Digital Signal Processing
Exercises

Jacek Cichoń
Faculty of Information and Communication Technology
Wrocław University of Science and Technology, 2023/24

Notation:

- digital unit impuls function: $\delta[n] = \begin{cases} 1: & n = 0 \\ 0: & n \neq 0 \end{cases}$
- digital unit step function: $u[n] = \begin{cases} 0: & n < 0 \\ 1: & n \geq 0 \end{cases}$
- digital ramp function: $r[n] = \begin{cases} 0: & n < 0 \\ n: & n \geq 0 \end{cases}$
- imaginary unit is denoted by j
- conjugate of complex number z is denoted by z^*

1 Signals

Ex. 1 — Draw the graph of the sequences

1. $x[n] = u[n-1] - u[n-5]$
2. $y[n] = u[n] - 2u[n-4] + u[n-6]$
3. $z[n] = r[n] - 2r[n-4] + r[n-8]$

Ex. 2 — Show that

1. $r[n] = \sum_{k=-\infty}^{n-1} u[k]$
2. $r[n] = \sum_{k=0}^{\infty} k\delta[n-k]$

Ex. 3 — Let

$$x[n] = \begin{cases} 1 & : n = 0 \\ 3 & : n = 1 \\ 2 & : n = 3 \\ 0 & : \text{else} \end{cases}$$

1. Express $x[m]$ in terms of shifted δ signals
2. Express $x[m]$ in terms of shifted $u(n)$ signals

Ex. 4 — Let

$$f[n] = \begin{cases} 0: & n \leq 0 \vee n \geq 6 \\ 1: & 1 \leq n \leq 2 \\ -1: & 3 \leq n \leq 5 \end{cases}$$

Express f in terms of unite step function.

Ex. 5 — Determine whether or not the following signals are periodic and for each signal which is periodic determine its fundamental period

1. $x[n] = \sin(0.25n)$
2. $x[n] = \cos(\pi + n)$
3. $x[n] = e^{\frac{j\pi n}{10}} \cos\left(\frac{\pi n}{5}\right)$

Ex. 6 — Find the even and odd parts of the following signals:

1. $x(n) = u(n)$
2. $x(n) = q^n u(n)$

Ex. 7 — Find the period of the sequence

$$x[n] = \cos\left(\frac{\pi n}{4}\right)\cos\left(\frac{\pi n}{5}\right) + \cos\left(\frac{\pi n}{10}\right) .$$

Ex. 8 — Let $x = \{\underline{2}, 4, 4, 3, 3, 5\}^*$ be a periodic signal with period 6 and let $y = \{\underline{0}, 0, 0, 3, 3, -2, 1, 1, 1, 4, -1, 2, 2, 2\}^*$ be a periodic signal with period 15. Find the fundamental period of the signal $x + y$.

Ex. 9 — Given the sequence $x[n] = (5 - n)(u[n] - u[n-5])$ make a sketch of

1. $y_0[n] = x[n]$
2. $y_1[n] = x[4 - n]$
3. $y_2[n] = x[2n - 3]$
4. $y_3[n] = x[n^2 - 2n + 1]$

Ex. 10 — The power in a signal $\{x[n]\}$ id defined as

$$P = \sum_{n=-\infty}^{\infty} x[n] \cdot x[n]^*$$

Show that if $x(n)$ is a real signal, x_e and x_o are the even and odd parts of x, P, P_e and P_o are the powers in x, x_e and x_o then

$$P = P_e + P_o .$$

Ex. 11 — Consider the sequence $x[n] = q^n u[-n]$ where $q > 1$.

1. Find $\sum_{n=-\infty}^{\infty} x[n]$
2. Compute the power in signal x
3. Let $y[n] = n \cdot x[n]$. Compute the power in signal y.

Ex. 12 — The average power in a signal x is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x[k] \cdot x[k]^* .$$

Calculate the average power of the following series

1. $x[n] = (1 + \frac{1}{n+1})u[n]$
2. $x[n] = \begin{cases} 1 : & 2|n \\ 2 : & \neg(2|n) \end{cases}$

2 Systems

Ex. 13 — Which of the following systems is homogeneous, additive and which are linear:

1. $F_0((x[n]) = e^{x[n]}$

2. $F_1((x[n]) = x[n+2] + 4x[n+1]$
3. $F_2((x[n]) = x(n+2) + 3x[n+1] + 1$
4. $F_3((x[n]) = x[n]\sin(\pi n)$
5. $F_4((x[n]) = \Re(x[n])$
6. $F_3((x[n]) = x[n] + x^*[n]$

Ex. 14 — Which of the following linear systems is time invariant?
1. $y(n) = 3x[n-2]$
2. $y(n) = 3x[2n]$

Ex. 15 — Show that convolution is associative.

Ex. 16 — Let $h[n] = \delta[n] - \delta[n-1]$. Find the convolution $x \star h$ for each of the following sequences
1. $x[n] = \cos(\omega n)$ (express your answers as $A\cos(\omega n + \theta)$ for some A and θ
2. $x[n] = (\frac{1}{2})^n u[n]$
3. $x[n] = u[n+1] - u[n-3]$

Ex. 17 — Express $S_k(\delta) \star S_l(\delta)$ in terms of shift of δ.

Ex. 18 — Let

$$\begin{aligned} h_1[n] &= \delta[n] + \delta[n-1] + \delta[n-2] \\ h_2[n] &= \delta[n+1] - \delta[n-1] \\ h_3[n] &= \delta[n-3] \end{aligned}$$

Simplify the expression $h = (h_1 \star h_2) + h_3$.

Ex. 19 — Let us fix $\alpha \in \mathbb{C}$. Consider the equation $y[n] = \alpha y[n-1] + x[n]$ (this is an example of moving - window average).
1. Suppose that $y[-1] = 0$. Calculate $y[k]$ for $k = 0 \ldots 5$.
2. Guess the general form of the formula for $y[n]$.
3. Show that if $|\alpha| < 1$ then considered system is BIBO.

Ex. 20 — Let us consider the following system $T(x)[n] = \frac{1}{5}\sum_{k=0}^{4} x[n-k]$.
1. Let $h = T(\delta)$. Calculate the function h.
2. Calculate $x \star h$
3. Design a Signal Graph Flow for considered system.

Ex. 21 — Consider equation $y[n] = y[n-1] + y[n-2] + x[n]$. Let us assume that $y[n] = 0$ for $n \leq 0$ and $y[1] = 1$.
1. Use Z transform to calculate

$$Y(z) = \sum_{n \geq 0} \frac{y[n]}{z^n}$$

2. Deduce from $Y(z)$ a general formula for $y[n]$.

Ex. 22 — The Backer-code of length 11 is the sequence

$$\mathbf{B} = [1, 1, 1, -1, -1, -1, 1, -1, -1, 1, -1] \ .$$

1. Design a filter to detect the presence of the code $\mathbf{B}$ in the signal.
2. Write the resulting transformation as a convolution.

3. Test the code detection performance on signals of the following form: $s = x + y$, where

$$x = \underbrace{[0,0,\dots,0]}_{100} ||\mathbf{B}|| \underbrace{[0,0,\dots,0]}_{100}$$

$$y = [y_1, \dots, y_{211}] \ ,$$

where (y_i) are independently generated real numbers from Gauss distribution with mean 0 and standard deviation 0.1 (this is so call Gausian noice).

4. Calculate the auto-correlation vector of $\mathbf{B}$.
5. Change the the sign in arbitrary position in Backer-code and calculate the autocorrelation vector of changed sequence.

2.1 Convolution and Z transform

Ex. 23 — Compute $z = x \star y$ where

1. $x[n] = \alpha^n u[n]$, $y[n] = \beta^n u[n]$
2. $x[n] = \alpha^n u[n]$, $y[n] = \alpha^{-n} u[-n]$, where $0 < \alpha < 1$
3. $y[n] = u[n]$ and x is arbitrary

Ex. 24 — Suppose that $\mathcal{L}$ is LTI such that $h = \mathcal{L}(\delta) = (\alpha^n u[n])_n$. Compute $\mathcal{L}(u)$.

Ex. 25 — Compute Z transform of the following sequences

1. $a[n] = \alpha^n u[n]$
2. $b[n] = \alpha^n u[-n]$
3. $c[n] = n u[n]$
4. $d[n] = n^2 u[n]$
5. $e = a \star c$
6. $f[n] = \sin(\alpha n) u[n]$
7. $g[n] = \cos(\alpha n) u[n]$

Hint: Try to solve the last two points together

3 Discrete Fourier Transform

From this point on, we will denote the imaginary unit by I.

Ex. 26 — Let $x_k(n) = \sin\left(\left(\frac{2\pi}{N} \cdot k\right) \cdot n\right)$ for $n = 0, \dots, N-1$. Use the formula

$$\sin(t) = \frac{1}{2I}(\exp(It) - \exp(-It))$$

to compute $\mathcal{F}(x_k)$.

Ex. 27 — Let $N = 256$. Compute Fourier transform of the following sequence

1.$x_1[n] = \delta[n]$

2.$x_2[n] = 1$

3.$y[n] = \cos(\frac{2\pi}{N} \cdot I \cdot 50 \cdot n)$

4.$z[n] = \sin(\frac{2\pi}{N} \cdot I \cdot 40 \cdot n)$

5.$w[n] = 3\cos(\frac{2\pi}{N} \cdot I \cdot 10 \cdot n) + 7\sin(\frac{2\pi}{N} \cdot I \cdot 300 \cdot n)$

Notation:

$$\Omega_N = \frac{1}{\sqrt{N}}[e^{-\frac{2\pi I}{N} i \cdot j}]_{i,j=0,\dots,N-1}$$

$$\Omega_N^* = \frac{1}{\sqrt{N}}[e^{\frac{2\pi I}{N} i \cdot j}]_{i,j=0,\dots,N-1}$$

Ex. 28 — Calculate $\Omega_N \cdot \Omega_N^*$ and $\Omega_N^* \cdot \Omega_N$, where $\cdot$ denotes matrix multiplication

Ex. 29 — Compute Ω_3 and use this matrix to calculate $\mathcal{F}(x)$ for $x = [1, I, 0]$.

Ex. 30 — Compute Ω_8.

Ex. 31 — Calculate $\mathcal{F}(\varepsilon_{N,a})$ for $a = 0, \dots, N-1$, where $(\varepsilon_{N,0}, \varepsilon_{N,1}, \dots, \varepsilon_{N,N-1})$ is the standard basis of $\mathbb{C}^N$, i.e. $\varepsilon_{N,0} = [1, 0, 0, \dots, 0]$, $\varepsilon_{N,1} = [0, 1, 0, \dots, 0]$, $\dots$, $\varepsilon_{N,N-1} = [0, 0, 0, \dots, 1]$.

Ex. 32 — Let $z \in \mathbb{C}^N$. Find a formula for $\mathcal{F}(\mathcal{F}(z))$.

Ex. 33 — Use any program or system to show the DFT of a sequence

$$x[n] = \cos\left(\frac{2\pi I}{64}(10 + \delta)n\right)$$

for $n = 0, \dots 63$ for various $\delta \in [0, 1]$. Your program should display real part, imaginary part and absolute value of obtained sequences. Interpret your observations.

4 Matrix representation

Ex. 34 — Let $A : \mathbb{C}^4 \to \mathbb{C}^4$ be defined by

$$Ax = 3S_1(x) + x$$

(where $(S_k(x))(n) = x(n - k \mod 4)$).

1. Show that A is translation invariant.
2. Find matrix of A in the standard basis of $\mathbb{C}^4$.
3. Find matrix of A is the Fourier basis of $\mathbb{C}^4$.

Ex. 35 — Show that a linear operator $A : \mathbb{C}^N \to \mathbb{C}^N$ is translation invariant if and only if A is S_1 invariant.

Ex. 36 — Find the circulant matrix of the linear operator $A : \mathbb{C}^4 \to \mathbb{C}^4$ defined by the formula

$$Ax = 2S_{-2}(x) + x + S_1(x)$$

Ex. 37 — Prove that a product of two circulant matrices is circulant.

Ex. 38 — Let $M \in M_{N\times N}(\mathbb{C})$ be an circulant matrix. Show that the linear mapping

$$Ax = M \cdot x^T$$

is translation invariant.

Ex. 39 — Let $b_1, b_2 \in \mathbb{C}^N$. Let C_b be the convolution operators defined by

$$C_b(x) = b \star x \ .$$

Show that

$$C_{b_1} \circ C_{b_2} = C_{b_1 \star b_2} \ .$$

Ex. 40 — Suppose that system H be described by equation $H(x) = h \star x$. The frequency respone of system H is defined as

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{I\omega n} \ ,$$

where $\omega \in \mathbb{R}$. Let $x[n] = e^{I\omega n}$. Show that

$$H(x) = H(\omega)x$$

Ex. 41 — Let system H be described by equation

$$H(x)[n] = \frac{1}{3}(\delta[n-1] + \delta[n] + \delta[n+1]) \ .$$

1. Represent H as a convolution, i.e. find h such that $H(x) = h \star x$.
2. Calculate system response function $H(\omega)$.
3. Let $x[n] = e^{I(\pi/3)n}$. Calculate $H(x)$.
4. Let $x[n] = \cos((\pi/3)n)$. Calculate $H(x)$.

Ex. 42 — Let system H be described by equation

$$H(x)[n] = \delta[n] - \delta[n-1] \ .$$

1. Represent H as a convolution, i.e. find h such that $H(x) = h \star x$).
2. Calculate system response function $H(\omega)$.
3. Let $x[n] = e^{I(\pi/3)n}$. Calculate $H(x)$.
4. Let $x[n] = e^{I(\pi/2)n}$. Calculate $H(x)$.

Ex. 43 — The function sinc is defined by equation

$$\mathrm{sinc}(x) = \begin{cases} 1 & : x = 0 \\ \frac{\sin(x)}{x} & : x \neq 0 \end{cases}$$

1. Draw a diagram of this function.
2. Show that sinc is continuous.
3. Calculate $\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \exp(I\omega k) d\omega$.
4. Show that
$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \exp(I\omega k)\exp(-I\omega n) d\omega = \begin{cases} 1 : & k = n \\ 0 : & k \neq n \end{cases}$$

to be continued
Good luck,
Jacek Cichoń