Zmiana oznaczeń $\begin{bmatrix} n \\ k \end{bmatrix} \left(= \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \right)$

↑

liczby cykliczne Stirlinga

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = [\![n = 0]\!]$$

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$$

$$\begin{bmatrix} n \\ n \end{bmatrix} = 1$$

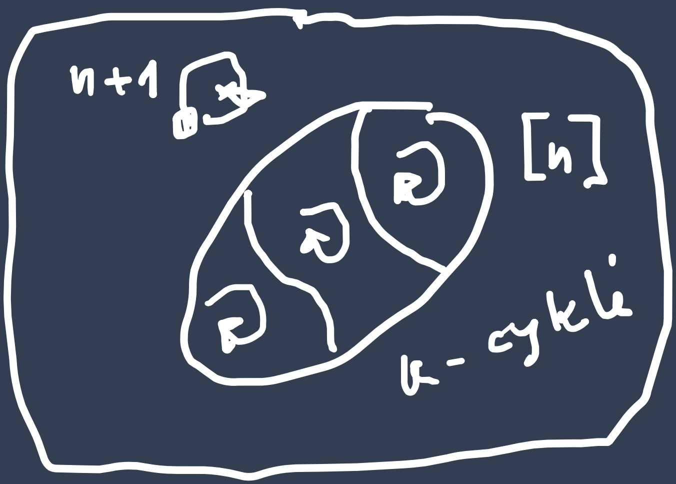
$$\boxed{\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!}$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)! \cdot H_{n-1}$$

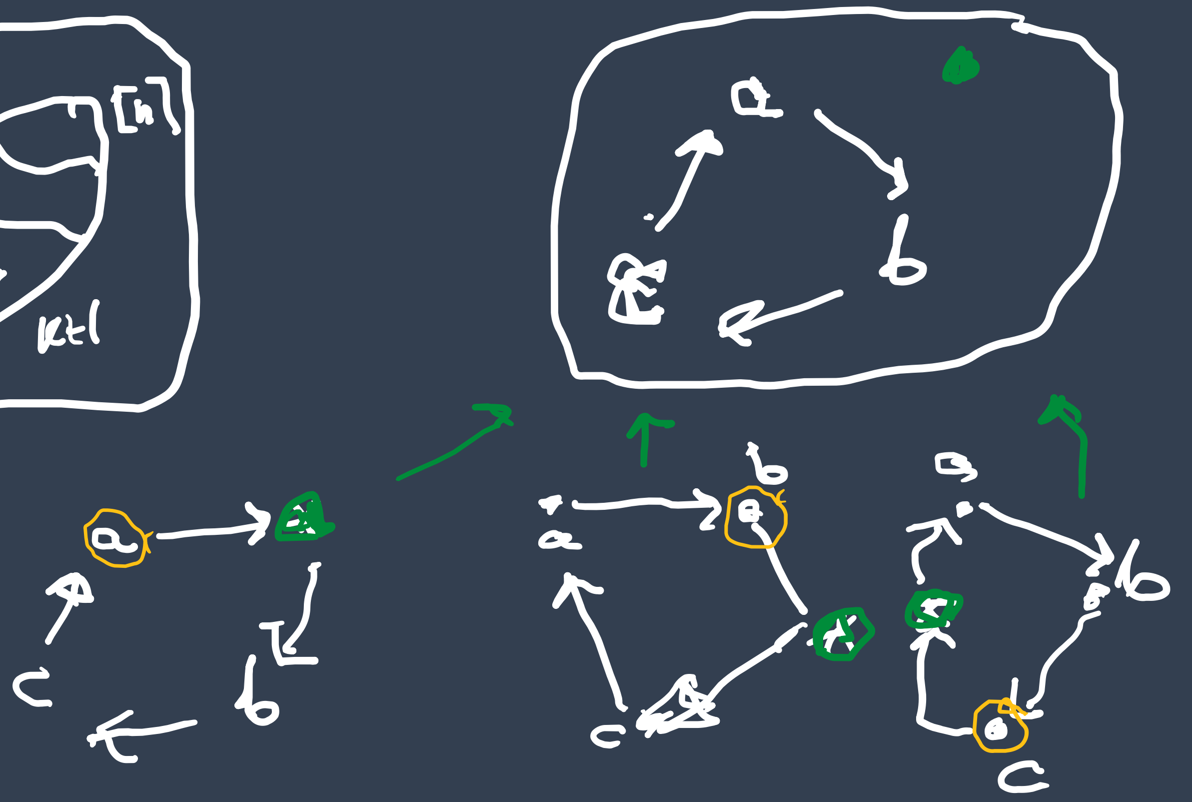
$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$$

Podst. wzór rekurencyjny

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + n \begin{bmatrix} n \\ k+1 \end{bmatrix}$$



$\begin{bmatrix} n \\ k \end{bmatrix}$



$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

$$\left\{ {n+1 \atop k+1} \right\} = \left\{ {n \atop k} \right\} + (k+1) \left\{ {n \atop k+1} \right\}$$

$$\left[{n+1 \atop k+1} \right] = \left[{n \atop k} \right] + n \left[{n \atop k+1} \right]$$

$x^{\overline{n}} = x(x+1) \cdot \ldots \cdot (x+(n-1))$ ← wielomian stopnia n

- $x^{\overline{0}} = 1$
- $n \geq 1 \longrightarrow$ $x^{\overline{n}} =$

$$= 0 + (n-1)! \cdot x + c_2 x^2 + \ldots + c_{n-1} x^{n-1} + x^n$$

$$x^{\overline{n}} = \sum_{k=0}^{n} c_{n,k} \cdot x^k$$

$n \geq 1$

$$\begin{cases} c_{n,0} = 0 \\ c_{n,1} = (n-1)! \\ c_{n,n} = 1 \end{cases}$$

$n \geq 1$

$$x^{\overline{n+1}} = x^{\overline{n}}(x+n) = \left(\sum_{k=0}^{n} c_{n,k} x^k\right)(x+n)$$

$$= \sum_{k=0}^{n} c_{n,k} \cdot x^{k+1} + \sum_{k=0}^{n} n \cdot c_{n,k} \cdot x^k = \sum_{k=0}^{n+1} \left(c_{n,k-1} x^k + n\, c_{n,k} x^k\right)$$

$(k=1 \ldots n)$

$$= \sum_{k=0}^{n+1} \underbrace{\left(c_{n,k-1} + n\, c_{n,k}\right)}_{c_{n+1,k}} x^k$$

$k < 0 \Rightarrow c_{n,k} = 0$

$$\sum_{k=0}^{n} c_{n,k} x^{k+1} = \sum_{l=1}^{n+1} c_{n,l-1} x^l$$

$l = k+1$

$$= \sum_{k=1}^{n+1} c_{n,k-1} x^k$$

$$C_{n+1,k} = C_{n,k-1} + C_{n,k} \cdot n \; ; \quad k \leftarrow k+1$$

$$C_{n+1,k+1} = C_{n,k} + n \cdot C_{n,k+1}$$

Ta sama rekur. co dla $\left[{n \atop k}\right]$

$$x^{\overline{n}} = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k$$

Z $\quad x^{\underline{n}} = (-1)^n (-x)^{\overline{n}}$

$$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}} = (-1)^n \sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-x)^k$$

$$\boxed{x^{\underline{n}} = \sum_k \underbrace{(-1)^{n+k} \begin{bmatrix} n \\ k \end{bmatrix}}_{s_{n,k}} x^k}$$

$$s_{n,k} = (-1)^{n+k} \begin{bmatrix} n \\ k \end{bmatrix} \leftarrow \text{znakowane licz. Stirl. I rodz.}$$

Def. Niech (Ω, P) będzie dyskr. przestrz. probab.
Zmienną losową na (Ω, P) nazywamy dowolną funkcję

$$X : \Omega \longrightarrow \mathbb{R}.$$

DEF: $X : \Omega \to \mathbb{R}$: wartość oczekiwana X

$$E(X) = \sum_{\omega \in \Omega} X(\omega) \cdot P[\omega]$$

expected value

$P[\{\omega\}]$

$\mathbb{P}$ $\quad \Omega = [n] \qquad P(A) = \dfrac{|A|}{n}$

$$X : [n] \longrightarrow \mathbb{R}$$

$$E(X) = \sum_{k=1}^{n} X(k) \cdot P(\{k\}) = \sum_{k=1}^{n} X(k) \frac{1}{n}$$

$$= \frac{X(1) + X(2) + \ldots + X(n)}{n}$$

wartość średnia

ogólniej:

$$\Omega \;; \quad P(A) = \frac{|A|}{|\Omega|}$$

$$E(X) = \frac{\sum_{\omega \in \Omega} X(\omega)}{|\Omega|}$$

Własności

(1) $X \equiv c \quad (\in \mathbb{R}) \qquad X: \Omega \to \mathbb{R}$

$E(X) = c$

$E(X) = \sum_{\omega} X(\omega) P[\{\omega\}] =$

$= \sum_{\omega} c \cdot P[\{\omega\}] = c \sum_{\omega \in \Omega} Pr[\{\omega\}]$

$= c \cdot P(\Omega) = c.$

(2) $E(a \cdot X) = \sum_{\omega} a \cdot X(\omega) P(\{\omega\}) = a \cdot E(X)$

(3) $E(X+Y) = \sum (X(\omega) + Y(\omega)) \cdot P(\{\omega\}) = E(X) + E(Y)$

$$(2)+(3):\quad \boxed{E(aX+bY) = a\cdot E(X) + b\cdot E(Y)}$$

$$X : \Omega \longrightarrow A \qquad \operatorname{rng}(X) \subseteq A$$

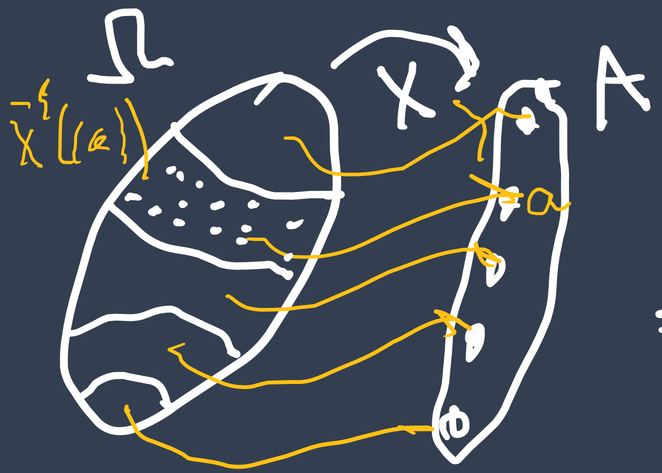
$$|A| \leq \aleph_0$$

$$E(X) = \sum_{\omega} X(\omega)P(\omega)$$

$$= \sum_{a\in A} \sum_{\omega \in X^{-1}(\{a\})} \overbrace{X(\omega)}^{X(a)} P(\omega) =$$

$$= \sum_{a\in A} a \cdot \sum_{\omega\in X^{-1}(\{a\})} P(\omega) = \sum_{a\in A} a\cdot P(X^{-1}(a))$$

$$= \sum_{a\in A} a \cdot P(X=a)$$



$$E(X) = \sum_{a \in A} a \cdot P(X = a)$$

w szczeg. $X : \Omega \longrightarrow \mathbb{N}$

$$E(X) = \sum_{n \in \mathbb{N}} n \cdot P(X = n)$$

P $\Omega = P([n])$ $P(\{A\}) = \frac{1}{2^n}$.

$S : \Omega \to \mathbb{N} : A \to |A|$

$E(S) = ?$

$$E(S) = \sum_{k \geq 0} k \cdot \Pr(|S| = k) =$$

$$= \sum_{k \geq 0} k \cdot \frac{\binom{n}{k}}{2^n} = \frac{1}{2^n} \sum_{k \geq 1} k \binom{n}{k} =$$

$$= \frac{1}{2^n} \sum_{k \geq 1} k \cdot \frac{n}{k} \binom{n-1}{k-1} = \frac{n}{2^n} \sum_{l=0}^{n-1} \binom{n-1}{l} =$$

$$= \frac{n}{2^n} \cdot 2^{n-1} = \frac{n}{2} .$$

② $\Omega = [n]^2$. $M(\{a, b\}) = \min(a, b)$

$$E(M) = ?$$

$X : \Omega \to \mathbb{N}$. zm. losowa

$$\varphi_X(x) = \sum_{n\geq 0} P[X=n] \cdot x^n$$ ← funkcja tworząca X

- $$\varphi_X(1) = \sum_{n\geq 0} P[X=n] \cdot 1 = 1$$
- $$\varphi'_X(x) = \sum_{n\geq 1} P[X=n] \quad n \cdot x^{n-1} = \sum_{n\geq 1} n \cdot P[X=n]\, x^{n-1}$$

$$\varphi'_X(1) = \sum_{n\geq 0} n P[X=n] = E(X).$$

$$\varphi'_X(1) = E(X)$$

Permutacje. $\Omega = \mathrm{Sym}([n])$ $n \geq 1$

$$P(\{\sigma\}) = \frac{1}{n!}$$

$C(\sigma)$ = ilość cykli perm. σ.

CEL: $E(C) = ?$

• $$\varphi_C(x) = \sum_k P[C=k]\, x^k = \sum_k \frac{\left[{n \atop k}\right]}{n!} x^k =$$

$$= \frac{1}{n!} \sum_{k=0}^{n} \left[{n \atop k}\right] x^k = \frac{1}{n!} x^{\overline{n}}.$$

CZYLI ±: ile cykli ma typowa permutacja

$$\left(x^{\overline{n}}\right)' = \left(x\cdot(x+1)\cdot(x+2)\cdot \ldots \cdot(x+(n-1))\right)'$$

$$= x'(x+1)\cdot\ldots\cdot(x+(n-1)) + x\cdot(x+1)'\cdot(x+2)\cdot\ldots\cdot(x+(n-1)) + \ldots$$

$$(fgh)' = f'gh + fg'h + fgh'$$

$$= x^{\overline{n}}\cdot\frac{1}{x} + x^{\overline{n}}\frac{1}{x+1} + \ldots = x^{\overline{n}}\left(\frac{1}{x}+\frac{1}{x+1}+\ldots+\frac{1}{x+(n-1)}\right)$$

$$\varphi_C'(1) = \frac{1}{n!}\cdot n!\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right) = H_n$$

$$E(C) = H_n$$

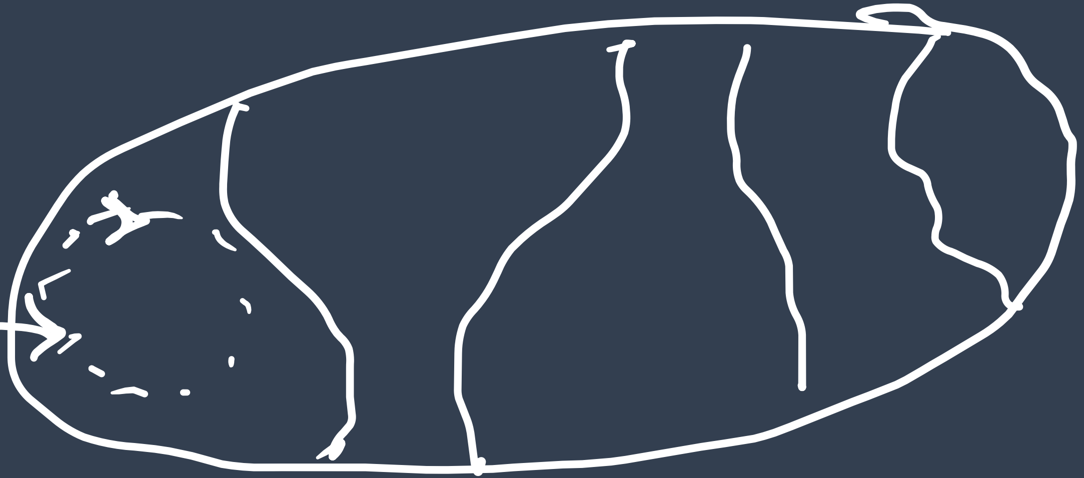
$$E(C) \approx \ln n$$

$n = 100$; $\ln 100 \triangleq 5$

[100]

$E(e) \simeq 5$

jeden Z



nich

ma

$\deg \geq \frac{100}{5} = 20$.