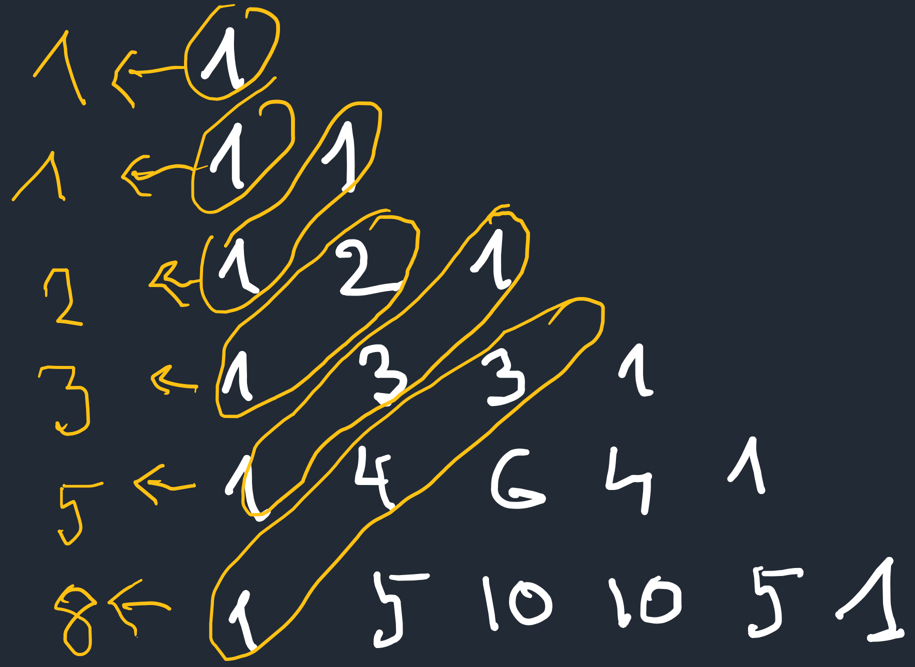
Cassini : ~1680

$$\begin{pmatrix} F_{n+m+1} & F_{n+m} \\ F_{n+m} & F_{n+m-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+m} =$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{m} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \cdot \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix}$$

Tw.

$$\boxed{F_{n+m} = F_{n+1} \cdot F_m + F_n \cdot F_{m-1}}$$



Tw $\sum_{a} \binom{n-a}{a} = F_{n+1}$

D-d: • $n=0$ $\binom{0}{0} = 1 = F_1$

• $n=1$: ok

• $\sum_{a} \binom{n+1-a}{a} =$

$$= \sum_{a} \left(\binom{n-a}{a} + \binom{n-a}{a-1} \right) = \sum_{a} \binom{n-a}{a} + \sum_{a} \binom{n-a}{a-1}$$

$$= F_{n+1} + \sum_{b} \binom{(n-1)-b}{b} = F_{n+1} + F_n = F_{n+2} \quad \blacksquare$$

$b = a-1$

$a = b+1$

LICZBY STIRLINGA II RODZAJU

$$\left\{ n \atop k \right\} = \left| \left\{ \mathcal{P} : |\mathcal{P}| = k \wedge \mathcal{P}\text{-rozbicie } [n] \right\} \right|$$

- $\mathcal{P}$-part $[n]$

1) $P \in \mathcal{P} \rightarrow P \neq \emptyset \wedge P \subseteq [n]$

2) $P, Q \in \mathcal{P},\ P \neq Q \rightarrow P \cap Q = \emptyset$

(P) $\left\{ 3 \atop 2 \right\} = 3$

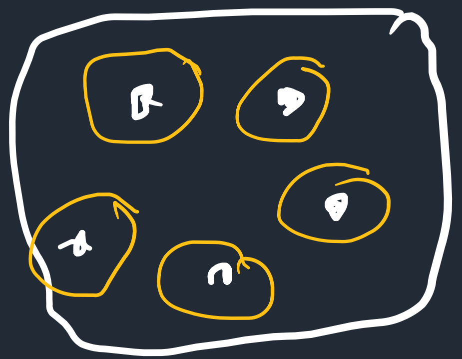
- $n \geq 1$ $\qquad \left\{ {n \atop 1} \right\} = 1$

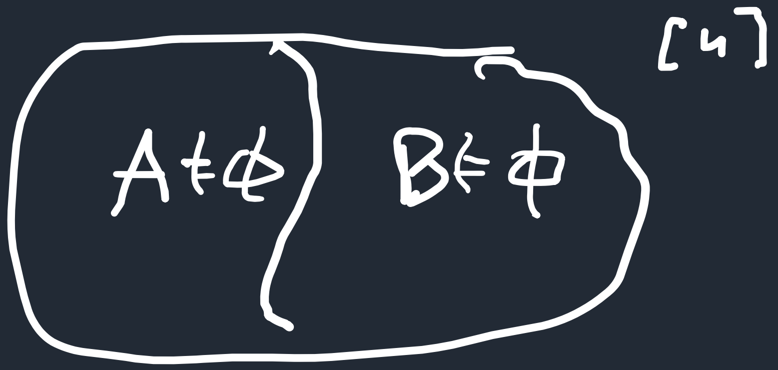
- $n \geq 1$ $\qquad \left\{ {n \atop n} \right\} = 1$

- $n \geq 1$ $\qquad \left\{ {n \atop n-1} \right\} = \binom{n}{2}$

- $n \geq 1$ $\qquad \left\{ {n \atop 2} \right\} = \frac{1}{2}\left(2^n - 2\right) = 2^{n-1} - 1.$

$\{A, B\} \Leftarrow (A, B),\ (B, A)$



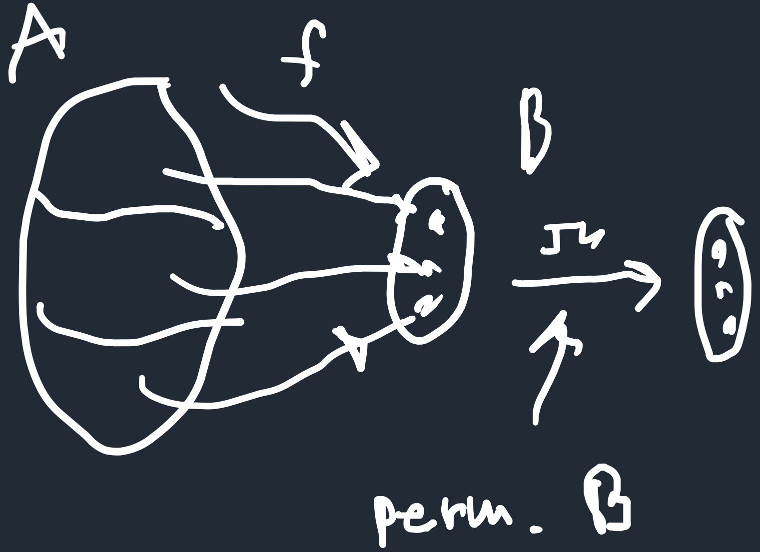


$$\mathrm{Sur}(A,B) = \{ f \in B^A : \mathrm{rng}(f) = B \}$$

$$f \in \mathrm{Sur}(A,B)$$

$$\downarrow$$

$$\{ f^{-1}(\{b\}) : b \in B \} \leftarrow \text{part. } A$$



$$\frac{1}{|B|!}\,|\mathrm{Sur}(A,B)| = \left\{ {|A| \atop |B|} \right\}$$

!!! FAKT: $\left\{ {n \atop k} \right\} = \frac{1}{k!} \left| \mathrm{Sur}([n],[k]) \right|$