Klasy kombinatoryczne

- $\mathcal{A} = (A, |\cdot|)$, $|\cdot| : A \longrightarrow \mathbb{N}$

  $(\forall n)\left(|\{a \in A : |a| = n\}| < \aleph_0\right)$

- $\mathcal{A}(x) = \sum_n a_n x^n$, $a_n = |\{a \in A : |a| = n\}|$

- $\mathcal{A}, \mathcal{B}$ – rozł. : $(\mathcal{A} + \mathcal{B})(x) = \mathcal{A}(x) + \mathcal{B}(x)$

- $\mathcal{A}, \mathcal{B}$ – K.K : $(\mathcal{A} \times \mathcal{B})(x) = \mathcal{A}(x) \cdot \mathcal{B}(x)$

wn. $\mathcal{A}, \mathcal{B}, \mathcal{C}$: $(\mathcal{A} \times \mathcal{B} \times \mathcal{C})(x) = \mathcal{A}(x) \cdot \mathcal{B}(x) \cdot \mathcal{C}(x)$

Oznaczenia:

- $\mathcal{E} = (\{\varepsilon\}, |\cdot|)$, $|\varepsilon| = 0$

  $\mathcal{E}(x) = 1 \quad (= 1 \cdot z^0)$

- $\mathcal{Z} = (\{a\}, |\cdot|)$, $|a| = 1$

  $\mathcal{Z}(x) = x \quad (= 1 \cdot x^1)$

CIĄGI

$\mathcal{A} = (A, |\cdot|)$

$$Seq(\mathcal{A}) = \bigcup_{n \geq 0} \mathcal{A}^n \quad \left(= \mathcal{E} + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) + \ldots \right)$$

Czy $Seq(\mathcal{A})$ jest K.K. ?

- zał. że jest $\varepsilon \in A$ t.że $|\varepsilon| = 0$.

$(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{A}^n$

$|(\varepsilon_1, \ldots, \varepsilon_n)| = |\varepsilon_1| + \ldots + |\varepsilon_n| = 0$

mamy ∞ wiele elem. rozm 0.

czyli: LIPA

- zał. że $(\forall a \in A)\ |a| \geq 1$

$$(a_1, \dots, a_k) \in Seq(A)$$

$$|(a_1, \dots, a_k)| = |a_1| + \dots + |a_k| \geq \underbrace{1 + \dots + 1}_{k} = k$$

CZYLI:

$$|(a_1, \dots, a_k)| = n \longrightarrow k \leq n$$

$$\longrightarrow |a_1|, \dots, |a_k| \leq n$$

$$\{\bar{a} : |\bar{a}| = n\} \subseteq (A_0 \cup \dots \cup A_n)^n \leftarrow \text{skończone}$$

WNIOSEK: $\begin{cases} \mathrm{Seq}(\mathcal{A}) \text{ jest K.K.} \\ \Updownarrow \\ a_0 = 0 \end{cases}$

$1+q+q^2+\ldots = \frac{1}{1-q}$

Zał. że $a_0 = 0$.

$$\mathrm{Seq}(\mathcal{A}) = \mathcal{E} + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) + \ldots$$
$$= \mathcal{E} + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \ldots$$

$$\mathrm{Seq}(\mathcal{A})(x) = 1 + A(x) + A^2(x) + A^3(x) + \ldots$$
$$= 1 + A(x)\Big(\underbrace{1 + A(x) + A^2(x) + \ldots}_{\mathrm{Seq}(\mathcal{A})(x)}\Big)$$

$$\mathrm{Seq}(\mathcal{A})(x) = 1 + \mathcal{A}(x) \cdot \mathrm{Seq}(\mathcal{A})(x)$$

$$\mathrm{Seq}(\mathcal{A})(x)\,(1 - \mathcal{A}(x)) = 1$$

$$\mathcal{A}(x) = \sum_n a_n x^n$$

$$\mathcal{A}(0) = a_0 = 0$$

$$\mathrm{Seq}(\mathcal{A})(x) = (1 - \mathcal{A}(x))^{-1}$$

$$\boxed{\mathrm{Seq}(\mathcal{A})(x) = \frac{1}{1 - \mathcal{A}(x)}}$$

$$1 + q + q^2 + \dots = \frac{1}{1-q}$$

P

$$\mathcal{D} = \left(\{\boxed{\cdot}, \boxed{\cdot\,\cdot}\}, |\cdot|\right), \quad |\boxed{\cdot}| = 1, \quad |\boxed{\cdot\,\cdot}| = 2$$

$$\mathcal{D}(x) = 1\cdot x + 1\cdot x^2 = x + x^2$$

$$\mathrm{Seq}(\mathcal{D})(x) = \frac{1}{1-x-x^2} = \sum_n d_n x^n$$

$$x \sum_n d_n x^n = \frac{x}{1-x-x^2} = \sum_n F_n x^n$$

l. Fib.

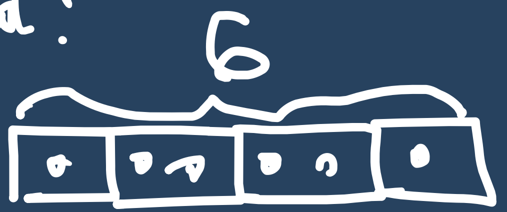
$$x \sum_n d_n x^n = \sum_n d_n x^{n+1} = \sum_{n \geq 1} d_{n-1} x^n$$

więc $(\forall n \geq 1)\ (F_n = d_{n-1})$

więc

$$(\forall n)\left(d_n = F_{n+1}\right).$$

Przykład:



n	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13

(P) $S = (\{\uparrow, \downarrow\}, |\cdot|)$, $|\uparrow| = |\downarrow| = 1$

$$S(x) = 2x$$

$$\mathrm{Seq}(S)(x) = \frac{1}{1-2x} = \sum_{n\geq 0} 2^n x^n$$

$$[x^n]\, \mathrm{Seq}(S)(x) = 2^n.$$

$$|\{\uparrow, \downarrow\}^n| = 2^n.$$

(P) $T = \left(\{ \boxdot, \odot, \boxed{\cdot \mid \boxdot} \}, |\cdot| \right), \quad |\boxdot| = |\odot| = 1, \quad |\boxed{\cdot \mid \boxdot}| = 2$

$$T(x) = 2x + x^2$$

$$\mathrm{Seq}(T)(x) = \frac{1}{1 - 2x - x^2}.$$

$$x^2 + 2x - 1 = 0 \qquad \Delta = 4 + 4 = 8$$

$$\omega_1 = \frac{-2 - 2\sqrt{2}}{2} = -(\sqrt{2} + 1)$$

$$\omega_2 = \frac{-2 + 2\sqrt{2}}{2} = (\sqrt{2} - 1)$$

$$t_n = \frac{(1+\sqrt{2})^{n+1} + (-1)^n (\sqrt{2}-1)^{n+1}}{2\sqrt{2}}$$

wn. $t_n \sim \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n+1} \sim C \cdot 2.25^n$

$$T(x) = \frac{1}{1-2x-x^2} = \sum_n t_n x^n$$

$$1 = T(x)(1-2x-x^2)$$

$$= \sum_n t_n x^n - 2x\sum_n t_n x^n - x^2 \sum_n t_n x^n =$$

$$1 = \sum_n t_n x^n - \sum_n (2t_n) x^{n+1} - \sum_n t_n x^{n+2}$$

$$= \sum_{n \geq 0} t_n x^n - \sum_{n \geq 1} (2 t_{n-1} x^n) - \sum_{n \geq 2} t_{n-2} x^n$$

$$= t_0 + t_1 x + \sum_{n \geq 2} t_n x^n - 2t_0 x - \sum_{n \geq 2} 2 t_{n-1} x^n - \sum_{n \geq 2} t_{n-2} x^n$$

$$= t_0 + (t_1 - 2t_0) x + \sum_{n \geq 2} \underbrace{(t_n - 2t_{n-1} - t_{n-2})}_{0} x^n$$

$t_0 = 1$

$t_1 - 2t_0 = 0$; $t_1 = 2$

$n \geq 2$:

$t_n = 2 \cdot t_{n-1} + t_{n-2}$

Int. kombu. 

$t_0 = 1$ // ciąg pusty

t_1 : 

$n \geq 2$

n kropek

Q. co stoi na końcu

1) 

n-1 kropek

2) 

3) 

n-2

$$\begin{cases} t_n = t_{n-1} + t_{n-1} \\ \quad + t_{n-2} \end{cases}$$

(P) Podziały liczby n:
ciąg $(x_1, \dots, x_k)$ t. że
1) $x_1 + \dots + x_k = n$
2) $x_1, \dots, x_k \geq 1$
(k-dowolne)

$$\mathcal{N}^+ = (\mathbb{N} \setminus \{0\}, |\cdot|), \quad |n| = n$$

$$\mathcal{N}^+(x) = x + x^2 + x^3 + \dots = x(1 + x + x^2 + \dots)$$

$$= \frac{x}{1-x}.$$

$$\mathrm{Seq}(\mathcal{N}^{+})(x) = \frac{1}{1-\mathcal{N}^{+}(x)} =$$

$$= \frac{1}{1-\frac{x}{1-x}} = \frac{1}{\left(\frac{1-x-x}{1-x}\right)}$$

$$= \frac{1-x}{1-2x} = \frac{1-2x+x}{1-2x} = 1+\frac{x}{1-2x}$$

$$= 1 - x\sum_{n\geq 0} 2^{n} x^{n} = 1 + \sum_{n\geq 0} 2^{n} x^{n+1} =$$

$$= 1 + \sum_{n \geq 1} 2^{n-1} x^n$$

$$p_n = \begin{cases} 1 & : n = 0 \quad // \text{ ciąg pusty} \\ 2^{n-1} & : n \geq 1 \end{cases}$$

Z Podaj interpr. kombinatoryczną tego wzorku.

MULTIZBIORY

Mamy $\mathcal{A} = (A, |\cdot|)$ t. że $a_0 = 0$.

Ustalmy $a \in A$. Wiemy, że $|a| \geq 1$.

$\mathcal{P}_a$: $0 \cdot a,\ 1 \cdot a,\ 2 \cdot a,\ 3 \cdot a, \ldots$

$$\mathcal{P}_a(x) = 1 + x^{|a|} + x^{2 \cdot |a|} + x^{3 \cdot |a|} + \ldots$$

$$= 1 + x^{|a|} + (x^{|a|})^2 + (x^{|a|})^3 + \ldots$$

$$= \frac{1}{1 - x^{|a|}}$$

$$\boxed{\mathrm{MULT}(\mathcal{A})(z) = \prod_{a \in A} \frac{1}{1 - z^{|a|}}}$$

P $\quad \mathcal{A} = (\{1, \dots, n\}, |\cdot|) \quad , \quad |i| = 1$

$$M(\mathcal{A})(z) = \prod_{i=1}^{n} \frac{1}{1 - z^1} = \frac{1}{(1-z)^n} = (1-z)^{-n}$$

$$= \sum_k \binom{-n}{k} (-z)^k = \sum_k \binom{k+n-1}{k} z^k$$

$$[z^k]\, M(\mathcal{A})(z) = \binom{k+n-1}{k} = \left(\!\!\binom{n}{k}\!\!\right).$$

(Z) $M = (\{1, 2, 5\}, |\cdot|)$

$|1| = 1, |2| = 2, |5| = 5$

Na ile sposobów mogę 100 zł przedst. jako sumę monet?

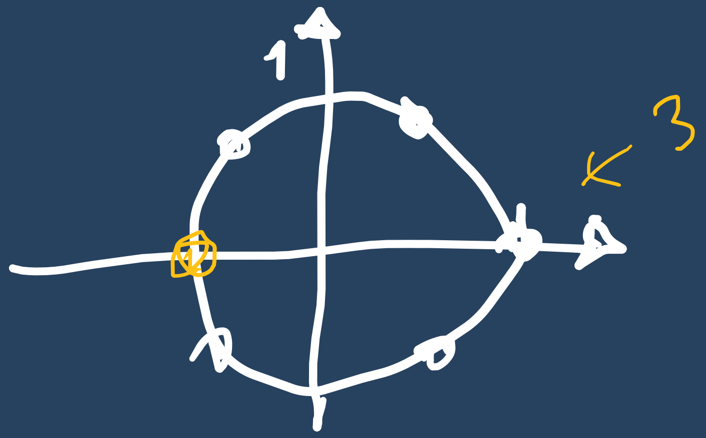
$$100 = \underbrace{1 + 1 + \ldots + 1}_{100} = 100 \cdot 1$$

$$= \underbrace{5 + 5 + \ldots + 5}_{20} = 20 \cdot 5$$

$$\mathrm{Mult}(\mathcal{M})(z) = \prod_{m \in \mathcal{M}} \frac{1}{1 - z^{|m|}} =$$

$$= \frac{1}{1-z} \cdot \frac{1}{1-z^2} \cdot \frac{1}{1-z^5} =$$

$$= \frac{1}{(1-z)(1-z^2)(1-z^5)} .$$



$$f(z) = \frac{1}{(1-z)(1-z^2)(1-z^5)}$$

WZÓR TAYLORA w $z=0$

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} x^n$$

Q: co stoi przy x^{100}

ODP: $\underline{\underline{541}}$

Dlaczego?
popatrz na następną stronę

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F[z_]:= 1/((1-z)(1-z^2)(1-z^5))
Series[F[z],{z,0,100}]
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Fragment sesji z programem MATHEMATICA

1+z+2 z^2+2 z^3+3 z^4+4 z^5+5 z^6+6 z^7+7 z^8+8 z^9+10 z^10+11 z^11+13 z^12+14 z^13+16 z^14+18 z^15+20 z^16+22 z^17+24 z^18+26 z^19+29 z^20+31 z^21+34 z^22+36 z^23+39 z^24+42 z^25+45 z^26+48 z^27+51 z^28+54 z^29+58 z^30+61 z^31+65 z^32+68 z^33+72 z^34+76 z^35+80 z^36+84 z^37+88 z^38+92 z^39+97 z^40+101 z^41+106 z^42+110 z^43+115 z^44+120 z^45+125 z^46+130 z^47+135 z^48+140 z^49+146 z^50+151 z^51+157 z^52+162 z^53+168 z^54+174 z^55+180 z^56+186 z^57+192 z^58+198 z^59+205 z^60+211 z^61+218 z^62+224 z^63+231 z^64+238 z^65+245 z^66+252 z^67+259 z^68+266 z^69+274 z^70+281 z^71+289 z^72+296 z^73+304 z^74+312 z^75+320 z^76+328 z^77+336 z^78+344 z^79+353 z^80+361 z^81+370 z^82+378 z^83+387 z^84+396 z^85+405 z^86+414 z^87+423 z^88+432 z^89+442 z^90+451 z^91+461 z^92+470 z^93+480 z^94+490 z^95+500 z^96+510 z^97+520 z^98+530 z^99+541 z^100+O[z]^101

to jest to co nas interesuje