Liczby Stirlinga I rodzaju

$$\left[{n \atop k}\right] = \left|\{\pi \in S_n : \pi \text{ rozkłada się na } k \text{ cykli}\}\right|$$

- $\left[{0 \atop 0}\right] = 1$; $\left[{n \atop 0}\right] = 0 \leftarrow n \geq 1$.
- $\left[{n \atop 1}\right] = (n-1)!$; $\left[{n \atop n}\right] = 1$
- $\left[{n \atop n-1}\right] = \binom{n}{2}$
- $\left[{n \atop 2}\right] = (n-1)! \cdot H_{n-1} \leftarrow n \geq 2$

$$H_k = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{k}$$

$$\sim \ln(k) + \gamma$$

Rekursja

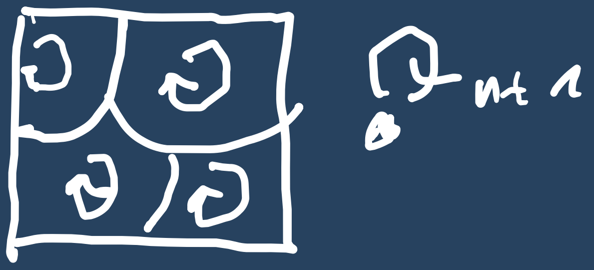
$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} +$$

$\pi \in Sym(\{1, .., n+1\})$; π ma $k+1$ cykli

1. $\pi(n+1) = n+1$

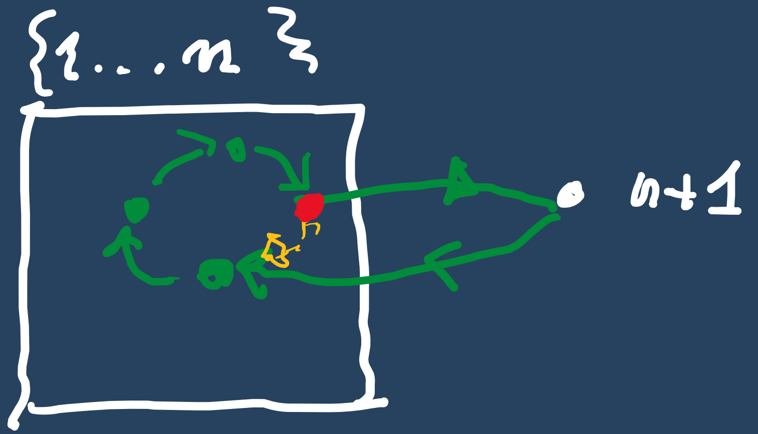
$\pi \restriction \{1, .., n\} \in S_n$

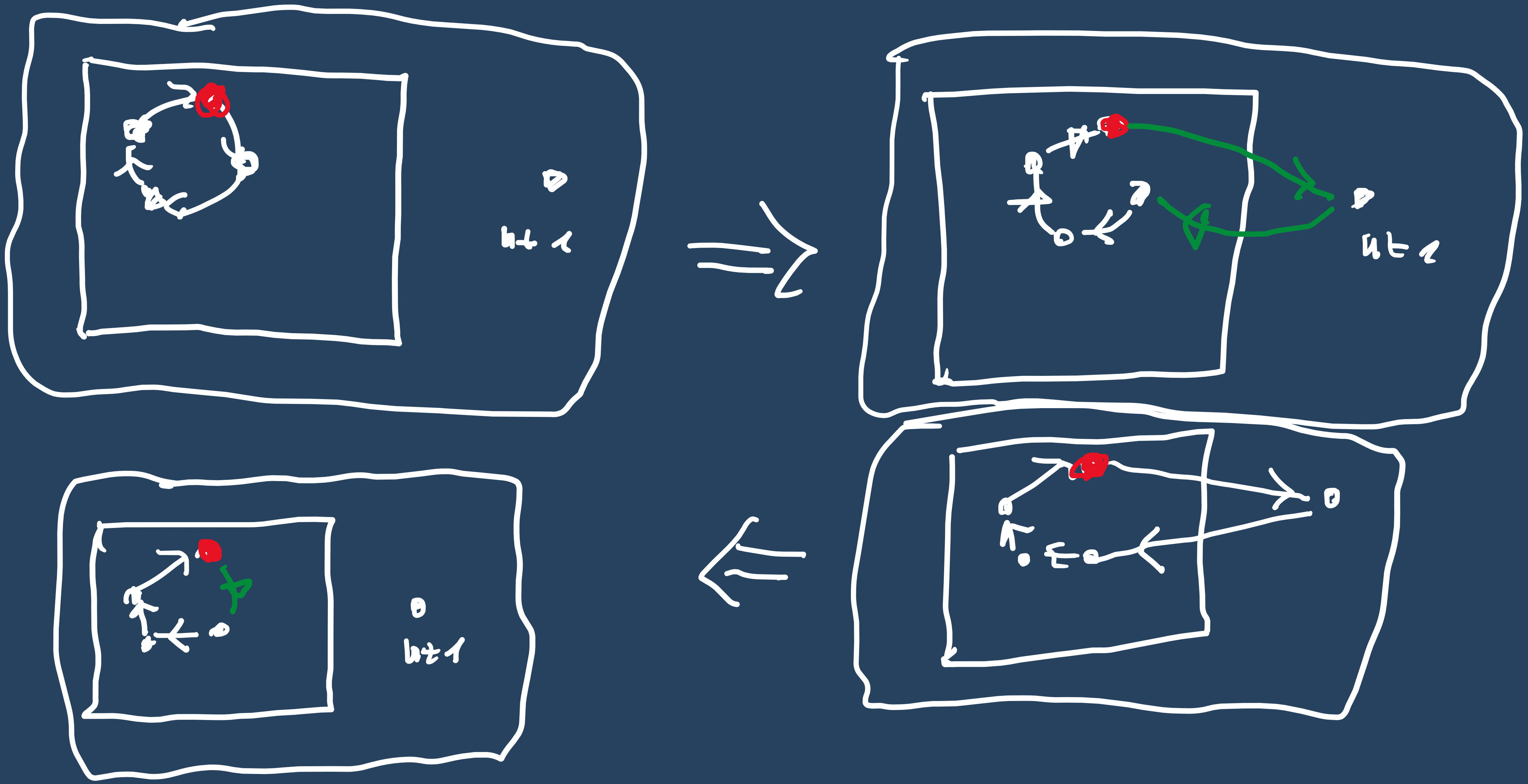


π

k-cykli

2. $\pi(n+1) \in \{1, .., n\}$

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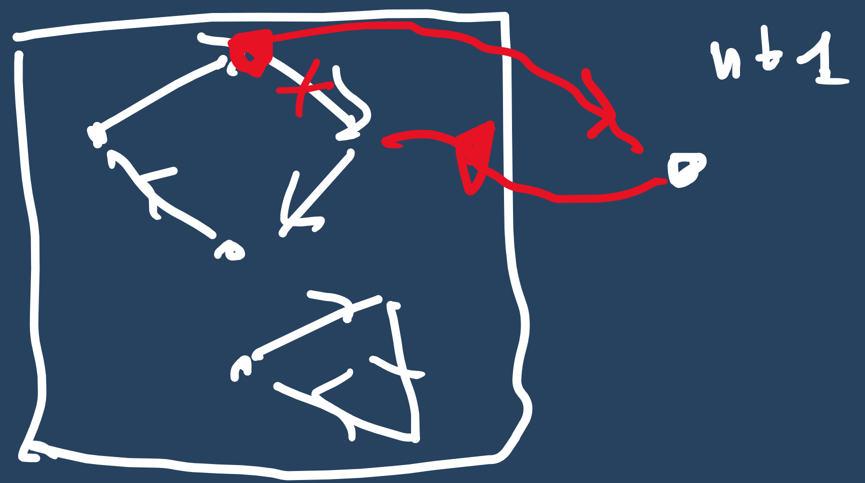
Na ile sposobów mogę wybrać 2

Odp: na n sposobów



Liczba cykli nie ulega zmianie

ZATEM!

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$$

TW. $$\boxed{\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + n \cdot \begin{bmatrix} n \\ k+1 \end{bmatrix}}$$

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

k

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 1 & & \\ & 0 & & 1 & & 1 & \\ 0 & & 2 & & 3 & & 1 \end{matrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1 + 2 \cdot 1 = 3$$

Def. $$x^{\overline{k}} = \prod_{i=0}^{k-1} (x+i)$$

(P) $$1^{\overline{n}} = \prod_{i=0}^{n-1} (1+i) = n!$$

(P) $$x^{\overline{1}} = x$$

$$\boxed{x^{\overline{0}} = 1}$$

Uwaga: $x^{\overline{k}}$ przy ustalonym k jest wielomianem stopnia k.

$$x^{\overline{n}} = \sum_{k=0}^{n} c_{n,k} \cdot x^{k}$$

$$\begin{cases} c_{1,0} = 0 & \text{bo} \\ c_{1,1} = 1 & x^{\overline{1}} = 0 + 1 \cdot x \end{cases}$$

P $\quad x^{\overline{2}} = x(x+1) = 0 + x + x^2$

$$c_{2,0} = 0 \qquad c_{2,1} = 1, \qquad c_{2,2} = 1$$

P $\quad x^{\overline{3}} = x(x+1)(x+2) = (x+x^2)(x+2) =$

$$= x^2 + x^3 + 2x + 2x^2 =$$

$$= 2x + 3x^2 + x^3$$

$c_{n,k}$:

0	1	2	3
0	2	3	1

$$x^{\overline{n}} = \sum_{k=0}^{n} c_{n,k}\, x^k$$

$n \geq 1$

$$x^{\overline{n+1}} = x^{\overline{n}}(x+n) = \left(\sum_k c_{n,k}\, x^k\right)(x+n)$$

$$= \sum_{k=0}^{n} c_{n,k}\, x^{k+1} + \sum_k n \cdot c_{n,k} \cdot x^k$$

$$= \sum_{l=0}^{n+1} c_{n,l-1}\, x^l + \sum_k n \cdot c_{n,k}\, x^k$$

$l = k+1$

$$= \sum_k \left(c_{n,k-1} + n\, c_{n,k}\right) x^k = \sum_{k=0}^{n+1} c_{n+1,k}\, x^k$$

$$C_{n+1,k} = C_{n,k-1} + n\, C_{n,k}$$

więc

$$C_{n+1,k+1} = C_{n,k} + n\, C_{n,k+1}$$

$$\left[{n+1 \atop k+1}\right] = \left[{n \atop k}\right] + n \left[{n \atop k+1}\right]$$

TA SAMA REKURR.

Wn. $C_{n,k} = \left[{n \atop k}\right]$

Wniosek.

$$\boxed{x^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{k}}$$

$$(-x)^{\overline{n}} = \prod_{i=0}^{n-1}(-x+i) = (-1)^{n} \prod_{i=0}^{n-1}(x-i) = (-1)^{n} x^{\underline{n}}$$

$$x^{\underline{n}} = (-1)^{n}(-x)^{\overline{n}} = (-1)^{n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-x)^{k}$$

$(-1)^{k} x^{k}$

$$x^{\underline{n}} = \sum_{k=0}^{n} \underbrace{\begin{bmatrix} n \\ k \end{bmatrix} \cdot (-1)^{n+k}}_{s_{n,k}} \cdot x^{k}$$

$$s_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n+k} \leftarrow$$ oryginalne liczby Stirlinga I rodzaju

DAWNA TERMINOLOGIA:

liczby stirlinga

$$= s_{n,k}$$

NOWA TERMIN.

Knuth

- $\begin{bmatrix} n \\ k \end{bmatrix}$: liczby stirlinga
- $s_{n,k}$: znakowane liczby Stirlinga

Traktujemy S_n jako konecb. przest. probab., czyli

$$A \subseteq S_n : \quad \mathbb{P}r[A] = \frac{|A|}{n!}.$$

Rozważamy zmienną losową:

$L(\pi)$ = liczba cykli permutacji π

CEL: $E[L] = ?$

$$E[L] = \sum_{k=0}^{n} k \cdot \Pr(L=k) =$$

$\operatorname{rng}(L) =: \{0, \ldots, n\}$

$$= \sum_{k=0}^{n} k \cdot \Pr(\{\pi \in S_n : L(\pi) = k\}) =$$

$$= \sum_{k=0}^{n} k \cdot \frac{|\{\pi \in S_n : L(\pi) = k\}|}{n!} = \sum_{k=0}^{n} k \cdot \frac{\left[{n \atop k} \right]}{n!}$$

$$= \frac{1}{n!} \sum_{k=0}^{n} k \cdot \left[{n \atop k} \right].$$

$$\boxed{E[X] = \sum_{a \in \operatorname{rng}(X)} a \cdot \Pr(X=a)}$$

$$x^{\overline{n}} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

$$\left(\sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k\right)' = \sum_{k=1}^{n} k \begin{bmatrix} n \\ k \end{bmatrix} x^{k-1}$$

$$\left(x^{\overline{n}}\right)' = \left(\prod_{i=0}^{n-1} (x+i)\right)' =$$

$$= \sum_{i=0}^{n-1} \frac{\prod_{j=0}^{n-1} (x+j)}{(x+i)}$$

$$(f \cdot g \cdot h)' =$$

$$= f' \cdot g h + f g' h +$$

$$f \cdot g \cdot h$$

$$(x+i)' = 1$$

$$(x^{\overline{n}})'\Big|_{x\leftarrow 1} = \sum_{i=0}^{n-1} \frac{\prod_{j=0}^{n-1}(x+j)\Big|_{x\leftarrow 1}}{1+i} =$$

$$= \sum_{i=0}^{n-1} \frac{n!}{1+i} = n! \sum_{l=1}^{n} \frac{1}{l} = n! \cdot H_n .$$

$$\sum_{k} k \begin{bmatrix} n \\ k \end{bmatrix} = n!\, H_n$$

$$E[L] = \frac{1}{n!}\sum_{k} k \begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{n!} \cdot n! \, H_n$$

TW.

$$\boxed{E[L] = H_n}$$

WNIOSEK!

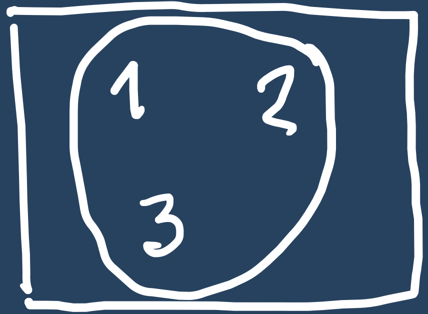
$$E[L] \sim \ln n + \gamma$$

WNIOSEK: typowa permutacja ma duży cykl $\left(\geqslant \frac{n}{H_n} \sim \frac{n}{\ln n} \right)$

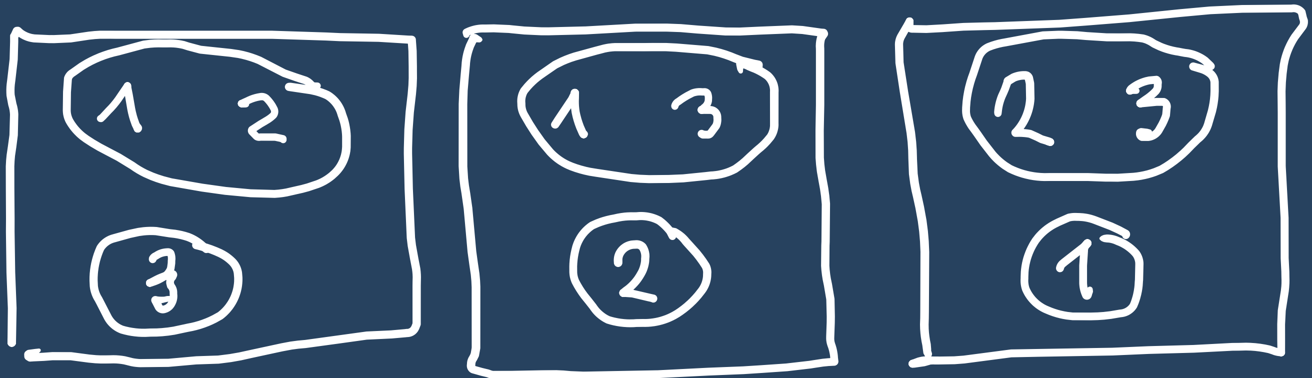
LICZBY STIRLINGA $\underline{\overline{II}}$ RODZAJU

$\left\{ n \atop k \right\}$ = liczba partycji zbioru $\{1, .., n\}$ na k (niepustych) części (składowych, bloków)

$(n = 3)$

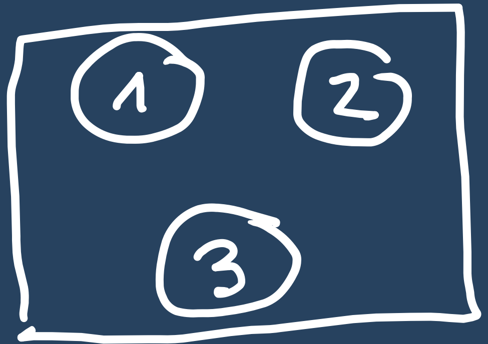


$\left\{ {3 \atop 1} \right\} = 1$



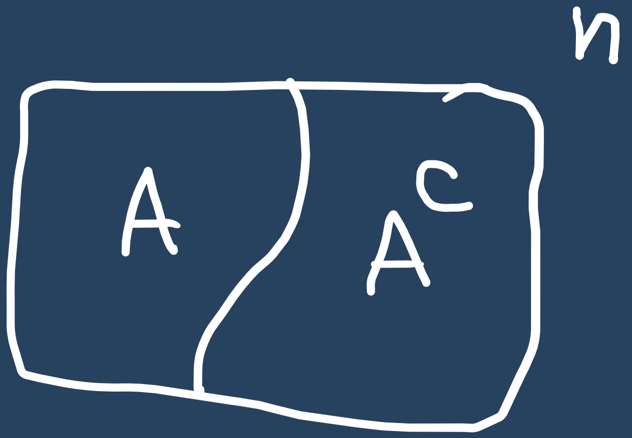
$\left\{ {3 \atop 2} \right\} = 3$

- $\left\{ {0 \atop 0} \right\} = 1$, $\left\{ {n \atop 0} \right\} = 0 \leftarrow n \geq 1$
- $\left\{ {n \atop 1} \right\} = 1$, $\left\{ {n \atop n} \right\} = 1$



$\left\{ {3 \atop 3} \right\} = 1$

$$\left\{ {n \atop 2} \right\} = \frac{1}{2}\left(2^n - 2\right)$$



$$1 \leq |A| \leq n-1$$

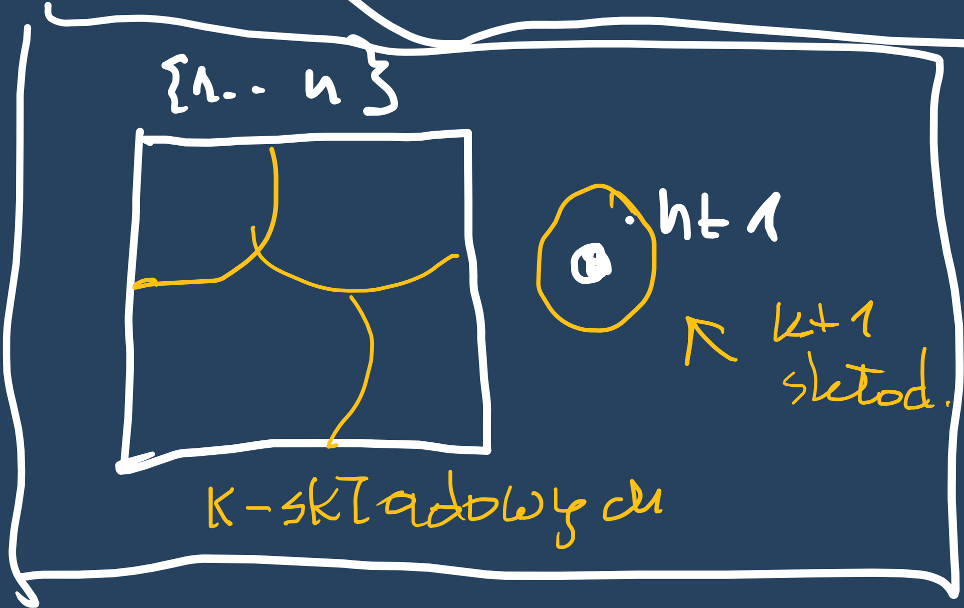
$$A \neq \phi, \quad A \neq \{1, \dots, n\}$$

$$\{A, A^c\} = \{A^c, A\}$$

$$\left\{ {n \atop 2} \right\} = 2^{n-1} - 1 \qquad n \geq 2$$

Rekurencja

$$\left\{ {n+1 \atop k+1} \right\} = \left\{ {n \atop k} \right\} + (k+1) \left\{ {n \atop k+1} \right\}$$



składowe ≡ bloki

zadanie: wyznacz 4 pocz. wiersze "trójk. Stirlinga II rodz"!!