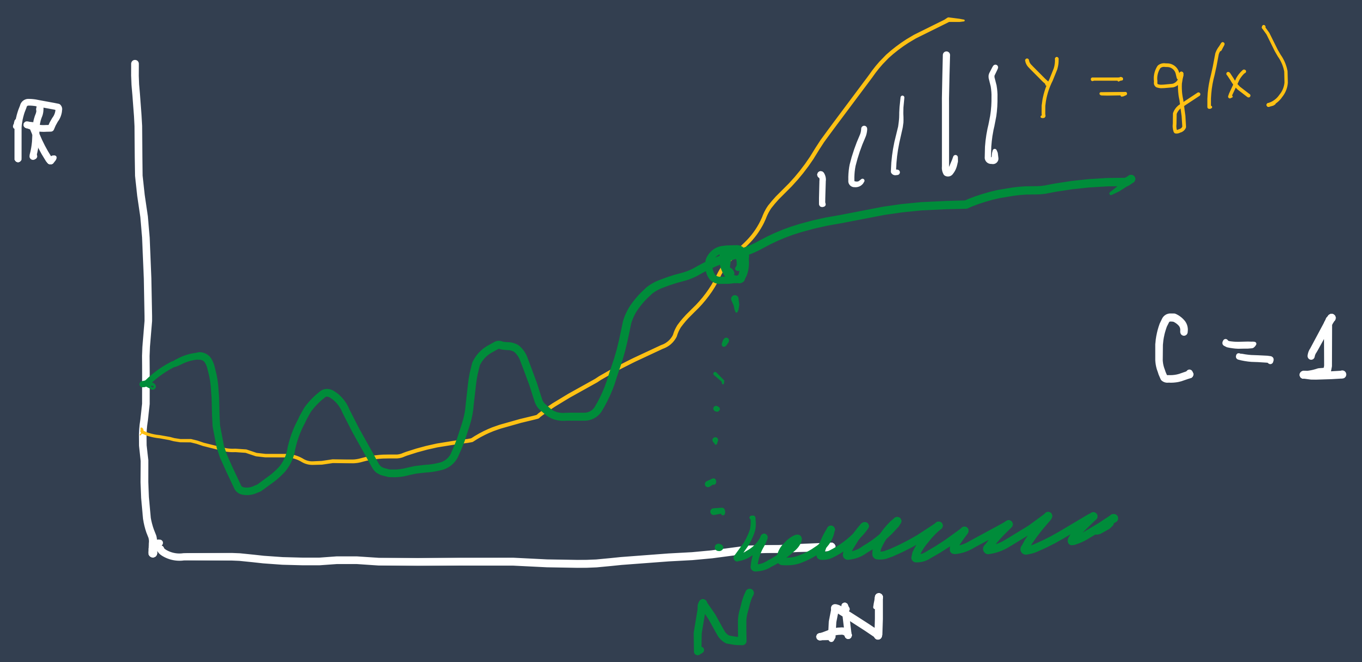
NOTACJA "DUŻE O" (Big O)

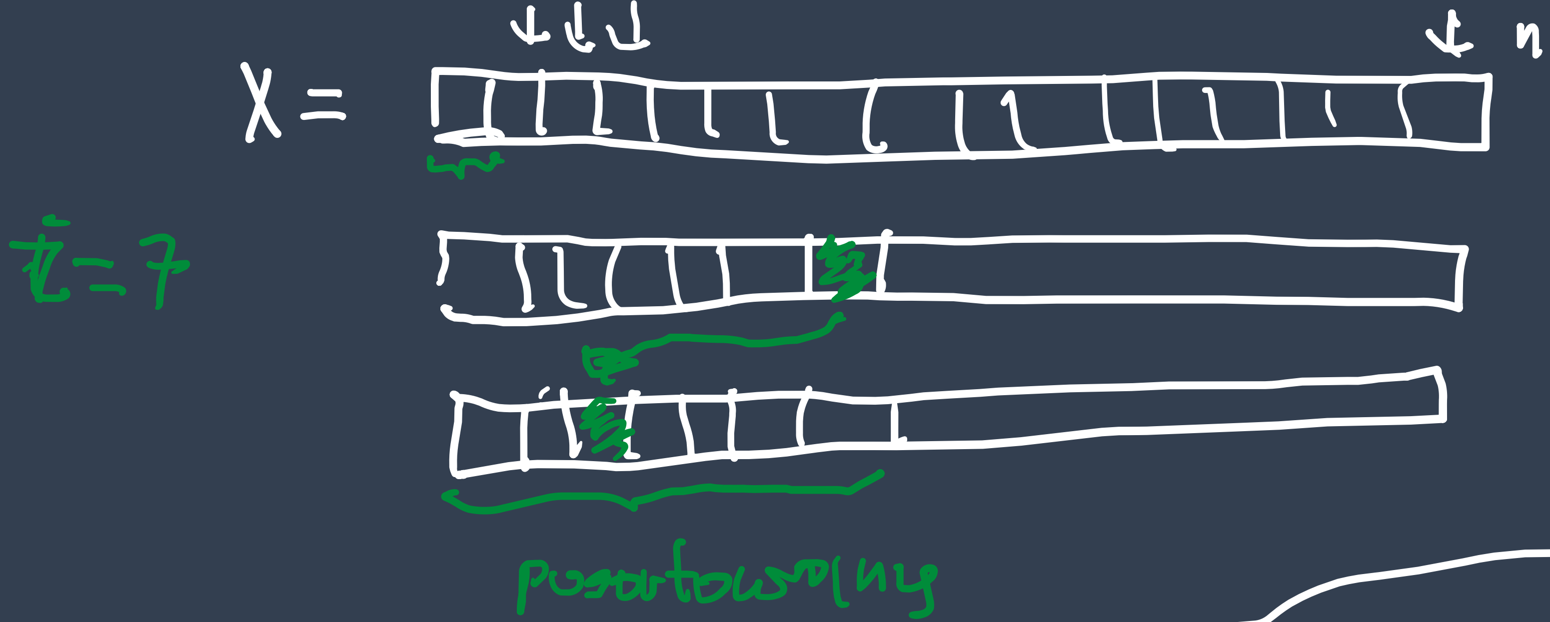
Def. Niech $f, g : \mathbb{N} \to \mathbb{R}$. Wtedy

$$f = O(g) \equiv (\exists C)(\exists N)(\forall n > N)\,(|f(n)| \leq C \cdot |g(n)|).$$

Intuicja



Ⓟ Sortowanie przez wstawianie



Najgorszy przypadek:

$n = 7$

7	6	5	4	3	2	1

1 2 3 4 5 6

L_n = Liczba porównań w najg. przypadku

$$L_n = 1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2}$$

$$L_n = \frac{1}{2} n(n-1) = \frac{1}{2}(n^2 - n) \leqslant \frac{1}{2} n^2$$

$$L_n = O(n^2).$$

Optym. alg. sortow. (przez porównanie)

$$L_n = O(n \lg n).$$

$$\underbrace{f = O(g)}_{\text{niechlujna notacja}} \equiv (\exists C)(\exists N)\ldots\ldots$$

Poprawnie: $f \in O(g)$

$$O(g) = \{ f \in \mathbb{R}^{\mathbb{N}} : (\exists C)(\exists N)(\forall n > N)(|f(n)| \leqslant C \cdot |g(n)| \}$$

Tw. Zał.-że $(\exists N)(\forall n > N)(g(n) \neq 0)$. wtedy

$$\boxed{f = O(g) \equiv \limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| < \infty}$$

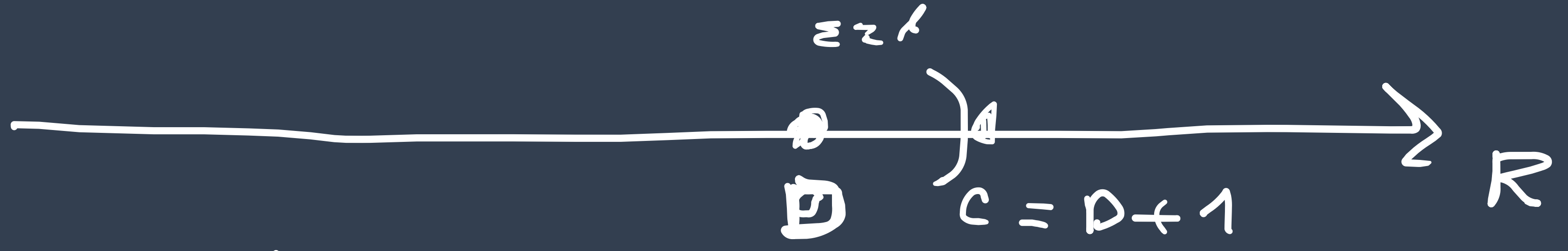
D-d. $(\Rightarrow)$ mamy C, N t.że $n > N \to |f(n)| \leq C \cdot |g(n)|$

możemy zał.-że $n > N \longrightarrow g(n) \neq 0$,

wtedy $n > N \longrightarrow \left|\frac{f(n)}{g(n)}\right| \leq C$

więc $\limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| \leq C < \infty$.

$\Leftarrow$ niech $D = \limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| \quad (< \infty)$



niech $C = D + 1$

wtedy jest N t. że $(\forall n > N)\ \left|\frac{f(n)}{g(n)}\right| < D + \underset{\uparrow\,\varepsilon}{1} = C$

więc

$n > N \longrightarrow |f(n)| \leq C \cdot |g(n)|$

$\blacksquare$

Wniosek: $\lim\limits_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| < \infty \Rightarrow f = O(g)$

P $\quad (\ln(n))^2 = O(n)$

$$\lim_{n \to \infty} \frac{(\ln(n))^2}{n} \overset{}{\underset{H}{=}} \lim \frac{2\ln(n) \cdot \frac{1}{n}}{1} = 2 \cdot \lim \frac{\ln(n)}{n} \overset{H}{=}$$

$$= 2 \cdot \lim_{n} \frac{\frac{1}{n}}{1} = 2 \cdot \lim_{n} \frac{1}{n} = 0$$

$\blacksquare$

Def. $f, g \in \mathbb{R}^{\mathbb{N}}$.

$$f = \Theta(g) \equiv f = O(g) \wedge g = O(f)$$

CZYLI: $f = \Theta(g) \equiv O(f) = O(g)$

Wn. zał. że $f, g > 0$. wtedy

Z $f = \Theta(g) \equiv 0 < \liminf_{n} \frac{f(n)}{g(n)} \leq \limsup_{n} \frac{f(n)}{g(n)} < \infty$

WNIOSEK: $0 < \lim \frac{f(n)}{g(n)} < \infty \longrightarrow f = \Theta(g)$

wr. $$\frac{1}{n} = \Theta\left(\frac{2}{n-1}\right)$$

BO: $$\lim \frac{\frac{1}{n}}{\frac{2}{n-1}} = \lim_{n} \frac{1}{2}\,\frac{n-1}{n} = \frac{1}{2}$$

wr. $$\frac{1}{n} = \Theta\left(\frac{2}{n}\right) \qquad O\left(\frac{1}{n}\right) = O\left(\frac{2}{n}\right)$$

$$\left.\begin{array}{l} f_1 = O(g_1) \\ f_2 = O(g_2) \end{array}\right\} \quad |f_1(n)+f_2(n)| \le |f_1(n)| + |f_2(n)|$$

$$\le c_1 \cdot |g_1(n)| + c_2 \cdot |g_2(n)| \le$$

$$\le c(|g_1(n)| + |g_2(n)|)$$

$c = \max(c_1, c_2)$

$$f_1 + f_2 = O(|g_1| + |g_2|)$$

CZYLI: $O(g_1) + O(g_2) \subseteq O(|g_1| + |g_2|)$

~~CZ~~ON: $O(g) + O(g) \subseteq O(2 \cdot |g|) =$

$= O(|g|).$

PRZYKŁAD: $a_n = H_{n\cdot 2} - \frac{1}{2} H_n$

① $H_n = \ln(n) + \gamma + \underbrace{O\left(\frac{1}{n}\right)}_{\varepsilon_n}$ $|\varepsilon_n| \leq \frac{c}{n}$

$$\boxed{H_{2n}-\tfrac{1}{2}H_n} = \left(\ln(2n)+\gamma+O\left(\tfrac{1}{2n}\right)\right) - \tfrac{1}{2}\left(\ln n+\gamma+O\left(\tfrac{1}{n}\right)\right)$$

$$= \ln n + \ln 2 + \overbrace{\gamma}^{+O\left(\frac{1}{2n}\right)} - \tfrac{1}{2}\ln n - \tfrac{1}{2}\gamma + O\left(\tfrac{1}{n}\right)$$

$-O(g) = O(g)$

$$= \tfrac{1}{2}\ln n + \left(\ln 2 + \tfrac{1}{2}\gamma\right) + O\left(\tfrac{1}{2n}\right) + O\left(\tfrac{1}{n}\right)$$

$$= \text{———} \quad " \quad \text{———} + O\left(\tfrac{1}{n}\right) + O\left(\tfrac{1}{n}\right)$$

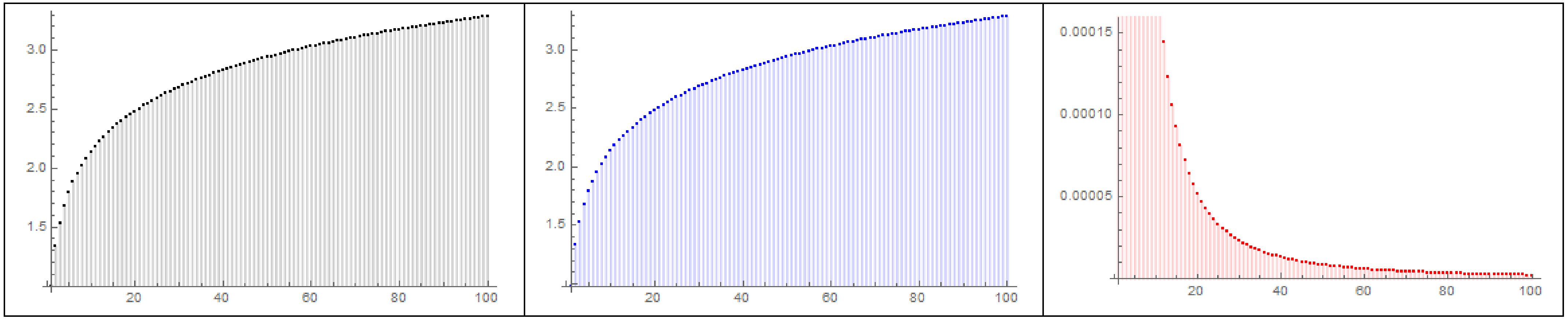
$$= \tfrac{1}{2}\ln n + \left(\ln 2 + \tfrac{1}{2}\gamma\right) + O\left(\tfrac{1}{n}\right).$$

2. Użyj dokł. wzoru: $H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right)$

$H_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$

Oto wyniki obliczeń numerycznych:

widzimy tutaj jakość przybliżenia



Chyba czujecie, że jest SUPER!!! ...

Ale można zrobić lepiej!!!

$$a_n = H_{2n} - \frac{1}{2}H_n$$

$$b_n = \frac{1}{2}\ln(n) + \left(\ln 2 + \frac{\gamma}{2}\right)$$

$$c_n = a_n - b_n$$

KOMBINATORYKA

$$\mathcal{A} = (\{a_1, \dots, a_n\}, |\cdot|) \; ; \; (\forall i)(|a_i| \neq 0)$$

$$\mathrm{MULT}(\mathcal{A})(z) = \prod_{i=1}^{n} \frac{1}{1 - z^{|a_i|}} \qquad (*)$$

uwaga (patrz Flajolet): ogólnie

$$\mathrm{MULT}(\mathcal{A})(z) = \prod_{a \in A} \frac{1}{1 - z^{|a|}}$$

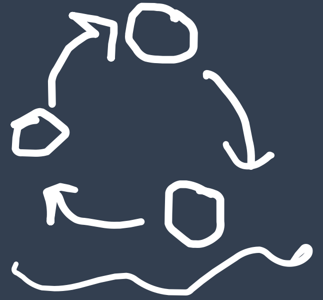
$$= \prod_{n \geq 1} \left(\frac{1}{1 - z^{n}} \right)^{|A_n|} . \qquad A_n = \{a \in A : |a| = n\}$$

CYKLE:

$\mathcal{A} = (\{\circ, \bullet\}, (\cdot 1)), \ |\circ| = |\bullet| = 1$

$n = 3$

$C_3 = 4$



1

(c,b,b) (b,c,b) (b,b,c)

1

1

$n = 4$

$$C_4 = 1 + 1 + 2 + 1 + 1 = 6$$

Mamy $\mathcal{A} = (A, |\cdot|)$, $|a| > 0$

$$\bar{a} = (a_0, \dots, a_{n-1}) \in \underbrace{A \times \dots \times A}_{n}$$

$$\bar{b} = (b_0, \dots, b_{n-1})$$

$$\bar{a} \sim_n \bar{b} \equiv (\exists k)(\forall i = 1..n)\left(b_i = a_{i+k \bmod n}\right)$$

$$\mathrm{CYCLE}(\mathcal{A}) = \cancel{\Sigma +} \mathcal{A} + \mathcal{A}^2/\sim_2 + \mathcal{A}^3/\sim_3 + \dots$$

TW.

$$\boxed{\mathrm{CYCLE}(\mathcal{A})(z) = \sum_{n \geq 1} \frac{\varphi(n)}{n} \ln \frac{1}{1 - \mathcal{A}(z^n)}}$$

$$\varphi(n) = \left| \{ k \in \{1..n\} : \mathrm{nwd}(k, n) = 1 \} \right|$$

Funkcja "phi" Eulera.

$$\mathrm{CYCLE}(\mathcal{B})(z) = \sum_{n \geq 1} \frac{\varphi(n)}{n} \ln \frac{1}{1 - A(z^n)}$$

$$A = (\{0, \bullet\}, 1 \cdot 1), \quad |\bullet| = |0| = 1$$

$$A(z) = 2z \qquad A(z^n) = 2z^n$$

$$\mathrm{CYCLE}(A)(z) = \sum_{n \geq 1} \frac{\varphi(n)}{n} \ln \frac{1}{1 - 2z^n} \qquad \ln \frac{1}{1-x} = \sum_{k \geq 1} \frac{1}{k} x^k$$

$$= \sum_{n \geq 1} \frac{\varphi(n)}{n} \sum_{k \geq 1} \frac{1}{k} (2 \cdot z^n)^k = \sum_{n \geq 1} \sum_{k \geq 1} \frac{\varphi(n)}{n \cdot k} 2^k z^{n \cdot k}$$

$$= \sum_{m} z^m \sum_{\substack{n \cdot k = m \\ n, k \geq 1}} \frac{\varphi(n)}{m} \cdot 2^k = \sum_{m} z^m \frac{1}{m} \sum_{n \cdot k = m} \varphi(n) \cdot 2^k$$

$$[z^5](*) = \frac{1}{5}\sum_{n\cdot k=5} \varphi(n)\, 2^k =$$

ZADANIE

$[z^7]\ (*)$

$$= \frac{1}{5}\Big(\underbrace{\varphi(1)\cdot 2^5}_{n=1} + \underbrace{\varphi(5)\cdot 2^1}_{n=5}\Big) =$$

ZADANIE

$[z^p]\ (*)$

$p \in$ PRIME

$$= \frac{1}{5}\left(2^5 + 4\cdot 2\right) = \frac{1}{5}\,(32+8) = \frac{1}{5}\cdot 40 = 8$$

$$(*) = \sum_m z^m \frac{1}{m} \sum_{n\cdot k=m} \varphi(n)\cdot 2^k$$