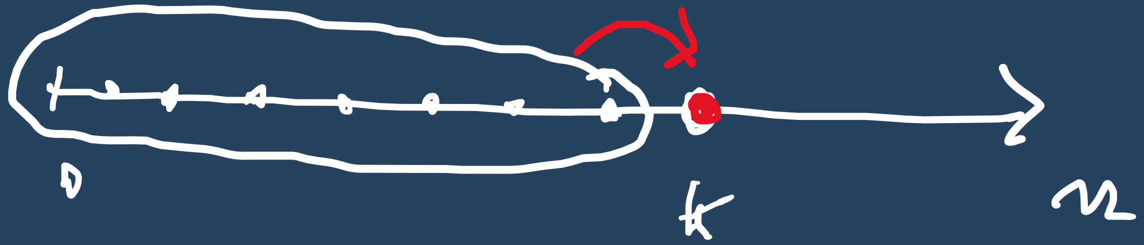
LICZBY CATALANA - c.d.

- $C_n = \frac{1}{n+1}\binom{2n}{n}$
- $\boxed{C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k}}$ $\left(= C_0 \cdot C_n + C_1 \cdot C_{n-1} + \ldots + C_n \cdot C_0\right)$
- $C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5, \ldots$

$C_4 = C_0 \cdot C_3 + C_1 C_2 + C_2 \cdot C_1 + C_3 \cdot C_0 =$

$= 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14$



$$C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k} = \sum_{\substack{a+b=n \\ a,b \geq 0}} C_a \cdot C_b .$$

P) mamy n-zmiennych $x_1, x_2, \dots x_n$ $(n \geq 1)$
mamy działanie $*$ binarne

$W_n \to$ { Ile różnych wyrażeń z $x_1, \dots x_n$
o z $*$ można zbudować ?

- $n=1$; x_1 : $w_1=1$
- $n=2$; $x_1 * x_2$: $w_2=1$
- $n=3$: $\left.\begin{matrix}(x_1 * x_2) * x_3 \\ x_1 * (x_2 * x_3)\end{matrix}\right\} w_3 = 2$
- $w_{n+1} = ?$

$x_1 * (\underbrace{\qquad\qquad}_{n\text{-zmiennych}})$

$(x_1 * x_2) * (\underbrace{\ldots\ldots}_{n-1\text{ - zmiennych}})$

W_{n+1} :

$$\underbrace{(\qquad)}_{a\text{ - zm}} * \underbrace{(\qquad)}_{b\text{ - zmiennych}}$$

$$a+b=n+1$$

$$a,b \geq 1$$

$$W_{n+1} = \sum_{\substack{a+b=n+1 \\ a,b\geq 1}} W_a \cdot W_b = \sum_{a=1}^{n} W_a \cdot W_{(n+1)-a} .$$

$L_n = W_{n+1}$ $\leftarrow$ robocza definicja $\quad n \geq 0$

- $L_0 = w_1 = 1$
- $L_1 = w_2 = 1$
- $L_{n+1} = w_{n+2} = \sum_{a=1}^{n+1} w_a \cdot w_{n+2-a} =$

$$= \sum_{a=1}^{n+1} L_{a-1} \cdot L_{n+1-a} = \sum_{k=0}^{n} L_k \cdot L_{n-k},$$

$k = a-1$

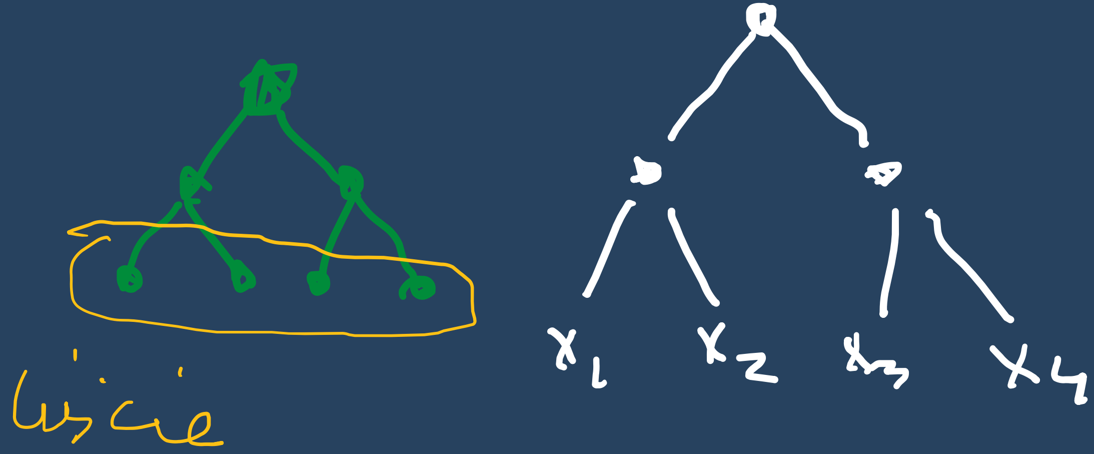
$a = k+1$

ZATEM: $L_n = C_n$

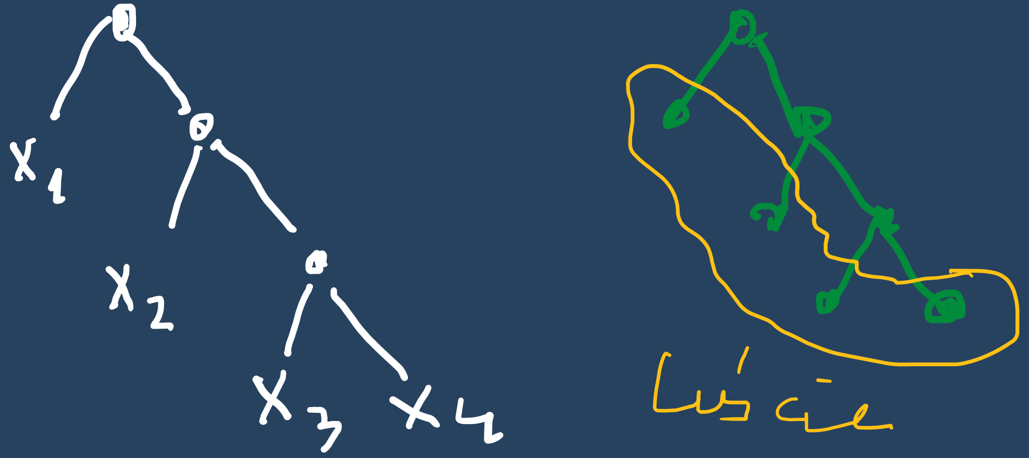
ZATEM: $\boxed{W_n = C_{n-1}}$!!!

Zliczanie drzew binarnych

$$(x_1 \cdot x_2) \cdot (x_3 \cdot x_4)$$



$$x_1 \cdot (x_2 \cdot (x_3 \cdot x_4))$$



WNIOSEK: liczba drzew binarnych o n liściach (T_n)

węzły wewn 4

5 liści

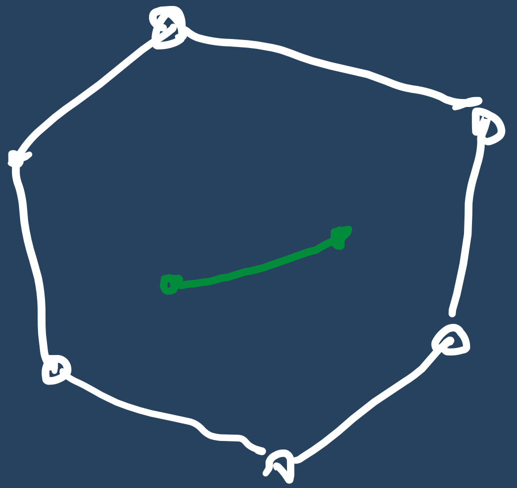
$$T_n = C_{n-1}$$

$$(x_1 \cdot x_2) \cdot ((x_3 \cdot x_4) \cdot x_5)$$

obserwacja:

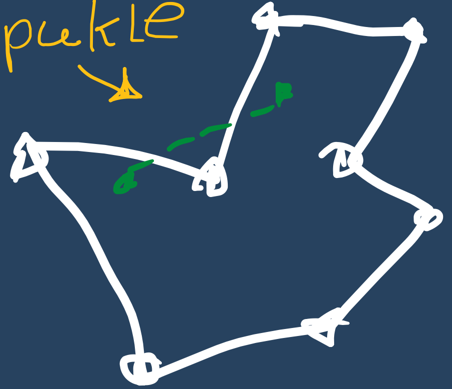
liczba węzłów wew. = liczbie liści -1.

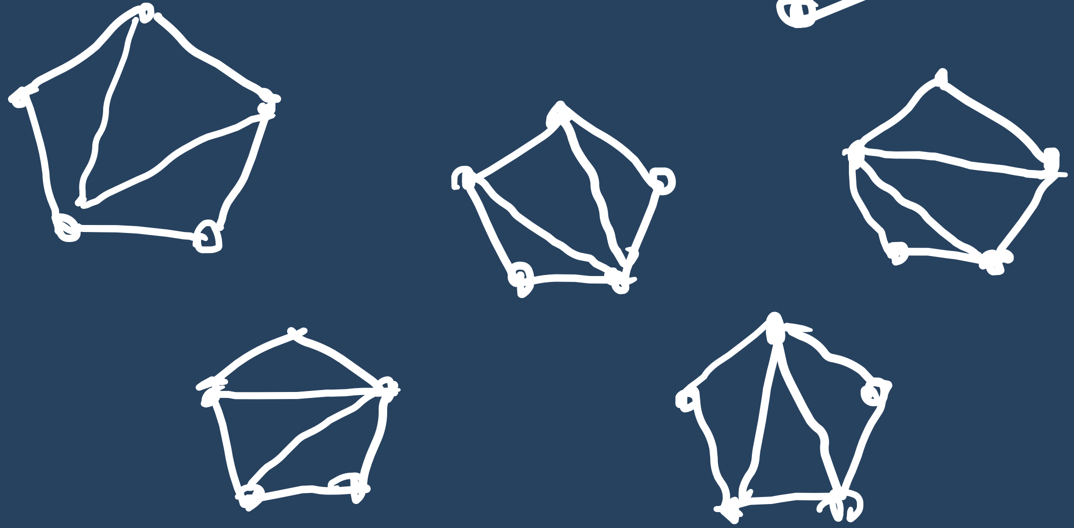
(P) Triangulacja wielokątów wypukłych.



wypukły

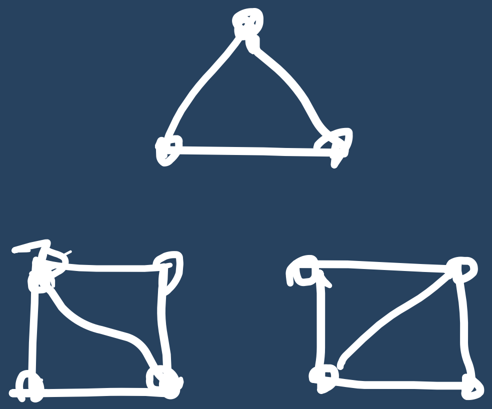
to nie jest wypukłe



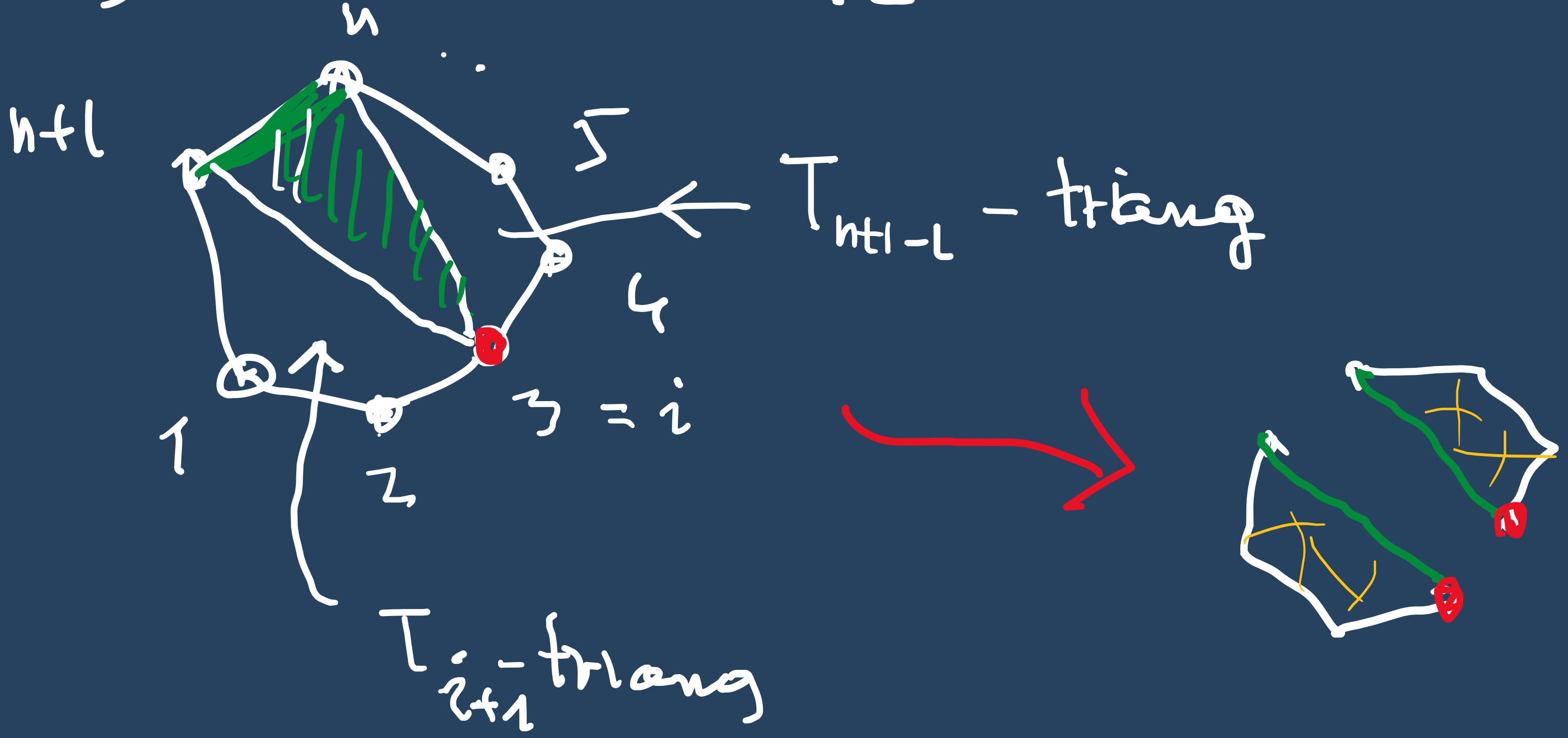


$T_3 = 1$

$T_4 = 2$



Pomysł na indukcję:



$$T_{n+1} = \sum_{a=2}^{n} T_a \cdot T_{n+2-a}$$

$T_2 = 1$

ZADANIE:

sprawdź to

robocze oznacz:

$$L_n = T_{n+2}$$

$$\vdots$$

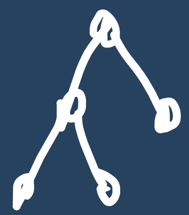
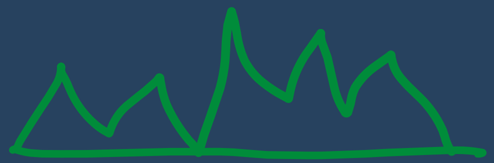
$$L_n = C_n$$

wn. $T_n = C_{n-2}$ $\quad n \geq 2$

↑ l. Catalana

pocz. l. Catalana

R. Stanley, Enumerative Combinatorics,
Appendix: 62 przypadki zastos. l. Catalana

$(x_1 \cdot x_2) \cdot x_3 \leftrightsquigarrow$  $\leftrightsquigarrow$ 

Q: szacowanie C_n

NOTACJA ASYMPTOTYCZNA

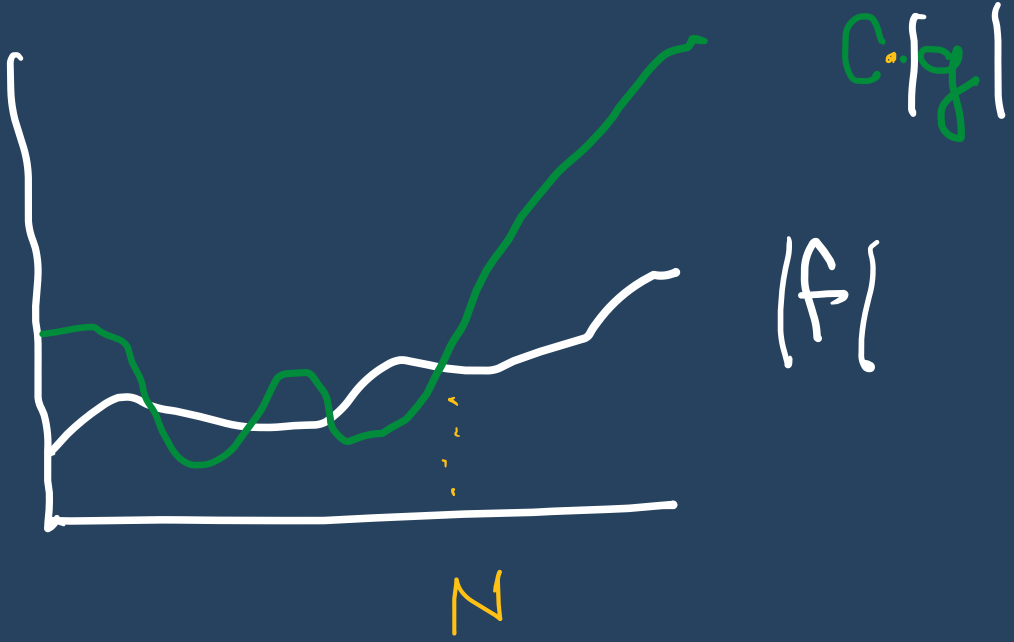
Rozważamy $f \in \mathbb{R}^{\mathbb{N}}$.

<u>Def</u>. $f = O(g)$

$$\Updownarrow$$

$$(\exists C)(\exists N)(\forall n \geq N)\left(|f(n)| \leq C \cdot |g(n)|\right)$$

$f = O(g)$



$$O(g) = \{ f \in \mathbb{R}^{\mathbb{N}} : (\exists C)(\exists N)(\forall n \geq N)\ |f(n)| \leq C \cdot |g(n)| \}$$

FAKT. Załóżmy, że $g > 0$. Wtedy

$$f = O(g) \equiv \left(\limsup_{n \to \infty} \frac{|f(n)|}{g(n)} < \infty \right).$$

<u>D-d</u>. $\Rightarrow$ Mamy C, N : $n \geq N \rightarrow |f(n)| \leq C \cdot |g(n)|$

$$n \geq N \rightarrow \frac{|f(n)|}{g(n)} \leq C$$

$$\rightarrow \limsup \frac{|f(n)|}{g(n)} \leq C < \infty.$$

$\Leftarrow \quad \overline{\lim_{n\to\infty}} \frac{|f(n)|}{g(n)} = C < \infty$

Jest N t.że

$$n > N \to \frac{|f(n)|}{g(n)} < C+1$$

$$n > N \to |f(n)| \le g(n)\cdot(C+1).$$

FAKT.

$$\lim_{n} a_n = g$$

$$\Downarrow$$

$$\overline{\lim_{n\to\infty}}\, a_n = g.$$

P

$$n \cdot \ln(n) \leq O(n^2)$$

czyli: jeśli $f(n) = n \ln n$

$g(n) = n^2$

$$\Rightarrow f = O(g)$$

$$\frac{n \cdot \ln n}{n^2} = \frac{\ln n}{n}$$

$$\lim_{n \to \infty} \frac{\ln n}{n} = \lim_{x \to \infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

$$\overline{\lim_{n \to \infty}} \frac{\ln n}{n} = 0 < \infty$$

$$n \ln(n) = O(n^2).$$

WZMOCNIENIE : $n(\ln(n))^k = O(n^2)$

$k \geq 1.$

P

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$= \frac{1}{6} n \cdot (2n^2 + 3n + 1) =$$

$$= \frac{1}{6}(2n^3 + 3n^2 + n) =$$

$$= \frac{1}{3} n^3 + \underbrace{\frac{1}{2} n^2 + \frac{1}{6} n}_{}$$

$$\frac{\frac{1}{2}n^2 + \frac{1}{6}n}{n^2} \xrightarrow[n\to\infty]{} \frac{1}{2}$$

$$\frac{1}{2}n^2 + \frac{1}{6}n = O(n^2)$$

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + O(n^2)}$$

czyli: jest $f = O(n^2)$ t. że

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + f(n).$$

DEF: $f \sim g \equiv \lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 1$

(dla $f, g > 0$)

DEF: $f = o(g) \equiv (\forall \varepsilon > 0)(\exists N)(\forall n > N)\big(|f(n)| \le \varepsilon \cdot |g(n)|\big)$.

Jeśli $f, g > 0$:

$f = o(g) \equiv \lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

WNIOSEK: $f = o(g) \longrightarrow f = O(g)$.

Wzór Stirlinga:

$$\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

czyli:

$$\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \xrightarrow[n\to\infty]{} 1.$$

Асимпт, ч. Каталана.

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}}$$

$$= \sqrt{\frac{4\pi n}{4\pi^2 n^2}}\; 2^{2n} = \frac{1}{\sqrt{\pi n}}\, 4^n$$

$$C_n \sim \frac{1}{n+1} \frac{1}{\sqrt{\pi n}} 4^n$$

$$\frac{1}{n+1} \sim \frac{1}{n} \qquad // \; \text{bo} \qquad \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \longrightarrow 1$$

$$C_n \sim \frac{1}{\sqrt{\pi}\, n^{3/2}} 4^n$$