Matematyka dyskretna

- struktury skończ.
- teoria grafów
- kombinatoryka
- ....

ZBIORY SKOŃCZONE

FAKTY. Załóżmy $|A| = n$, $|B| = m$.

- $A \cap B = \emptyset \to |A \cup B| = n + m$
- $|A \times B| = n \cdot m$
- $|B^A| = m^n$
- $|P(A)| = 2^n$

INDUKCJA MATEMAT.

Zbiór potęgowy $|P(A)| = 2^{|A|}$

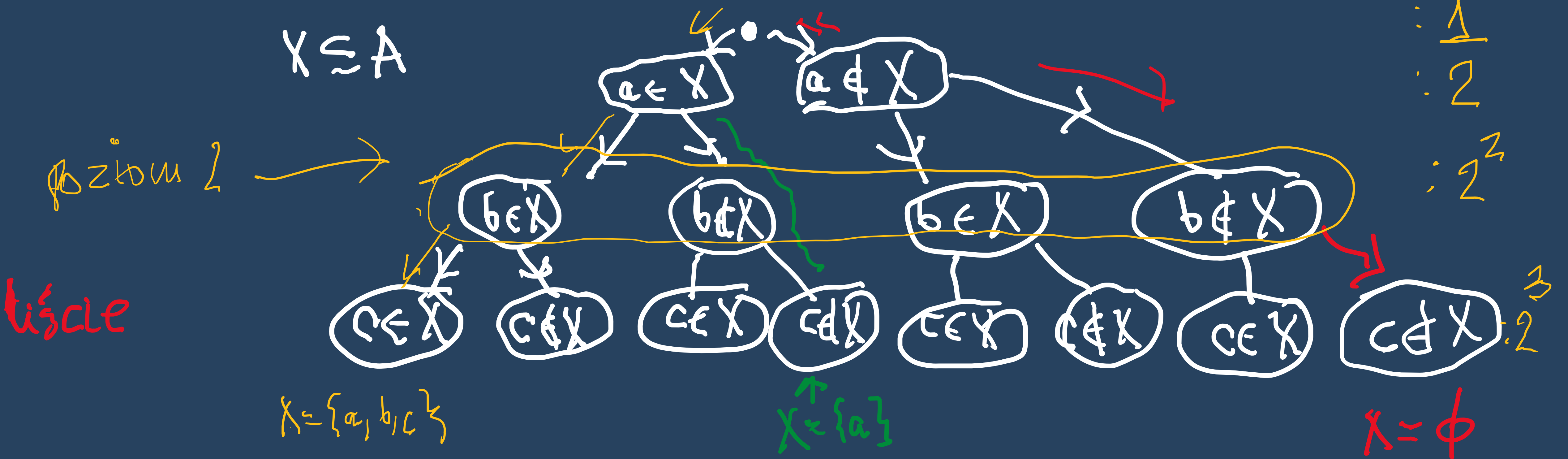
(1) ind. po n:

(2) $A = \{a, b, c\}$

$X \subseteq A$

korzeń

drzewo decyzyjne



(3) $X \subseteq A = \{a_1, \dots, a_n\}$ $\quad P(A) \sim \{0,1\}^A$

$c(X) \in \{0,1\}^{\{1,\dots,n\}}$: $c(X)(i) = \begin{cases} 1: & a_i \in X \\ 0: & a_i \notin X \end{cases}$

$A = \{a_1, a_2, a_3\}$

$X = \{a_1, a_3\}$ $\quad c(X) = \text{"}101\text{"} \rightsquigarrow (101)_{(2)}$

$$h(X) = \sum_{i=0}^{n-1} c(X)_{i+1} \cdot 2^i \; ; \quad = 1 + 0 \cdot 2^1 + 1 \cdot 2^2 = 5$$

$A = \{a_1, \ldots, a_n\}$

$h(\emptyset) = 0$

$h(A) = 1\cdot 2^0 + 1\cdot 2^1 + \ldots + 1\cdot 2^{n-1} =$

$= 1 + 2 + 2^2 + \ldots + 2^{n-1} = \frac{1-2^n}{1-2} = 2^n - 1.$

$0 \leq h(X) \leq 2^n - 1$

$h : P(A) \xrightarrow[\text{na}]{1-1} \{0, \ldots, 2^n - 1\}$

$1 + q + \ldots + q^n = \frac{1-q^{n+1}}{1-q} \quad : q \neq 1$

Jak wyznaczyć $\overline{b}_i$?

$0 \leq k < 2^n$:

1) przedstaw $k = (i_{n-1} i_{n-2} \ldots i_1 i_0)_{(2)}$
w zapisie dwójkowym

2) odczytaj z tego X :

Q: jaki to ma związek z tab.
0 - 1 ?

DEF. $[A]^k = \{X \subseteq A : |X| = k\}$

Q: jak obliczyć $|[A]^k|$?

DEF. $\mathrm{inj}(B, A) = \{f \in A^B : f \text{ jest injekcją}\}$

Dla $n, m \in \mathbb{N}$:

$\mathrm{inj}(m, n) = |\mathrm{inj}(B, A)|$, $|B| = m$, $|A| = n$.

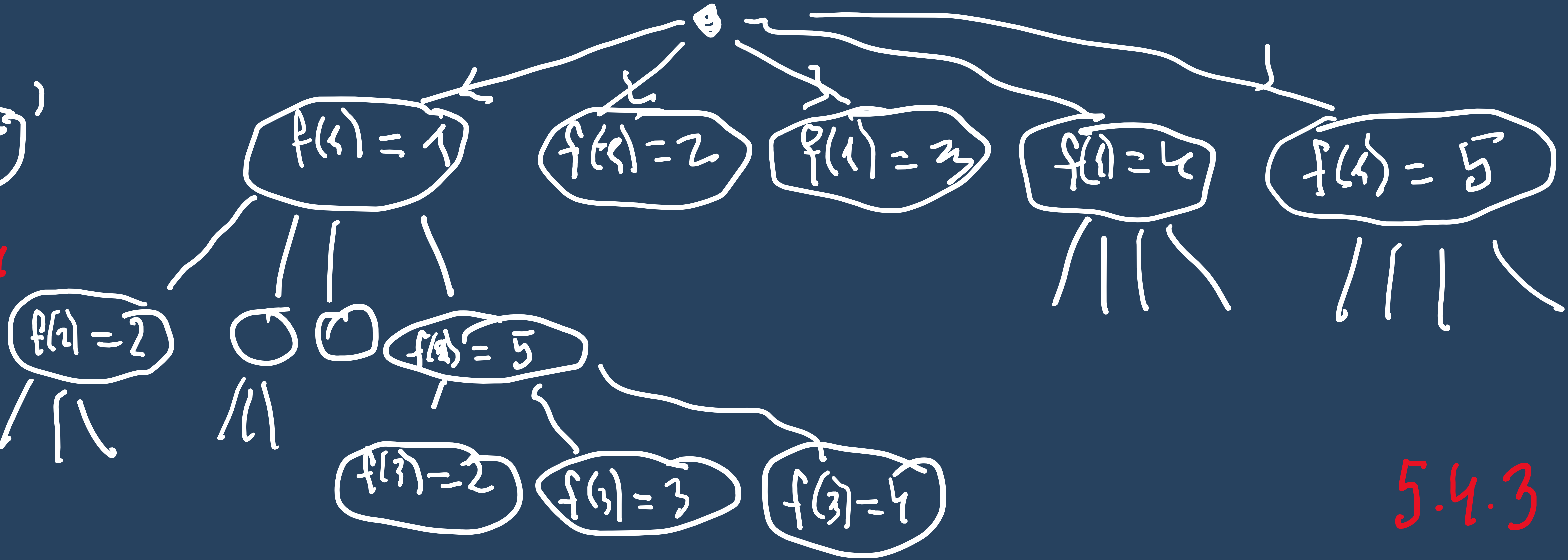
(P) $inj(3,5) = ?$

$f : \{1,2,3\} \xrightarrow{1-1} \{1, \dots, 5\}$

drzewo decyzyjne

$5'$

$5 \cdot 4$



$5 \cdot 4 \cdot 3$

$inj(3,5) = 5 \cdot (5-1) \cdot (5-2)$

$$\mathrm{inj}(k,n) = (n-0)\cdot(n-1)\cdot\ldots\cdot(n-(k-1))$$

$$= \prod_{i=0}^{k-1}(n-i)$$

Def. $x^{\underline{k}} = \prod_{i=0}^{k-1}(x-i)$

silnia malejące

potęga malejąca

$$\mathrm{inj}(k,n) = n^{\underline{k}}$$

$$\mathrm{inj}(k,n) = n(n-1)\cdot\ldots\cdot(n-(k-1)) =$$

$$= \frac{n(n-1)\cdot\ldots\cdot(n-(k-1))\cdot\overset{(n-k)}{\ldots}\cdot 2\cdot 1}{1\cdot 2\cdot\ldots\cdot(n-k)}$$

$$= \frac{n!}{(n-k)!}$$

CEL : $|[n]^k| = ?$

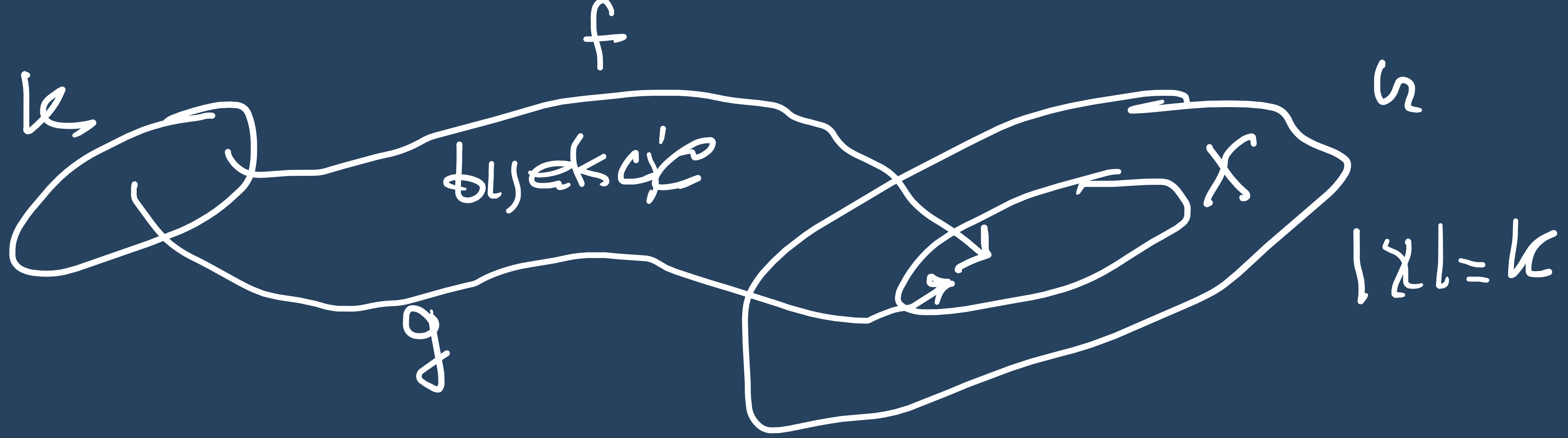
Ustalmy n. Rozw. $\{1, \dots, n\}$.

$$[n]^k = \{ X \subseteq \{1, \dots, n\} : |X| = k \}.$$

$X \subseteq \{1 \dots, n\}$, $|X| = k$,

mamy $f: \{1, \dots, k\} \xrightarrow[\text{na}]{1-1} X$.

Q. Kiedy $f, g: \{1, \dots, k\} \xrightarrow[\text{na}]{1-1} X$



$$g^{-1}\circ f : \{1,\dots,k\} \xrightarrow[\text{na}]{1-1} \{1,\dots,k\}$$

$$g^{-1}\circ f = \sigma \quad : \quad f = g\circ\sigma$$

$$\mathrm{inj}(k,k) = k^{\underline{k}} = k!$$

$$\operatorname{inj}(\{1,..k\},\{1,..n\}) = \biguplus_{X \in [\{1,..,n\}]^k} \operatorname{inj}(\{1,..k\}, X)$$

$$\operatorname{inj}(k,n) = \sum_{X \in [\{1..n\}]^k} |\operatorname{inj}(\{1..k\}, X)| =$$

$$= \sum_{X \in [\{1..n\}]^k} k^{\underline{k}} = k! \, |[\{1..n\}]^k|$$

$$|\{1...n\}^k| = \frac{n!}{(n-k)!} \frac{1}{k!} = \frac{n!}{(n-k)! \, k!}$$

DEF (wsp. dwum. Newtona)

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

WNIOSEK. $P(\{1..n\}) = \bigcup_{k=0}^{n} [\{1..n\}]^{k}$

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Q. Jak wygenerować wszystkie zbiory z $[\{1, ..., n\}]^2$ 2-elementowe

1 2 3 4 5 6 7

```
for i=1 to n do
    for j=i+1 to n do
        print([i,j])
```

Ile razy print jest wykonywane?

odp: $\binom{n}{2} = \frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n = O(n^2)$

później

$$\left\{\begin{array}{l}\text{for } I = 1 \text{ to } n \text{ do} \\ \quad \text{for } J = I+1 \text{ to } n \\ \quad\quad OP(i,j)\end{array}\right.$$

koszt czasowy $OP(i,j) = C$.

cała pętla kosztuje nas

$$\binom{n}{2} \cdot C = \frac{n(n-1)}{2} \cdot C.$$

Pętle zagłęb. 3 ?

```
for i = 1 to n do
    for j = i+1 to n do
        for k = j+1 to n do
            OP(i,j,k)
```

OP będzie wykonywana : $\binom{n}{3} =$

$$= \frac{n(n-1)(n-2)}{6} = \frac{1}{6}\left(n^3 - 3n^2 + 2n\right)$$

ZADANIE: dane n, k.

wygeneruj $\left[\{1, \ldots, n\}\right]^{K}$.

SUMA: $A \cap B = \emptyset \rightarrow |A \cup B| = |A| + |B|$.

ogólniej.

$|A| = a + c$

$|B| = b + c$

$|A \cup B| = a + b + c$

$|A \cap B| = c$

$|A| + |B| =$

$a + b + 2 \cdot c$

$= (a + b + c) + c$

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

$$|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| -$$

$$- |A \cap (B \cup C)| =$$

$$= |A| + |B| + |C| - |B \cap C| - \left(|(A \cap B) \cup (A \cap C)| \right) =$$

$$= \text{—} \quad \text{''} \quad \text{——} \quad - \left(|A \cap B| + |A \cap C| - |A \cap B \cap C| \right)$$

$$|A \cup B \cup C| = |A| + |B| + |C|$$

$$- (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{T \in [\{1..n\}]^i} \left|\bigcap_{j \in T} A_j\right|$$

zasada włączeń - wyłączania.