Kwantyfikatory

$\Omega \neq \emptyset$ ustalona przestrzeń

$\varphi: \Omega \longrightarrow \{\mathbb{0}, \mathbb{1}\}$

$D_\varphi = \{\omega \in \Omega : \varphi(\omega) = \mathbb{1}\}$

$$\begin{cases} (\forall x)\varphi(x) \equiv (D_\varphi = \Omega) \\ (\exists x)\varphi(x) \equiv (D_\varphi \neq \emptyset). \end{cases}$$

UWAGA. Zał. że $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ $(n \in \mathbb{N})$

$$\big((\forall x)\varphi(x)\big) \equiv \varphi(\omega_1) \wedge \varphi(\omega_2) \wedge \dots \wedge \varphi(\omega_n) \quad \Big(\equiv \bigwedge_{i=1}^{n} \varphi(\omega_i)\Big)$$

$$\equiv$$

$$(\forall \omega)\varphi(\omega)$$

$$(\exists x)(\varphi(x)) \equiv \varphi(\omega_1) \vee \varphi(\omega_2) \vee \dots \vee \varphi(\omega_n) \quad \Big(\equiv \bigvee_{i=1}^{n} \varphi(\omega_i)\Big)$$

UWAGA (terminologia)

$(\forall x)\,\varphi(x)$; $\bigwedge_x \varphi(x)$; $\forall$ Ⓐ forall ; $\exists x\, \varphi(x) \equiv \bigvee_x \varphi(x)$

$\exists$ Ⓔ exists

Tw. $\neg(\exists x)\varphi(x) \equiv (\forall x)(\neg\varphi(x))$
$\neg(\forall x)\varphi(x) \equiv (\exists x)(\neg\varphi(x))$ } prawa de Morgana.

Uwaga. Zał. że $\Omega = \{\omega_1, \dots, \omega_n\}$.

$$\neg(\exists x)\varphi(x) \equiv \neg \bigvee_{i=1}^{n} \varphi(\omega_i) \equiv \neg(\varphi(\omega_1) \vee \dots \vee \varphi(\omega_n)) \overset{\text{de Morgan}}{\equiv}$$

$$\equiv (\neg\varphi(\omega_1)) \wedge (\neg\varphi(\omega_2)) \wedge \dots \wedge (\neg\varphi(\omega_n)) \equiv \bigwedge_{i=1}^{n} (\neg\varphi(\omega_i))$$

$$\equiv (\forall x)(\neg\varphi(x)).$$

Zad. 1c. $\varphi, \psi : \Omega \to \{0,1\}$. $(\varphi \wedge \psi)(x) = \varphi(x) \cdot \psi(x)$

$D_\varphi, D_\psi \subseteq \Omega$

- $(\forall x)((\varphi \wedge \psi)(x)) \overset{def}{\equiv} (D_{\varphi \wedge \psi} = \Omega)$

$$\equiv (D_\varphi \cap D_\psi = \Omega)$$

$$\equiv (D_\varphi = \Omega) \wedge (D_\psi = \Omega)$$

$$\equiv (\forall x)\varphi(x) \wedge (\forall x)\psi(x)$$

$$\boxed{(\forall x)(\varphi(x) \wedge \psi(x)) \equiv (\forall x)\varphi(x) \wedge (\forall x)\psi(x)}$$

UWAGA: $\Omega = \{\omega_1, \dots, \omega_n\}$

$$(\forall x)(\varphi(x) \wedge \psi(x)) \equiv (\varphi(\omega_1) \wedge \psi(\omega_1)) \wedge (\varphi(\omega_2) \wedge \psi(\omega_2)) \wedge \dots \wedge (\varphi(\omega_n) \wedge \psi(\omega_n))$$

$$\equiv (\varphi(\omega_1) \wedge \dots \wedge \varphi(\omega_n)) \wedge (\psi(\omega_1) \wedge \dots \wedge \psi(\omega_n)) \equiv (\forall x)\varphi(x) \wedge (\forall x)\psi(x)$$

FAKT. $\underbrace{(\forall x)\varphi(x) \vee (\forall x)\psi(x)}_{L} \to \underbrace{(\forall x)(\varphi(x) \cup \psi(x))}_{P}$

D-d.

$$L \equiv (D_\varphi = \Omega) \vee (D_\psi = \Omega).$$

zał. że $D_\varphi = \Omega$.

wtedy $D_{\varphi \cup \psi} = \underbrace{D_\varphi}_{\Omega} \cup D_\psi = \Omega$, więc

$(\forall x)(\varphi(x) \cup \psi(x))$ $\square$

(P) $\Omega = \mathbb{N}$; $\varphi(n) =$ "$2 | n$"; $\psi(n) =$ "$\neg(2 | n)$"

$(\forall x)(\varphi(x) \cup \psi(x)) \equiv 1$ (T) $\leftarrow$

$(\forall x)\varphi(x) \equiv 0$; $(\forall x)\psi(x) \equiv 0$ $\equiv \perp$

FAKT. $(\exists x)(\varphi(x) \vee \psi(x)) \equiv (\exists x)\varphi(x) \vee (\exists x)\psi(x)$

D-d. $(\exists x)(\varphi(x) \vee \psi(x)) \equiv D_{\varphi \cup \psi} \neq \emptyset$

$$\equiv D_\varphi \cup D_\psi \neq \emptyset$$

$$\equiv D_\varphi \neq \emptyset \vee D_\psi \neq \emptyset$$

$$\equiv (\exists x)\varphi(x) \vee (\exists x)\psi(x).$$

Alternatywny dowód

$$(\exists x)(\varphi(x) \vee \psi(x)) \equiv \neg(\forall x)(\neg(\varphi(x) \vee \psi(x))) \equiv$$

$$\equiv \neg(\forall x)((\neg\varphi(x)) \wedge (\neg\psi(x))) \equiv \neg((\forall x)\neg\varphi(x) \wedge (\forall x)\neg\psi(x))$$

$$\equiv \neg(\forall x)\neg\varphi(x) \vee \neg(\forall x)\neg\psi(x) \equiv$$

$$\equiv (\exists x)\varphi(x) \vee (\exists x)\psi(x).$$

FAKT. $(\exists x)(\varphi(x) \wedge \psi(x)) \longrightarrow (\exists x)\varphi(x) \wedge (\exists x)\psi(x)$

D-d. $L \equiv (D_{\varphi \wedge \psi} \neq \emptyset) \equiv (D_\varphi \cap D_\psi \neq \emptyset)$

$\to (D_\varphi \neq \emptyset \wedge D_\psi \neq \emptyset) \equiv (\exists x)\varphi(x) \wedge (\exists x)\psi(x).$

$\equiv P$

P $\Omega = \mathbb{N}$; $\varphi(u) =$ "$2 | u$" ; $\psi(u) =$ "$\neg 2 | u$".

$L \equiv \perp$

$P \equiv \top$

Fakt. $\varphi : \Omega \to \{0,1\}$; $\eta : \Omega \to \{0,1\}$

η - stała. Wtedy

$$(\forall x)(\varphi(x) \overset{\vee}{\wedge} \eta(x)) \equiv ((\forall x)\, \varphi(x)) \overset{\vee}{\wedge} c_\eta$$

$$(\exists x)(\varphi(x) \underset{\vee}{\wedge} \eta(x)) \equiv ((\exists x)\, \varphi(x)) \underset{\vee}{\wedge} c_\eta$$

gdzie $c_\eta \leftarrow$ stała wartość η.

Niech $\varphi : \Omega \times \Omega \longrightarrow \{0,1\}$.

Dla $\omega \in \Omega$:

$$\varphi_\omega(y) = \varphi(\omega, y)$$
$$\varphi^\omega(x) = \varphi(x, \omega)$$

wtedy $\varphi_\omega, \varphi^\omega : \Omega \to \{0,1\}$.

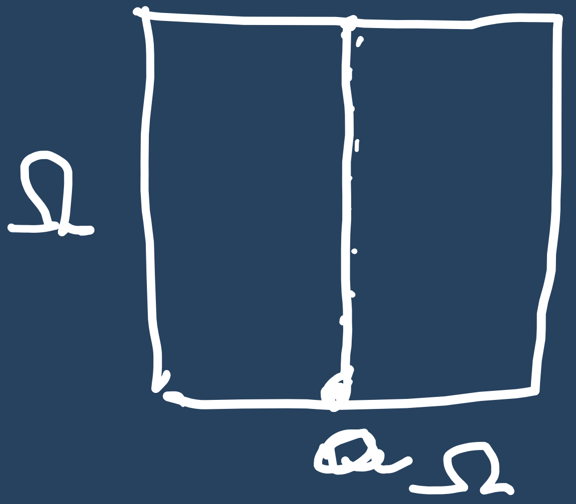
<u>Def</u>.

$$\begin{cases} (\forall x)(\forall y)\varphi(x,y) \overset{\text{def}}{\equiv} (\forall x)\big((\forall y)\varphi_x(y)\big) \\ (\forall x)(\exists y)\varphi(x,y) \equiv (\forall x)\big((\exists y)\varphi_x(y)\big) \\ (\exists x)(\exists y)\varphi(x,y) \equiv (\exists x)\big((\exists y)\varphi_x(y)\big) \\ (\exists x)(\forall y)\varphi(x,y) \equiv (\exists x)\big((\forall y)\varphi_x(y)\big) \end{cases}$$

$$(\forall y)(\forall x)\varphi(x,y) \equiv (\forall y)\big((\forall x)\varphi^y(x)\big)$$

- $(\forall x)(\forall y)\,\varphi(x,y) \equiv (\forall x)\big((\forall y)\,\varphi_x(y)\big) \equiv$

$$\equiv (\forall x)\left(D_{\varphi_x} = \Omega\right) \equiv \left(D_\varphi = \Omega \times \Omega\right)$$

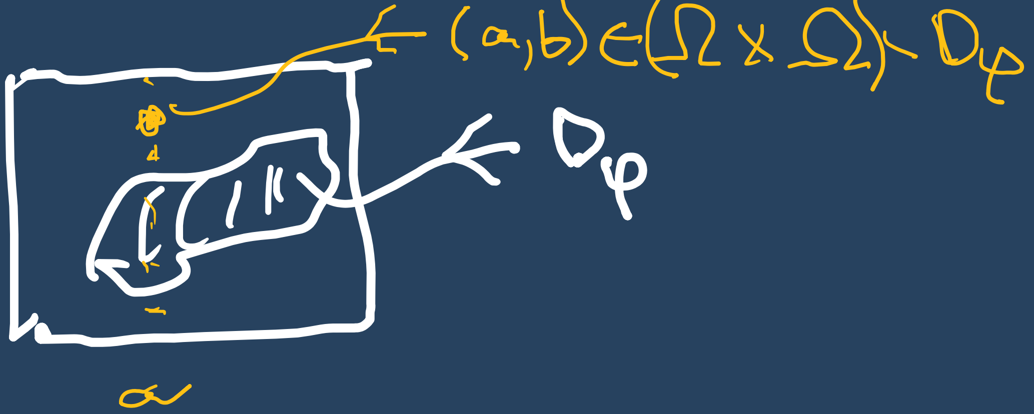


$$D_\varphi = \{(x,y) \in \Omega \times \Omega : \varphi(x,y) = 1\}$$

$$D_{\varphi_a} = \{y \in \Omega : \varphi_a(y) = 1\} =$$

$a \in \Omega$

$$= \{y \in \Omega : \varphi(a,y) = 1\}$$



Wn. $(\forall x)(\forall y)\varphi(x,y) \equiv \left(D_\varphi = \Omega^2 \; (=\Omega \times \Omega)\right)$

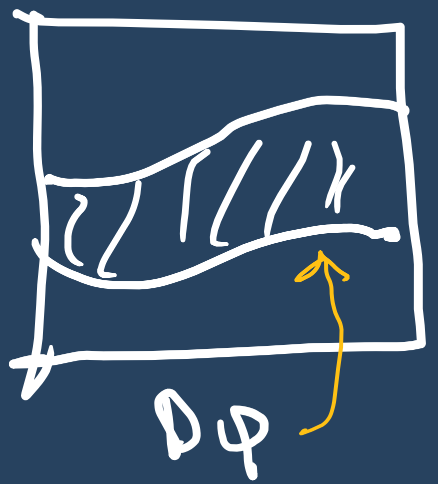
Interpretacja $(\forall x)(\exists y)\varphi(x,y)$

czyli $(\forall x)\left((\exists y)\varphi_x(y)\right)$

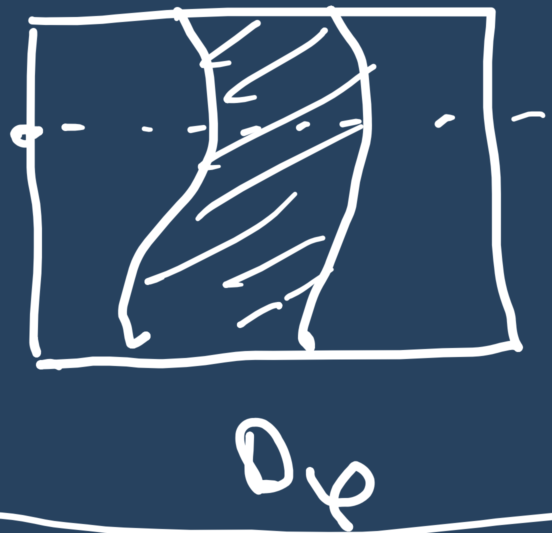
czyli $(\forall x)\left(D_{\varphi_x} \neq \emptyset\right)$

$$D_{\varphi_x} = \{y : \varphi_x(y) = 1\} = \{y : \varphi(x,y) = 1\}$$

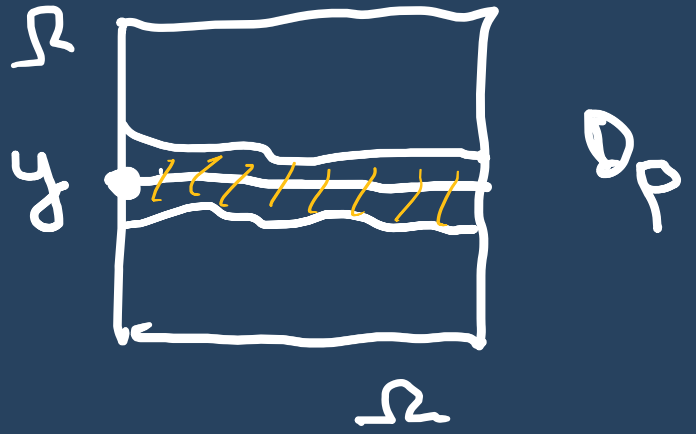
$(\forall x)(\exists y)\varphi(x,y)$



$(\forall y)(\exists x)\varphi(x,y)$



$(\exists y)(\forall x)\varphi(x,y)$



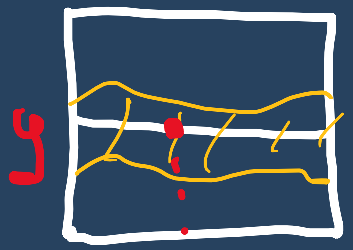
$$\equiv (\exists y)\big((\forall x)\varphi^y(x)\big)$$

$$\equiv (\exists y)(D_{\varphi^y} = \Omega)$$

$$\equiv (\exists y)(\{x\in\Omega : \varphi^y(x) = 1\} = \Omega)$$

$$\equiv (\exists y)(\{x\in\Omega : \varphi(x,y) = 1\} = \Omega\}$$

FAKT. $(\exists y)(\forall x)\varphi(x,y) \to (\forall x)(\exists y)\varphi(x,y)$



Ⓟ nie ma implikacji odwrotnej:

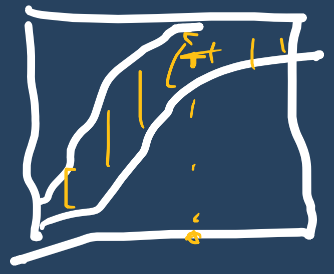
$\Omega = \mathbb{R}$; $\varphi(x,y) = $ "$x < y$"

$(\forall x)(\exists y)(x<y) \equiv T$

np. mamy $x \in \mathbb{R}$

wtedy $x < \underbrace{x+1}_{y}$

$(\exists y)(\forall x)(x<y) \equiv \perp$



D-d. $(\exists y)(\forall x)\varphi(x,y) \equiv (\exists y)\big((\forall x)\varphi^y(x)\big)$

$\equiv (\exists y)(\underbrace{D_{\varphi^y} = \Omega}_{\eta(y)}) \equiv (D_\eta \neq \emptyset)$.

Bierzemy $b \in \Omega$ t. że $\eta(b) = 1$

czyli $D_{\varphi^b} = \Omega$, czyli $\{x : \varphi^b(x) = 1\} = \Omega$

czyli

$\{x : \varphi(x,b) = 1\} = \Omega$.

czyli dla każd. x mamy y t. że $\varphi(x,y)$. CZYLI

$(\forall x)(\exists y)\varphi(x,y)$.

P

$$(\forall x)(\forall y)\varphi(x,y) \equiv D_\varphi = \Omega^2$$

$$(\forall y)(\forall x)\varphi(x,y) \equiv D_\varphi = \Omega^2$$

$$(\forall x)(\forall y)\varphi(x,y) \equiv (\forall y)(\forall x)\varphi(x,y)$$

$$(\exists x)(\exists y)\varphi(x,y) \equiv (\exists y)(\exists x)\varphi(x,y)$$