Kwantyfikatory

$(\forall x)(\exists y)$, $(\forall x)(\forall y)$, ... $(\exists x)(\exists y)$

- wiemy, że
  - $(\forall x)(\forall y)\varphi(x,y) \equiv (\forall y)(\forall x)\varphi(x,y)$

    $\parallel \qquad\qquad\qquad \parallel$

    $D_\varphi = \Omega^2 \qquad\qquad D_\varphi = \Omega^2$

  - $(\exists x)(\exists y)\varphi(x,y) \equiv (\exists y)(\exists x)\varphi(x,y)$

    $\parallel \qquad\qquad\qquad \parallel$

    $D_\varphi \neq \emptyset \qquad\qquad D_\varphi \neq \emptyset$.

$\boxed{\Omega \neq \emptyset}$

- FAKT: $(\forall x)(\forall y)\varphi(x,y) \Rightarrow (\exists y)(\forall x)\varphi(x,y)$

<u>D-d</u>. Weźmy dowolne, ale ustalone $b \in \Omega$.

Zatem $(\forall x)\,(x,b) \in D_\varphi$, czyli $(\forall x)\,\varphi(x,b)$,

czyli $(\forall x)(D_{\varphi b} \neq \emptyset)$, czyli ~~[illegible]~~ $(\exists y)(\forall x)\varphi(x,y)$ $\boxtimes$

- FAKT. $(\exists y)(\forall x)\varphi(x,y) \Rightarrow (\forall x)(\exists y)\varphi(x,y)$

D-d. Załóżmy, że $(\exists y)(\forall x)\varphi(x,y)$. Jest $b \in \Omega$ t. że $(\forall x)\varphi(x,b)$. Więc $(\forall x)(\exists y)\varphi(x,y)$.

- FAKT. $\boxed{(\forall x)(\exists y)\varphi(x,y) \Rightarrow (\exists x)(\exists y)\varphi(x,y)}$

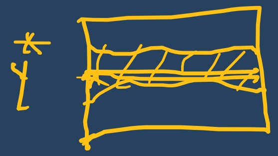
<u>D-d</u>. Załóżmy, że $(\forall x)(\exists y)\varphi(x,y)$. Weźmy $a \in \Omega$. Wtedy $(\exists y)\varphi(a,y)$. Jest $b \in \Omega$ t. że $\varphi(a,b)$. Zatem $(\exists x)(\exists y)\varphi(x,y)$.

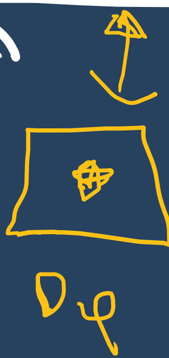
Z Pokaż, że $(\forall y)(\exists x)\varphi(x,y) \Rightarrow \cancel{(\exists x)(\exists y)}\varphi(x,y)$.

HINT. Zdefiniuj $\psi(x,y) = \varphi(y,x)$. ∎

Podsumowanie

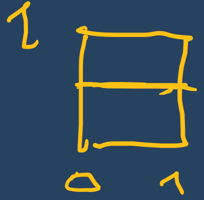
$$
\begin{tikzcd}[column sep=small]
 & (\exists y)(\forall x)\varphi(x,y) \arrow[r, Rightarrow, "(2)"] & (\forall x)(\exists y)\varphi(x,y) \arrow[dr, Rightarrow, "(3)"] & \\
\boxed{\begin{array}{c}(\forall x)(\forall y)\varphi(x,y)\\ \equiv \\ (\forall y)(\forall x)\varphi(x,y)\end{array}} \arrow[ur, Rightarrow, "(1)"] \arrow[dr, Rightarrow] & & & \boxed{\begin{array}{c}(\exists x)(\exists y)\varphi(x,y)\\ \equiv \\ (\exists y)(\exists x)\varphi(x,y)\end{array}} \\
 & (\exists x)(\forall y)\varphi(x,y) \arrow[r] & (\forall y)(\exists x)\varphi(x,y) \arrow[uul, "\text{NIE!!}"'] \arrow[ur, Rightarrow] &
\end{tikzcd}
$$





$\Omega = [0,1]$ $\qquad$ $\varphi : [0,1]^2 \longrightarrow \{0,1\}$

(1) $(\forall x)(\forall y)\varphi(x,y) \Longrightarrow (\exists y)(\forall x)\varphi(x,y)$



$D_\varphi = [0,1]\times\{\frac{1}{2}\}$

$\varphi(x,y) =$ "$y = \frac{1}{2}$".

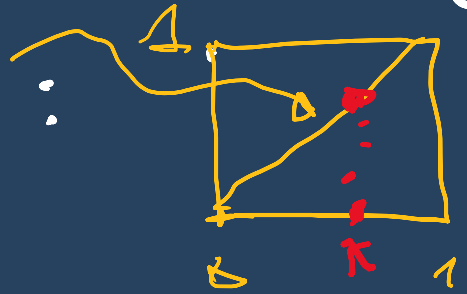
wtedy $(\exists \underset{\uparrow \frac{1}{2}}{y})(\forall x)\varphi(x,y)$

ALE: $(\forall x)(\forall y)\varphi(x,y)$ nie jest prawdziwe

(2) $(\exists y)(\forall x)\varphi(x,y) \Rightarrow (\forall x)(\exists y)\varphi(x,y)$ $\quad$ weźmy $y = \frac{1}{3}$

$\varphi(x,y) =$ "$x = y$"

D_φ:



(3) $\forall x \exists y\ \varphi(x,y) \to (\exists x)(\exists y)\varphi$

$\varphi(x,y) =$ "$x = \frac{1}{2} \wedge y = \frac{1}{2}$"

$D_\varphi = \{(\frac{1}{2}, \frac{1}{2})\}$.

Przykład: Czy $(\exists y)(\forall x)\varphi(x,y) \Leftarrow (\forall y)(\exists x)\varphi(x,y)$?

$\Omega = [0,1]$ $\qquad \varphi(x,y) = $ "$x = y$".

wtedy $(\forall y)(\exists x)\varphi(x,y) \equiv \top$

ale $(\exists y)(\forall x)\varphi(x,y) \equiv \bot$

Kwantyfikatory ograniczone.

mamy $\Omega \neq \emptyset$. mamy $A \subseteq \Omega$.

<u>Def</u>. $(\forall x \in A)\varphi(x) \stackrel{def}{\equiv} (\forall x)(x \in A \to \varphi(x))$

$(\exists x \in A)\varphi(x) \equiv (\exists x)(x \in A \wedge \varphi(x))$.

Oznaczenia: $(\forall x \in A)\varphi(x) \longleftrightarrow (\forall x)_A \varphi(x) \longleftrightarrow (x)_A \varphi(x)$

$\longleftrightarrow \bigwedge_{x \in A} \varphi(x)$.

dziwne

FAKT (de Morgan)

(1) $\neg(\forall x\in A)\varphi(x) \equiv (\exists x\in A)(\neg\varphi(x))$

(2) $\neg(\exists x\in A)\varphi(x) \equiv (\forall x\in A)(\neg\varphi(x))$.

D-d. (1) $\neg(\forall x\in A)\varphi(x) \overset{def}{\equiv} \neg(\forall x)(x\in A \to \varphi(x))$

$\underset{dM}{\equiv} (\exists x)\neg(x\in A \to \varphi(x)) \underset{F}{\equiv} (\exists x)\neg(\neg x\in A)\vee\varphi(x))$

$\underset{dM}{\equiv} (\exists x)(x\in A \wedge \neg\varphi(x)) \overset{def}{\equiv} (\exists x\in A)\neg\varphi(x)$.

(2) zadanie

$p \to q \equiv \neg p \vee q$

FAKT. Zał. że $(\forall x)(\varphi(x) \to \psi(x))$. Wtedy

$$(\forall x)\varphi(x) \Rightarrow (\forall x)\psi(x).$$

PRZYKŁAD. Niech $f: \mathbb{R} \to \mathbb{R}$.

Def. f jest jednostajnie ciągła, jeśli

$$(*) \equiv (\forall \varepsilon>0)(\exists \delta>0)(\forall x,y)(|x-y|<\delta \to |f(x)-f(y)|<\varepsilon).$$

$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ (\forall \varepsilon \in \mathbb{R}^+) & (\exists \delta \in \mathbb{R}^+) & (\forall x)(\forall y) \end{array}$$

$$(*) \equiv (\forall \varepsilon>0)\boxed{(\exists \delta>0)\ (\forall x)}\overbrace{(\forall y)(\ \ldots\ )}^{\varphi(\delta,x)}$$

$$\Rightarrow (\forall \varepsilon>0)(\forall x)(\exists \delta>0)(\forall y)(|x-y|<\delta \to |f(x)-f(y)|<\varepsilon)$$

$$\equiv (\forall x)(\forall \varepsilon>0)(\exists \delta>0)(\forall y)\big(|y-x|<\delta \rightarrow |f(y)-f(x)|<\varepsilon\big)$$

$$\stackrel{def}{\equiv} (\forall x)(f \text{ jest ciągła w } x)$$

$$\stackrel{def}{\equiv} \text{„} f \text{ jest ciągła na } \mathbb{R}\text{''}.$$

WNIOSEK: Jeśli $f: \mathbb{R} \to \mathbb{R}$ jest jednostajnie ciągła, to jest ciągła.

UWAGA: $(\forall x)\varphi(x)$

$(\forall x)(\exists y)\psi(x,y)$

$(\forall x)(\exists y)(\forall z)\,\eta(x,y,z)$

$\vdots$

ZADANIE. Rozważamy grę.

- mamy 30 zapałek
- mamy 2 graczy. grają naprzemiennie.
- pojedynczy ruch: może zabrać 1 lub 2 zapałki.
- wygrana: wygrywa ten kto weźmie ostatnią zapałkę (doprowadzi do 0 zapałek).

$$30 \xrightarrow{\mathrm{I}} 28 \xrightarrow{\mathrm{II}} 26 \xrightarrow{\mathrm{I}} 25 \xrightarrow{\mathrm{II}} \ldots \rightarrow 5 \xrightarrow{\mathrm{I}} 3 \xrightarrow{\mathrm{II}} 2 \xrightarrow{\mathrm{I}} 0$$

KTO MA STRATEGIĘ ZWYCIĘZKĄ?