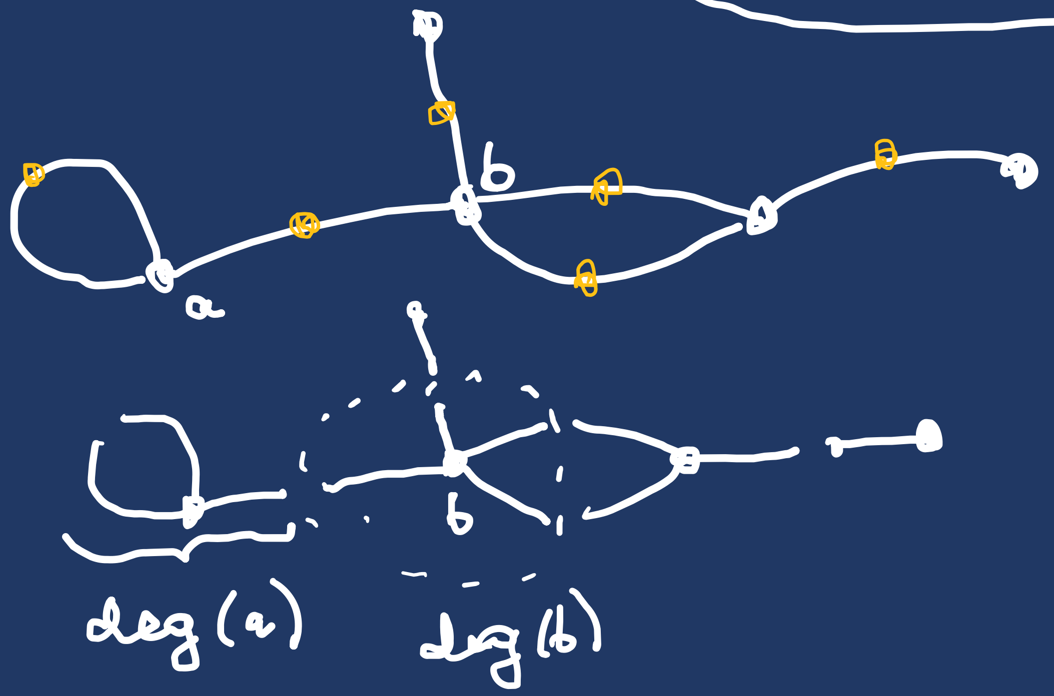
$\mathcal{G} = (V, E, \gamma)$

$$\boxed{\sum_{v \in V} \deg(v) = 2 \cdot |E|}$$



o – $|E|$

Uwagi: (V, E) - graf proste,

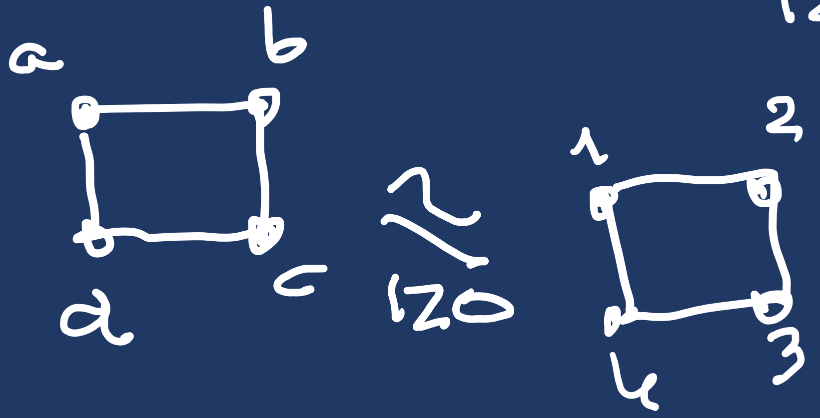
ustalony V.

Ile mamy możliwych E?

ODP: $2^{\binom{|V|}{2}}$

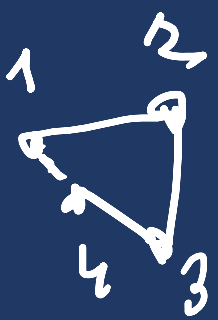
<u>Def</u>. $(V_1, E_1) \underset{\text{IZO}}{\simeq} (V_2, E_2)$ iff istnieje $\varphi : V_1 \xrightarrow[\text{na}]{1-1} V_2$

t.że $(\forall x,y \in V_1)\left(\{x,y\} \in E_1 \Leftrightarrow \{\varphi(x), \varphi(y)\} \in E_2\right)$

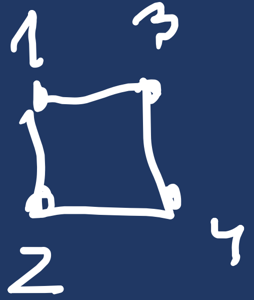


wystarczy zajmować się grafami t.że $V = \{1,\dots,n\}$ $(=[n])$ $(n \geq 1)$.

Q. Ile jest nieizomorficznych grafów na $\{1,\dots,n\}$?



$\simeq$

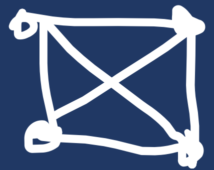


$$t_n \sim \frac{2^{\binom{n}{2}}}{n!}$$

UWAGA: $\sim \dfrac{2^{n^2/2}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = \dfrac{1}{\sqrt{2\pi n}} \dfrac{2^{n^2/2} \cdot e^n}{n^n} \longrightarrow \infty$ Z

Przykłady:

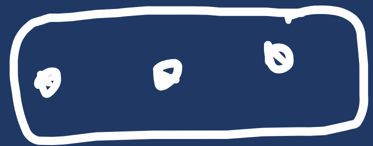
$K_n = ([n], [n]^2)$ graf zupełny



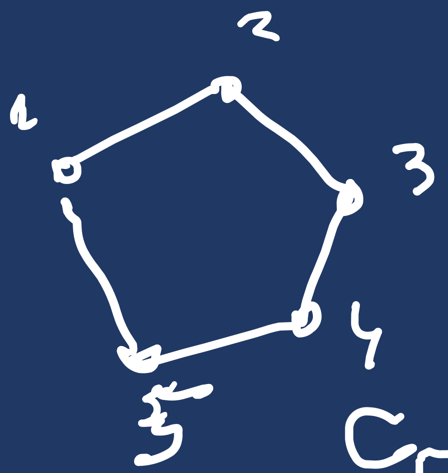
K_4

K_3

$E_n = ([n], \emptyset)$ pusty graf



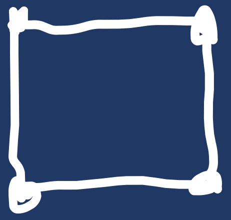
E_3



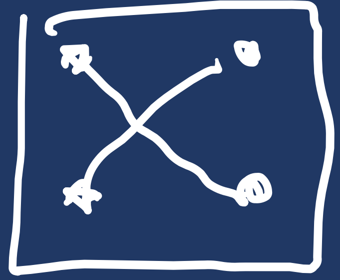
C_5 cykliczny

$\overline{K_n} = E_n$

Def. $G = (V, E)$:
$\overline{G} = (V, [V]^2 \setminus E)$ dopełnienie grafu



C_4

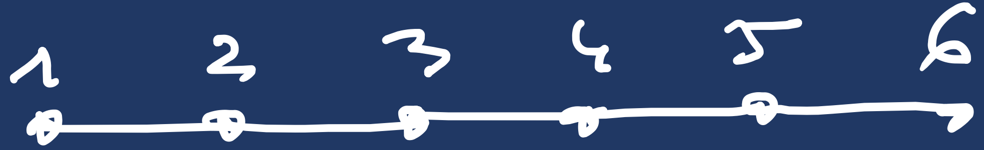


$\overline{C_4}$

$\simeq$

$$\overline{C_4} \simeq P_2 + P_2$$

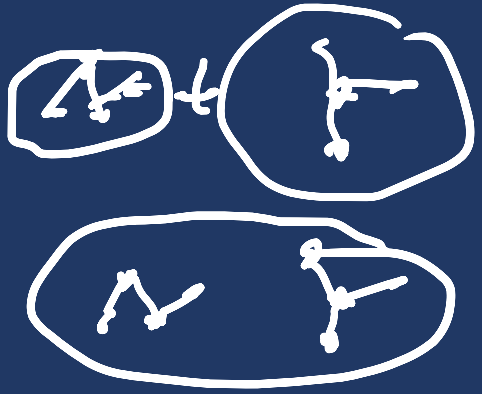
P_n :



ścieżka n-elem (graf liniowy)

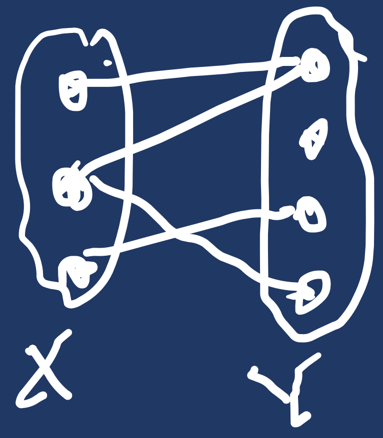
<u>Def</u>. Zał. że $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ oraz $V_1 \cap V_2 = \emptyset$.

$$G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$



Def. Graf $G=(V,E)$ nazywamy dwudzielnym jeśli istnieją zbiory X, Y niepuste t. że

1) $X \cap Y = \emptyset$
2) $X \cup Y = V$
3) $E \subseteq \{\{x,y\} : x \in X \land y \in Y\}$

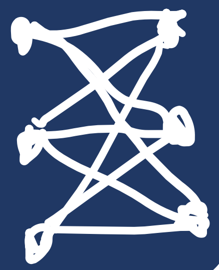


$K_{n,m} \cong (V_1 \cup V_2, E)$

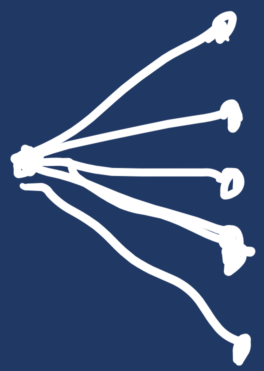
$|V_1| = n$
$|V_2| = m$

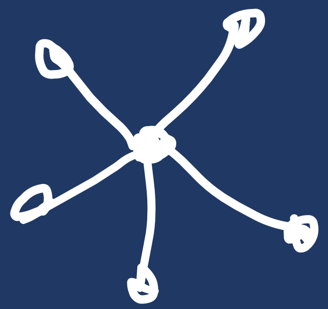
$E \subseteq \{\{x,y\} : x \in V_1 \land y \in V_2\}$

$K_{3,3}$:



$K_{1:n}$

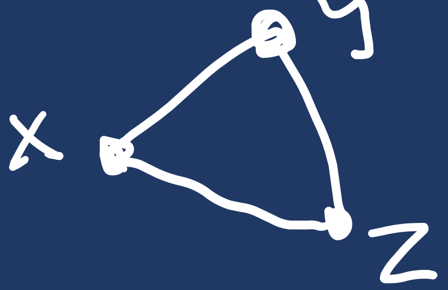




S_n

gwiazda o n liściach

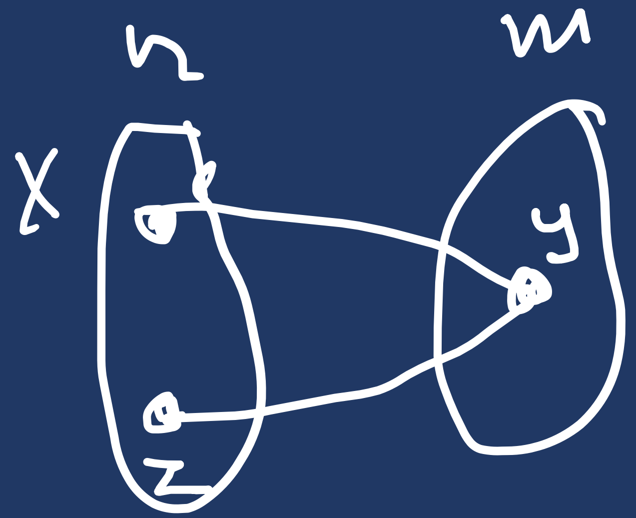
Trójkąt w grafie:



3 różne wierzch. x, y, z t. że $\{x,y\}, \{y,z\}, \{z,x\} \in E$.

liść : wierzchołek o rzędzie 1

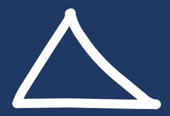
FAKT: w $K_{n,m}$ nie ma trójkątów



$\{x,y\} \in E$

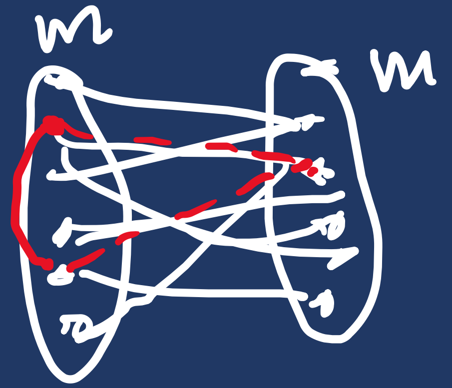
$\{y,z\} \in E$

$x \neq z$; $\{x,z\} \notin E$



P $n = 2m$

$K_{m,m}$:



$|V| = n \ (= 2m)$

$|E| = m^2 = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4}$

Tw. Zał. że (V,E) jest grafem prostym

Wtedy

$$\sum_{\{x,y\}\in E} \underbrace{\left(\deg(x)+\deg(y)\right)} = \sum_{x\in V} \deg^2(x)$$

Q: jak to zwizualizować

D-d:

$$L = \frac{1}{2}\cdot\sum_{x,y\in V} \|\{x,y\}\in E\| \left(\deg(x)+\deg(y)\right) =$$

$$= \frac{1}{2}\sum_{x,y} \|\{x,y\}\in E\|\cdot\deg(x) + \frac{1}{2}\sum_{x,y} \|\{x,y\}\in E\| \deg(y) =$$

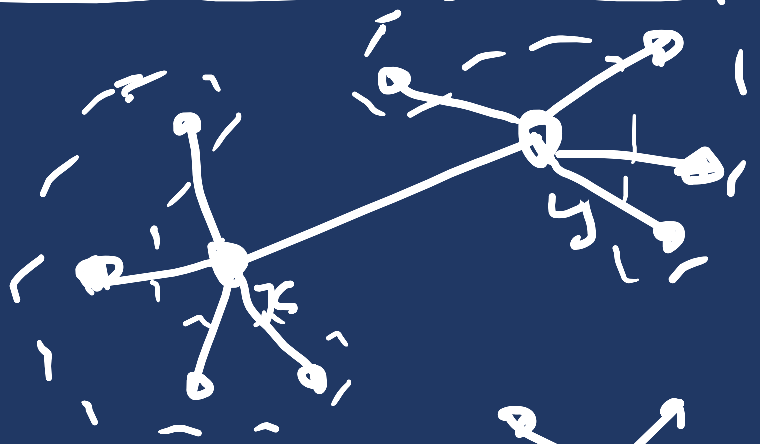
$$= \sum_{x,y} \|\{x,y\}\in E\| \cdot \deg(x) = \sum_{x} \deg(x)\cdot\sum_{y} \|\{x,y\}\in E\| = \sum_{x} \deg^2(x) \quad \blacksquare$$

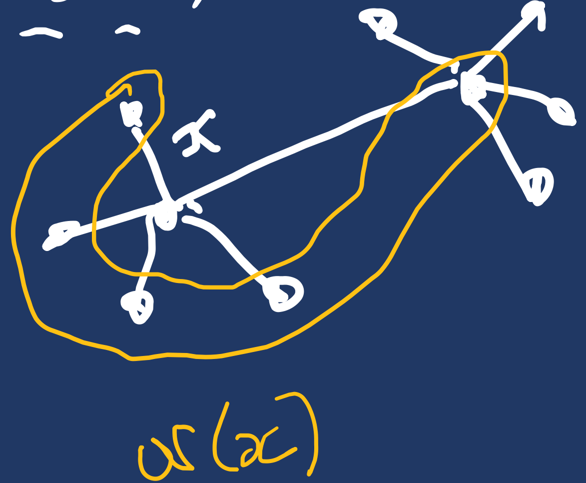
Zał. że (V,E) nie zawiera trójkątów.

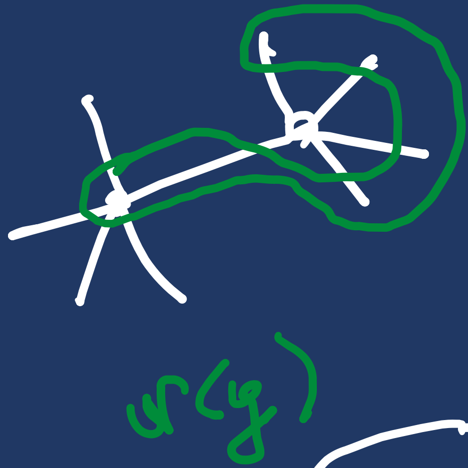
Wb. że $x,y \in V$ i $\{x,y\} \in E$

$$\mathcal{N}(x) = \{a \in V : \{x,a\} \in E\}$$

$$|\mathcal{N}(a)| = \deg(a)$$







$$\mathcal{N}(x) \cap \mathcal{N}(y) = \emptyset$$

$$\boxed{\deg(x) + \deg(y) \leq |V|}$$

(V,E) graf bez trójkątów $\qquad n = |V|$

$$2\cdot|E| = \sum_{x\in V}\deg(x) = \sum_{x} 1\cdot\deg(x) \leqslant \sqrt{\sum_{x\in V}1^2}\cdot\sqrt{\sum_{x\in V}\deg^2(x)} =$$

$$= \sqrt{n}\cdot\sqrt{\sum_{\{x,y\}\in E}(\deg(x)+\deg(y))} \leqslant \sqrt{n}\sqrt{\sum_{\{x,y\}\in E} n} = \sqrt{n}\cdot\sqrt{n\cdot|E|}$$

$$= n\cdot\sqrt{|E|}$$

$$4\cdot|E|^2 \leqslant n^2\cdot|E|$$

$$\boxed{|E| \leqslant \frac{n^2}{4}}$$

Tw. Jeśli w (V,E) nie ma $\triangle$ to $|E| \leqslant \frac{n^2}{4}$

$n := |V|$

$$\left|\sum_{i=1}^{n}(a_i\cdot b_i)\right| \leqslant \sqrt{\sum_{i=1}^{n}a_i^2}\cdot\sqrt{\sum_{i=1}^{n}b_i^2}$$

nier. Cauchy'ego

ekstremalna teoria grafów

TERMINOLOGIA:

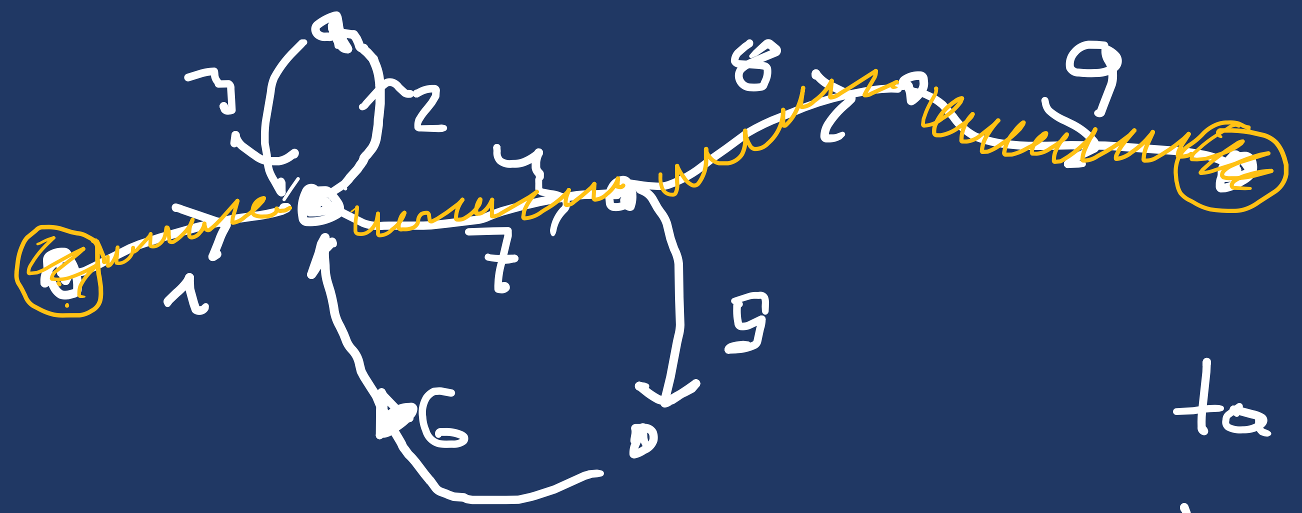
(V, E, γ)

1) trasa:

w. pocz $\rightarrow$ $x_1, e_1, x_2, e_2, x_3, e_3, \ldots, x_{n-1}, e_{n-1}, x_n$ $\leftarrow$ końcowy

$x_1, \ldots, x_n \in V$, $e_1, \ldots, e_{n-1} \in \gamma$

$\gamma(e_i) = \{x_i, x_{i+1}\}$

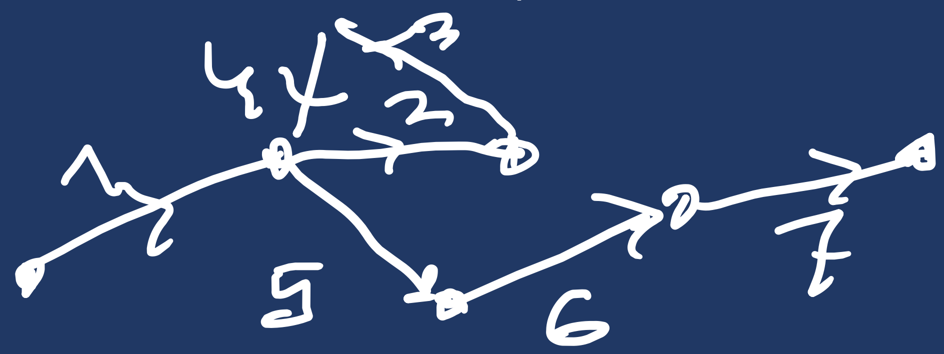


$e_4 = e_7$

ta sama krawędź może występować wiele razy

2) ścieżka:

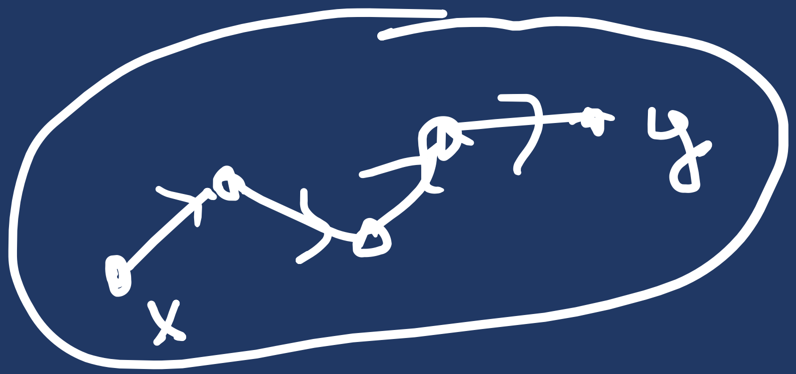
trasa bez powtórzonych krawędzi



3) droga (ścieżka prosta)

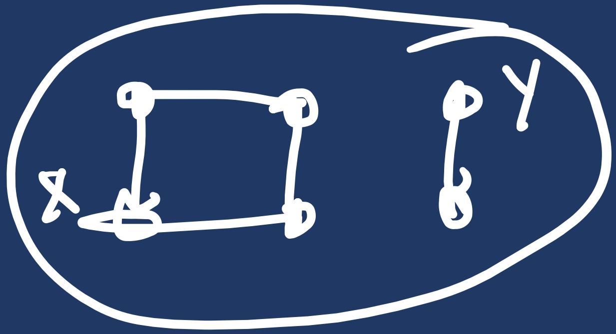
ścieżka bez powtórzonych wierzchołków

(V, E)



$d(x,y) =$ ~~dł~~ najmniejsza długość drogi od x do y.
lub ∞ jeśli nie ma drogi

$d(x,x) = 0$



$d(x,y) = \infty$.

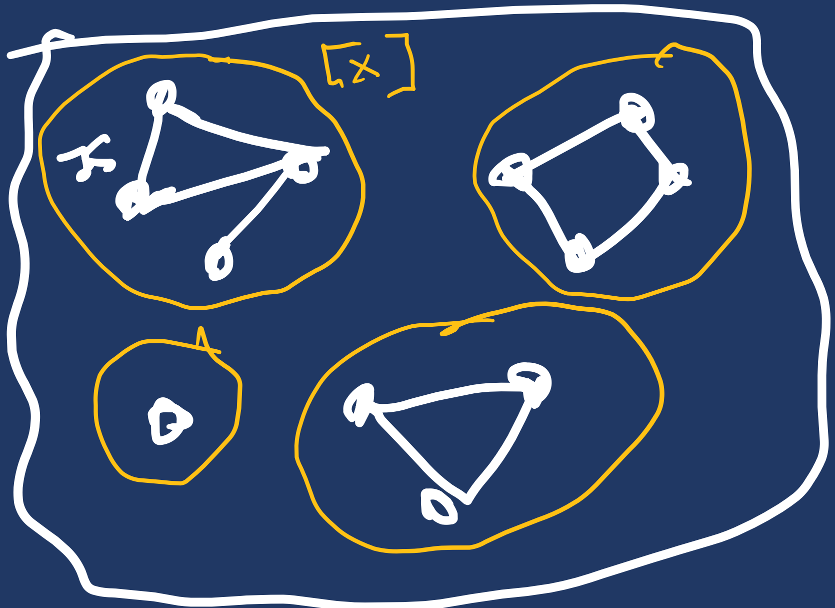
DEF. graf jest **spójny** jeśli $(\forall x,y \in V)(d(x,y) < \infty)$

Uwaga: (V,E):

na V określamy relację

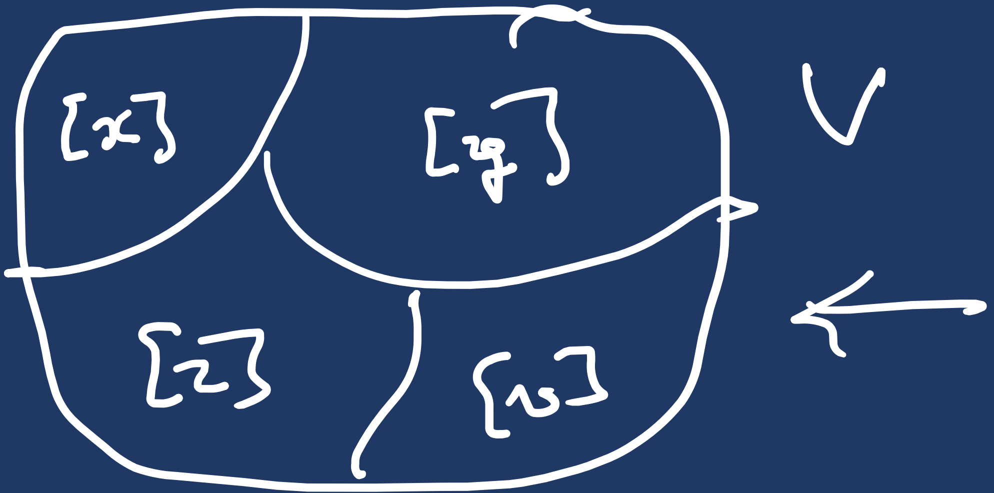
$$x \approx y \equiv d(x,y) < \infty$$

FAKT: $\approx$ jest rel. równoważności.



$$d(x,z) \le d(x,y) + d(y,z)$$

$(V,E)\ ,\ \simeq$



$V/\!\simeq$ składowe spójne grafu

FAKT. G - spójny

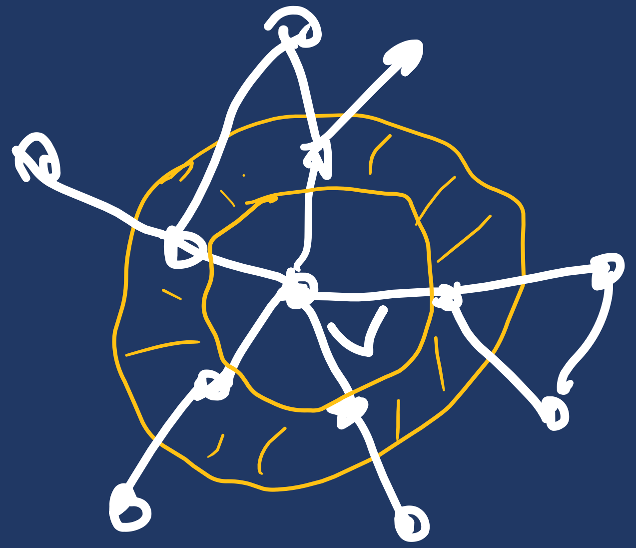
$\Updownarrow$

d jest odległością na V

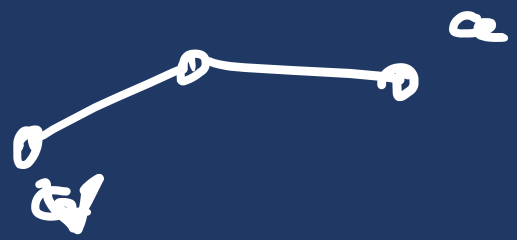
FAKT. (V,E), $v \in V$

$$\mathcal{N}(x) = \{a \in V : \{a,x\} \in E\} =$$

$$= \{a \in V : d(x,a) = 1\}$$



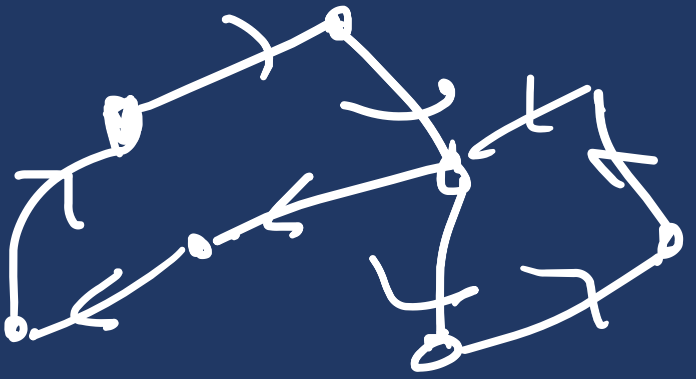
$$\{a \in V : d(x,a) = 2\}$$



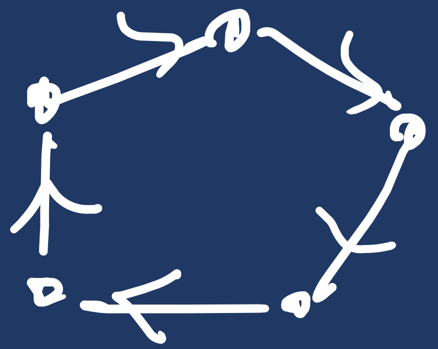
PĘTLE: $x_1 e_1 x_2 e_2 x_3 \dots x_{n-1} e_{n-1} x_n$

ścieżka

t. że $x_n = x_1$



CYKL: pętla bez powtórzonych wierzchołków poza x_1 i x_n



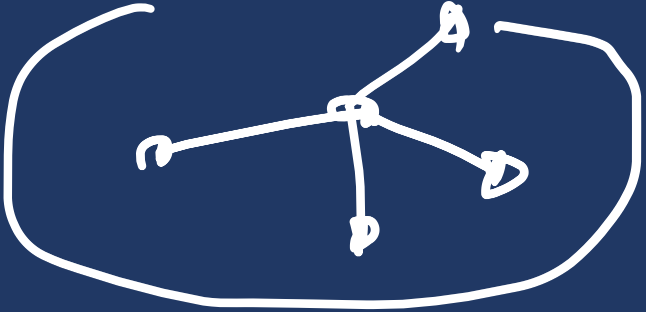
DEF. Lasem nazywamy graf bez cykli

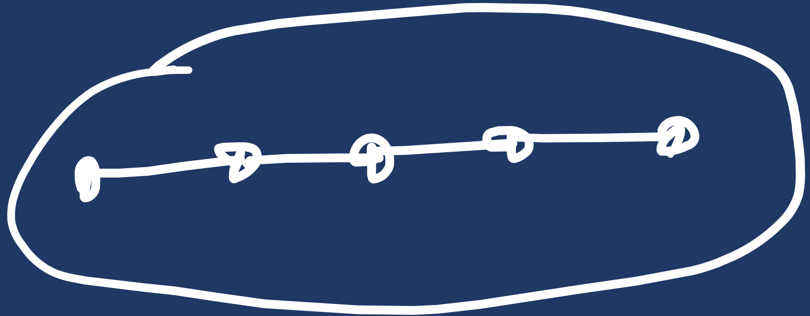
DRZEWEM nazywamy spójny las

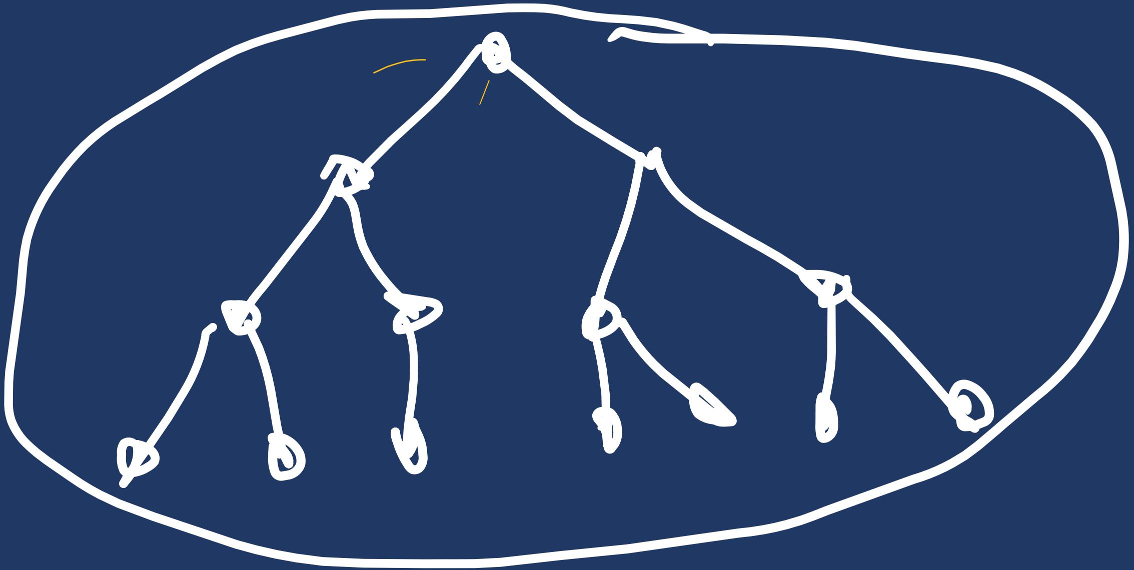
drzewo : graf spójny, acykliczny

las : graf acykliczny

drzew







las ≡ suma rozłączna drzew