

# Subsets of the Real Line

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# Introduction

Modern mathematics presents a large number of abstract notions and sophisticated constructions. It is enough to mention two branches of mathematics - topology and algebra - to see how they compete in inventing and formulating more and more complicated notions and general concepts. For instance, in topology many classes of spaces remote from our everyday intuitions are defined and investigated. Also in algebra we may point to categories and functors with their numerous diagrams to see how abstract objects we deal with.

However, we shall not make a big mistake if we state that the real line  $\mathbb{R}$  still remains the fundamental object for the whole mathematics. Notice that it is equipped with three basic mathematical structures - *ordering*, *topologic* and *algebraic* structures - and is used as an initial model for further constructions and investigations. Evidently, the real line is placed in the center of modern mathematical analysis and its various applications.

It is worth reminding here how several problems and questions from the classical mathematical analysis unavoidably brought the nineteen century mathematicians to the study of different subsets of  $\mathbb{R}$  (for example, to the study of the set of convergence points of a given sequence of functions, or to the study of the set of points at which a given function has a derivative). Moreover, the complexity of some problems from analysis implied complication of the structure of the considered sets. Qualitative changes took place by the end of the nineteenth century, when there appeared Cantor's idea, concerning both the general theory of infinite sets and the theory of point-sets. Essentially new constructions, which gave entirely different point-sets, having no analogs in mathematical analysis of previous times, were carried (for instance, it is enough to mention the construction of the famous Cantor discontinuum on  $\mathbb{R}$ ). At the beginning of the twentieth century, with the appearance of the theory of real functions which originated from the works of three French classicists - Baire, Borel and Lebesgue - still new families of subsets of  $\mathbb{R}$  constructed with complicated methods became the subject of investigations. Such sets turned out to be very useful for solving several important problems and questions from analysis. Finally, the first reasoning which used uncountable forms of the Axiom of Choice has brought new subsets of  $\mathbb{R}$  with a more complicated and sophisticated nature. These sets, too, have gradually become objects of intense study.

At present we possess a rather developed theory of subsets of the real line. This theory helps us to see the structure of  $\mathbb{R}$  more deeply. Of course, when we study subsets of the real line it is convenient to classify them with respect to some common properties. For instance, instead of studying one particular first category set on  $\mathbb{R}$  it is sometimes better to focus on the whole ideal of first category sets and investigate the properties of this ideal. The same remark may be applied to the sets of Lebesgue measure zero and to many other ideals containing small subsets of the real line. We do not mention the ideals of the first category sets and Lebesgue

measure zero sets accidentally. A large number of very deep analogies between these ideals can be observed. Moreover, the similarity between the category and measure can also be noticed in considerations of two fundamental  $\sigma$ -algebras of subsets of  $\mathbb{R}$ : namely, the  $\sigma$ -algebra of sets with the Baire property and the  $\sigma$ -algebra of sets measurable in the Lebesgue sense. It is worth remarking here that one of the most important properties of these algebras is the so called countable chain condition from which the completeness of the appropriate quotient algebras follows. This fact plays an essential role in constructing various special models of set theory.

In our book we consider both a variety of remarkable subsets of the real line and certain interesting classes of subsets of  $\mathbb{R}$ .

For the reader's convenience we try to present the material in a comprehensive and detailed form. For this reason the first part of the book has a more general character and contains a large number of basic facts from set theory, topology, descriptive set theory and measure theory. After these preliminary facts we consider some structural properties of  $\mathbb{R}$  and then give detailed constructions of some important classical subsets of  $\mathbb{R}$ . Among them there are the Cantor discontinuum mentioned above, a Lebesgue measurable set which together with its complement intersects every non-empty open interval on a set of strictly positive measure, a Vitali non-measurable set, a Hamel basis for  $\mathbb{R}$ , a totally imperfect Bernstein set, a Luzin set, a Sierpiński set and so on. In this part of the book we begin the study of first category sets and Lebesgue measure zero sets and discuss not only analogies between category and measure but also some essential differences between them. Besides, we present here an important result of Choquet, namely, the Choquet theorem on capacities with several applications.

The second part of our book is devoted to more recent and technically more complicated constructions of subsets of the real line. We start with deeper information from set theory: we discuss axioms of set theory, forcing techniques and absoluteness of various formulas. Basing on these concepts we continue investigations and study, as before, of properties of individual subsets of the real line as well as properties of certain special classes of subsets of  $\mathbb{R}$ . We present Mycielski's two theorems concerning measure and category in the product space and Mokobodzki's one theorem about essential difference between measure and category. Then we give some applications of these results to the subsets of  $\mathbb{R}$ . Further, we consider more thoroughly the ideal of first category sets, the ideal of Lebesgue measure zero sets and the appropriate  $\sigma$ -algebras: of sets with the Baire property and of the Lebesgue measurable sets. Our main purpose here is to describe some cardinal-valued functions (characteristics) related to the classes of sets mentioned above and to some other auxiliary classes. In particular, we demonstrate how these cardinal-valued characteristics behave and change in different models of set theory. Among other results presented in this part worth noticing are: Kunen's theorem on the existence of a Lebesgue non-measurable set in  $\mathbb{R}$  with cardinality less than the cardinality of continuum, provided the cardinality of continuum is a real-valued measurable cardinal and related, stronger results; Raisonier's theorem on the existence (without uncountable forms of the Axiom of Choice) of Lebesgue non-measurable subset of  $\mathbb{R}$ , provided the first uncountable cardinal is comparable to the cardinality of continuum.

The book ends with three appendices. The first one contains some fundamental facts from infinitary combinatorics which have numerous applications in general topology and analysis. The second one is devoted to the proof of two basic theorems from measure theory: Maharam's theorem on representation of Boolean algebras

with measures and von Neumann's theorem on the existence of a multiplicative lifting. The last appendix contains a few results on measurable selectors such as, e.g., the famous Kuratowski - Ryll-Nardzewski theorem, and some applications of these results.

We will end our book with historical remarks about authors of some results and theorems presented in the main text and exercises.

Finally, let us notice that the present book includes a large number of various exercises. One can meet here exercises of a standard type but there are many original and rather difficult exercises, too. We recommend these exercises to the reader even if he is not going to solve them. We can say that the reader will get a possibility to gain additional information about the subject of this book, if he sometimes looks through these exercises.



# Part I

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# Chapter 1

## Preliminary Facts from Set Theory

In this Chapter we fix the notation and introduce some elementary facts from set theory which we need in Part 1 of our book.

Our presentation is done in the so called "naive set theory" however, this is actually the theory that is commonly used by the mathematicians. This kind of treatment allows us to present several basic definitions and facts about ordinals, cardinals, infinitary combinatorics and some extra axioms of set theory without getting too deeply into the logical structure of these notions.

The most popular system of axioms of set theory is the so called *Zermelo-Fraenkel* set theory usually denoted by **ZF**. The basic notions of this theory are sets and the membership relation denoted by  $\in$ . Theory **ZF** consists of several axioms which formalize properties of sets in terms of relation  $\in$ . In this Chapter we are not going to present a full list of these axioms since, as stated above, we shall work in the naive set theory, which is sufficient for most of our purposes in Part 1 of this book. A construction of the formal Zermelo-Fraenkel axiomatic system will be discussed in Chapter 1 of Part 2 of the book.

Until now it is unknown whether theory **ZF** is consistent. In other words, we do not know if it is possible to deduce a logically false statement from the axioms of **ZF**. This explains why the notion of relative consistency is used in a lot of results of the modern set theory. However, we believe that theory **ZF** is consistent and we shall tactically assume this in our book.

In our considerations we shall apply standard notations and terminology commonly used in many branches of mathematics. However, we shall briefly recall some of these notations.

We constantly use the following logical symbols

$$\neg, \vee, \&, \rightarrow, \longleftrightarrow, \exists, \forall,$$

which we expect to be well-known to the reader.

The empty set will be denoted by  $\emptyset$ . The set of all elements  $x$  of a given set  $X$  such that the condition  $\phi(x)$  holds, will be denoted by

$$\{x \in X : \phi(x)\}.$$

We write  $Y \subseteq X$ , if  $Y$  is a subset of  $X$ . We will use the standard notations  $\cup$  and  $\cap$  for unions and intersections of the families of sets. We say that two sets  $X$  and  $Y$  are **disjoint** if  $X \cap Y = \emptyset$ .



The difference between two sets  $X$  and  $Y$  will be denoted by  $X \setminus Y$ . The symmetric difference between sets  $X$  and  $Y$  will be denoted by  $X \Delta Y$ , i.e.

$$X \Delta Y = (X \setminus Y) \cup (Y \setminus X).$$

We denote the power set of a set  $X$ , i.e. the set of all subsets of  $X$ , by  $P(X)$ .

The **ordered pair** of  $x$  and  $y$  is denoted by  $(x, y)$ . Recall that

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

The **ordered n-tuple**  $(x_1 \dots x_n)$  is a natural generalization of the notion of an ordered pair and is defined by an easy recursion. The **Cartesian product**  $X \times Y$  of two sets  $X$  and  $Y$  is the set of all ordered pairs  $(x, y)$  such that  $x \in X$  and  $y \in Y$ . Analogously,  $X_1 \times \dots \times X_n$  is the set of all n-tuples  $(x_1, \dots, x_n)$  such that

$$x_i \in X_i (i = 1, \dots, n).$$

A **binary relation** is any set of ordered of ordered pairs. If  $\Phi$  is a binary relation then we put

$$\begin{aligned} pr_1(\Phi) &= \{x : (\exists y)((x, y) \in \Phi)\}, \\ pr_2(\Phi) &= \{y : (\exists x)(x, y) \in \Phi\}, \\ \Phi^{-1} &= \{(x, y) : (y, x) \in \Phi\}. \end{aligned}$$

If  $\Psi$  is another binary relation, then we put

$$\Psi \circ \Phi = \{(x, z) : (\exists y)((x, y) \in \Phi \ \& \ (y, z) \in \Psi)\}.$$

If for  $\Phi$  we have the inclusion  $\Phi \subseteq X \times Y$ , then we say that  $\Phi$  is a binary relation between elements of sets  $X$  and  $Y$ . Moreover, if  $X = Y$  then we say that  $\Phi$  is a binary relation on  $X$ . The important example of a binary relation on a given set  $X$  is any **equivalence** relation on this set, i.e. a binary relation  $\Phi$  on  $X$  which satisfies the following three conditions:

$$\begin{aligned} \{(x, x) : x \in X\} &\subseteq \Phi, \\ \Phi^{-1} &= \Phi, \\ \Phi \circ \Phi &= \Phi. \end{aligned}$$

We will treat a **function** (a **mapping**) as a special binary relation  $f$  satisfying the following condition:

$$((x, y) \in f \ \& \ (x, y') \in f) \rightarrow (y = y').$$

For any function  $f$  and any pair  $(x, y) \in f$  we will write  $y = f(x)$  and say that  $y$  is the value of  $f$  on an element  $x$ . We will say that  $f$  is a function defined on  $X$  if  $X = pr_1(f)$ . In such a case we also write  $X = dom(f)$  and say that  $X$  is the **domain** of  $f$ . The set  $pr_2(f)$  is called the **range** of  $f$ . As a rule we denote the range of  $f$  by the symbol  $ran(f)$ .

A function  $f$  defined on a set  $X$  is into a set  $Y$  if  $ran(f) \subseteq Y$ . In such a case we write

$$f : X \rightarrow Y$$

and say that  $f$  is a mapping from  $X$  into  $Y$ . Sometimes we will also use the following notation:

$$x \rightarrow f(x) \quad (x \in dom(f)).$$

We say that a mapping  $f : X \rightarrow Y$  is an **injection** if

$$(\forall x \in X)(\forall y \in X)(x \neq y \rightarrow f(x) \neq f(y)).$$

We say that a mapping  $f : X \rightarrow Y$  is a **surjection** if  $Y = f(X)$  and, finally, we say that  $f$  is a **bijection** if  $f$  is both an injection and a surjection.

The restriction of a function  $f : X \rightarrow Y$  to a set  $A \subseteq X$  is denoted by  $f|_A$ . We will denote the set of all functions from  $X$  into  $Y$  by  $Y^X$ .

An indexed family of sets  $(X_i)_{i \in I}$  is a function with domain  $I$  such that its value on every  $i \in I$  is  $X_i$ . In the usual way we define the union  $\bigcup_{i \in I} X_i$  and the intersection  $\bigcap_{i \in I} X_i$  of this family. The **Cartesian product** of a family  $(X_i)_{i \in I}$ , denoted  $\prod_{i \in I} X_i$ , is the set of all mappings  $f$  from  $I$  into  $\bigcup_{i \in I} X_i$  such that for each  $i \in I$  we have  $f(i) \in X_i$ .

A **partial ordering** on a set  $X$  is a binary relation  $\preceq$  contained in  $X \times X$  which is **reflexive** (i.e.  $x \preceq x$  for all  $x \in X$ ), **transitive** (i.e.  $x \preceq y$  and  $y \preceq z$  imply  $x \preceq z$ ) and **weakly antisymmetric** (i.e.  $x \preceq y$  and  $y \preceq x$  imply  $x = y$ ). If  $x \in X$ ,  $y \in X$  and  $x \preceq y$ , then we sometimes say that  $y$  is **above**  $x$ . Sometimes instead of  $x \preceq y$  it is more convenient to write  $y \succeq x$ .

Let  $X$  be a set partially ordered by  $\preceq$ . The pair  $(X, \preceq)$  is called a **partially ordered set**. In some considerations it is convenient to use the **strict** ordering  $\prec$  instead of  $\preceq$ . Of course, we define the strict ordering in the usual way as follows

$$x \prec y \iff (x \preceq y \ \& \ x \neq y).$$

An element  $x$  from the partially ordered set  $(X, \preceq)$  is called **maximal** in  $X$  if

$$(\forall y \in X)(y \succeq x \rightarrow x = y).$$

An element  $x$  from the partially ordered set  $(X, \preceq)$  is called **minimal** in  $X$  if

$$(\forall y \in X)(y \preceq x \rightarrow x = y).$$

For any two elements  $a, b$  from the partially ordered set  $(X, \preceq)$  we put:

$$]a, b[ = \{x \in X : a \prec x \prec b\},$$

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A partial ordering  $\preceq$  on a set  $X$  is **linear** if it is **connected** (i.e.  $x \preceq y$  or  $y \preceq x$  for any  $x, y \in X$ ).

Any subset of a partially ordered set  $(X, \preceq)$  linearly ordered by  $\preceq$  will be called a **chain** in  $X$ . A set  $Y \subseteq X$  is called an **antichain** in  $X$  if any two distinct elements from  $Y$  are **incomparable** with respect to  $\preceq$ , i.e.

$$(\forall x \in Y)(\forall y \in Y)(x \neq y \rightarrow (\neg(x \preceq y) \ \& \ \neg(y \preceq x))).$$

A set  $(X, \preceq)$  is **well ordered** by  $\preceq$  if each non-empty subset  $Y$  of  $X$  has the **least element**  $y \in Y$ , i.e.

$$(\forall z \in Y)(y \preceq z).$$

**Ordinal numbers** (or simply **ordinals**) are sometimes defined as the isomorphism types of well ordered sets. We prefer another, more convenient for us, way. We define an ordinal number as any set  $\alpha$  satisfying the following two conditions:

- 1)  $\alpha$  is transitive, i.e.  $\bigcup \alpha \subseteq \alpha$ ;

2)  $\alpha$  is well ordered by the relation

$$\{(x, y) \in \alpha \times \alpha : x = y \vee x \in y\}.$$

If  $\alpha$  and  $\beta$  are ordinals, then we write  $\alpha < \beta$  if  $\alpha \in \beta$ . It is easy to check that

$$\alpha \subseteq \beta \longleftrightarrow (\alpha \in \beta \vee \alpha = \beta).$$

It is clear that the empty set  $\emptyset$  is an ordinal which is also denoted by the usual symbol 0. By the method of transfinite induction it is not difficult to prove that for any ordinal  $\alpha$  the equality

$$\alpha = \{\xi : \xi < \alpha\}$$

holds. The above definition of ordinal numbers seems to be a bit artificial but it simplifies the notation and many considerations. Moreover, this definition shows us that ordinal numbers are concrete mathematical objects which can be constructed recursively beginning from the empty set. Extending this idea we may also assert that all sets can be constructed recursively starting with the empty set (in this way we obtain the so called **von Neumann Universe**, which will be discussed in detail in Part 2 of the book).

Of course, one can prove that every well ordered set is isomorphic with an ordinal in our sense. Hence, ordinal numbers defined as above are the canonical members of the equivalence classes of isomorphism types of well ordered sets. Let us recall to the reader that for any two well ordered sets one (of them) is isomorphic with an initial segment of the second one. From these facts we immediately deduce that the relation  $\in$  is a well ordering on the class of all ordinals (more precisely, the relation  $\in$  is a well ordering on any set of ordinals).

If  $\alpha$  is an ordinal and  $\alpha = \bigcup \alpha$ , then we say that  $\alpha$  is a **limit** ordinal. Notice that if an ordinal  $\alpha$  is not a limit ordinal, then we have  $\alpha = \beta \cup \{\beta\}$  for some ordinal  $\beta$ . In this case we write  $\alpha = \beta + 1$  and call  $\alpha$  a **successor** ordinal number or (more exactly) the successor of  $\beta$ .

Addition and multiplication of ordinals are defined by the method of transfinite recursion. Namely, for addition we have

- 1)  $\alpha + 0 = \alpha$ ,
- 2)  $\alpha + (\beta + 1) = (\alpha + \beta) + 1$ ,
- 3)  $\alpha + \lambda = \bigcup_{\zeta < \lambda} (\alpha + \zeta)$ , for limit ordinal  $\lambda$ .

Respectively, for multiplication we have

- 1)  $\alpha \cdot 0 = 0$ ,
- 2)  $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ ,
- 3)  $\alpha \cdot \lambda = \bigcup_{\zeta < \lambda} \alpha \cdot \zeta$ , for limit ordinal  $\lambda$ .

We denote the least infinite ordinal number by  $\omega$  (or by  $\omega_0$ ). This ordinal will be usually identified with the set  $\mathbb{N}$  of all natural numbers. Any ordinal  $\alpha$  can be uniquely represented in the form

$$\alpha = \beta + n,$$

where  $\beta$  is a limit ordinal and  $n < \omega$ . If in this representation  $n$  is an odd natural number, then  $\alpha$  is called an **odd ordinal**. If  $n$  is an even natural number, then  $\alpha$  is called an **even ordinal**. In particular, all limit ordinals are even ordinals.

Ordinal numbers are very useful in a lot of fields of modern mathematics, especially when the method of transfinite induction or transfinite recursion must be used. Strong results can be obtained if we assume that any set of objects may be well ordered. This is guaranteed by the famous **Axiom of Choice**. An equivalent version of the Axiom of Choice is the following sentence:

for every set  $X$  there exists a bijection from  $X$  onto some ordinal.

The Axiom of Choice is denoted by **AC**. The famous result of Gödel says that theory **(ZF)** & **(AC)** is consistent. To establish this result Gödel distinguished from the von Neumann Universe a certain subclass of sets, which is now called the **Constructible Universe**, and showed that in this class all axioms of theory **(ZF)** & **(AC)** hold. The second important result is due to Cohen, who proved that theory **(ZF)** &  $(\neg \mathbf{AC})$  is also consistent. To construct a required universe Cohen used an essentially different technique which is known now as the **method of forcing**. These two results say together that the Axiom of Choice is independent of theory **ZF**.

Theory **(ZF)** & **(AC)** is usually denoted by **ZFC**. In this book we shall use theory **ZFC** mainly. Only few, specially interesting results will be formulated in theory **ZF** and we shall stress these parts of considerations separately.

The Axiom of Choice is commonly used in most branches of mathematics. For example, it is even used in the classical proof of the equivalence of two definitions of continuous functions from the real line into the real line. One of these definitions is due to Cauchy and the other to Heine. It is worth remarking here that in the proof of the equivalence of these two definitions we need only some weak version of **AC**.

At present a lot of equivalent formulations of the Axiom of Choice are known. Probably the simplest one is the following :

the Cartesian product of any family of non-empty sets is also non-empty.

But the most usual form of **AC** is known as the **Zorn Lemma** (proved, in fact, several years before Zorn by Kuratowski). The formulation of the Zorn lemma is the following:

Let  $(X, \preceq)$  be a partially ordered set such that for each chain  $Y \subseteq X$  there exists an element  $x \in X$  which is above all elements from  $Y$ ; then  $X$  has a maximal element.

As we have already said, the Axiom of Choice implies that any given set  $X$  can be bijectively mapped onto some ordinal. Therefore, it is natural to define the **cardinality** of a set  $X$  as the least ordinal  $\alpha$  such that there exists a bijection from  $X$  onto  $\alpha$ . Equivalently, the cardinality of  $X$  is the least order-type of all well-orderings on  $X$ . The cardinality of a set  $X$  is denoted by  $card(X)$  (in contemporary works on set theory more popular is the notation  $|X|$ ).

We start the discussion about cardinal arithmetic with one useful result which will have a lot of further applications.

**Theorem 1.1 (Banach)** *Suppose that  $X, Y$  are two sets and that*

$$f : X \rightarrow Y, \quad g : Y \rightarrow X$$

*are two injections. Then the sets  $X$  and  $Y$  can be represented as disjoint unions*

$$X = X_1 \cup X_2, \quad Y = Y_1 \cup Y_2$$

in such a way that the functions

$$f : X_1 \rightarrow Y_1, \quad g : Y_2 \rightarrow X_2$$

are bijections.

**Proof.** Let  $\phi : P(X) \rightarrow P(X)$  be a mapping defined by the formula

$$\phi(Z) = X \setminus g(Y \setminus f(Z)),$$

where  $Z \in P(X)$ . It is easy to check that for any family  $(Z_i)_{i \in I}$  of subsets of  $X$  the equality

$$\phi\left(\bigcup_{i \in I} Z_i\right) = \bigcup_{i \in I} \phi(Z_i)$$

holds. Let us denote

$$A = \emptyset \cup \phi(\emptyset) \cup \phi^2(\emptyset) \cup \phi^3(\emptyset) \dots$$

and observe that

$$\phi(A) = A.$$

So we have

$$A = X \setminus g(Y \setminus f(A)), \quad X \setminus A = g(Y \setminus f(A)).$$

Thus, we may put

$$X_1 = A, \quad X_2 = X \setminus A, \quad Y_1 = f(A), \quad Y_2 = Y \setminus f(A).$$

The Banach theorem is proved.

We want to pay the reader's attention not only to the theorem just proved but also to the method of the proof. It will be important for us that the constructed sets  $X_1$ ,  $X_2$ ,  $Y_1$  and  $Y_2$  are obtained, applying functions  $f$  and  $g$ , by taking images, differences of sets and countable unions of sets. Moreover, the proof of the Banach theorem was done effectively, i.e. without the use of the Axiom of Choice, so this theorem was proved in theory **ZF**.

From Theorem 1 we easily get the classical **Cantor theorem**, which says that if there are injections  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$ , then there exists a bijection between  $X$  and  $Y$ . The Cantor theorem (called also the Cantor - Bernstein theorem) is the basic fact of the theory of cardinal numbers and is one of the few important results about cardinalities which can be proved without the Axiom of Choice.

We say that an ordinal  $\alpha$  is a **cardinal number** (or simply **cardinal**) if  $\alpha$  is the least ordinal of its own cardinality, i.e. if  $\alpha = \text{card}(\alpha)$ . Equivalently, an ordinal number  $\alpha$  is a cardinal if it is equal to  $\text{card}(X)$  for some set  $X$ .

The first cardinal after  $\omega$  is denoted by  $\omega_1$ , the first cardinal after  $\omega_1$  is denoted by  $\omega_2$ , and so on. Hence, for any ordinal  $\alpha$  the  $\alpha^{\text{th}}$  cardinal number after  $\omega$  is denoted by  $\omega_\alpha$ . We want to remark here that we usually reserve symbols

$$\kappa, \lambda, \mu, \nu, \dots$$

to denote arbitrary cardinals.

The **successor of a cardinal**  $\kappa$ , denoted by  $\kappa^+$ , is the least cardinal bigger than  $\kappa$ . Hence, for example

$$\omega^+ = \omega_1, \quad (\omega_1)^+ = \omega_2, \dots$$

Addition and multiplication of cardinal numbers  $\kappa$  and  $\lambda$  are defined by the formulas

$$\kappa + \lambda = \text{card}((\kappa \times \{\emptyset\}) \cup (\lambda \times \{1\})),$$

$$\kappa \cdot \lambda = \text{card}(\kappa \times \lambda).$$

We want to pay the reader's attention to an essential difference between cardinal and ordinal arithmetics (for example,  $\omega+1 = \omega$  in cardinal arithmetic and  $\omega+1 > \omega$  in ordinal arithmetic).

Addition and multiplication of cardinal numbers have usual well known properties of addition and multiplication defined in the set  $\mathbb{N}$  of natural numbers. The following theorem essentially simplifies the calculations with infinite cardinals and has no analogue for natural numbers.

**Theorem 1.2** *If  $\kappa$  is an infinite cardinal number, then  $\kappa \cdot \kappa = \kappa$ .*

**Proof.** The theorem is obvious for  $\kappa = \omega$ . Suppose now that the theorem is true for all infinite cardinals  $\lambda < \kappa$ . In particular, we have  $\lambda \cdot \lambda < \kappa$  for all cardinals  $\lambda < \kappa$ . Let us define the ordering  $\ll$  on  $\kappa \times \kappa$ . Let us put

$$(\xi, \eta) \ll (\xi', \eta')$$

if and only if one of the next three conditions holds:

- 1)  $\max\{\xi, \eta\} < \max\{\xi', \eta'\}$ ,
- 2)  $\max\{\xi, \eta\} = \max\{\xi', \eta'\}$  and  $\xi < \xi'$ ,
- 3)  $\max\{\xi, \eta\} = \max\{\xi', \eta'\}$  and  $\xi = \xi'$  and  $\eta < \eta'$ .

It is easy to see that the ordering  $\ll$  defined in this way is a well ordering on  $\kappa \times \kappa$ . For each  $(\alpha, \beta) \in \kappa \times \kappa$  we have

$$\{(\xi, \eta) : (\xi, \eta) \ll (\alpha, \beta)\} \subseteq (\max\{\alpha, \beta\} + 1) \times (\max\{\alpha, \beta\} + 1).$$

From this fact we obtain

$$\text{card}(\{(\xi, \eta) : (\xi, \eta) \ll (\alpha, \beta)\}) \leq \omega \cdot \text{card}(\alpha) \cdot \text{card}(\beta) < \kappa.$$

Hence,  $\kappa \cdot \kappa \leq \kappa$  and it is obvious that  $\kappa \leq \kappa \cdot \kappa$ . Consequently, we have  $\kappa \cdot \kappa = \kappa$ .

From Theorems 1 and 2 it immediately follows that for any two infinite cardinals  $\kappa$  and  $\lambda$  the equality

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$$

holds. The next theorem (essentially due to Sierpiński) will play an important role in many further constructions.

**Theorem 1.3** *Suppose that  $\kappa$  is an infinite cardinal number,  $(X_\alpha)_{\alpha \in \kappa}$  is a family of sets and each of these sets has cardinality  $\kappa$ . Then there exists a family  $(Y_\alpha)_{\alpha \in \kappa}$  of pairwise disjoint sets such that*

$$Y_\alpha \subseteq X_\alpha \quad \& \quad \text{card}(Y_\alpha) = \kappa$$

for any  $\alpha < \kappa$ .

**Proof.** Let us fix any bijection  $\phi : \kappa \rightarrow \kappa \times \kappa$ . Of course, we can write  $\phi = (\phi_1, \phi_2)$ , where  $\phi_1 : \kappa \rightarrow \kappa$  and  $\phi_2 : \kappa \rightarrow \kappa$ . We may define by transfinite recursion an injective family  $(x_\alpha)_{\alpha \in \kappa}$  such that  $x_\alpha \in X_{\phi_1(\alpha)}$  for every  $\alpha < \kappa$ . This can be done, since at each step  $\alpha < \kappa$  we have

$$\text{card}(\{x_\xi : \xi < \alpha\}) < \kappa = \text{card}(X_{\phi_1(\alpha)}).$$

Now we put

$$Y_\alpha = \{x_\beta : \phi_1(\beta) = \alpha\}$$

and the proof is finished since

$$\text{card}(\{\beta : \phi_1(\beta) = \alpha\}) = \kappa$$

for every  $\alpha < \kappa$ .

The **exponentiation** of cardinals is defined in the following way:

$$\kappa^\lambda = \text{card}(X^Y),$$

where  $X$  and  $Y$  are any two sets such that  $\text{card}(X) = \kappa$  and  $\text{card}(Y) = \lambda$ .

Exponentiation of cardinals has a lot of properties which are similar to properties of exponentiation of natural numbers. For example,

$$\begin{aligned}(\kappa \cdot \lambda)^\nu &= \kappa^\nu \cdot \lambda^\nu, \\ \kappa^\lambda \cdot \kappa^\nu &= \kappa^{\lambda+\nu}, \\ (\kappa^\lambda)^\nu &= \kappa^{\lambda \cdot \nu}.\end{aligned}$$

Moreover, it is easy to check that  $\kappa^0 = 1$  for any  $\kappa$  (in particular,  $0^0 = 1$ ) and  $1^\kappa = 1$ . If  $\kappa$  is infinite and  $0 < \lambda < \omega$ , then by induction we have  $\kappa^\lambda = \kappa$ . If  $2 \leq \kappa \leq \lambda$  and  $\lambda$  is infinite, then  $\kappa^\lambda = 2^\lambda$ .

**Example 1.1** Let us consider the set  $\mathbb{Q}$  of all rational numbers and the set  $\mathbb{R}$  of all real numbers. These two sets are the main objects in the our entire book. At first we have

$$2^\omega = \text{card}(\mathbb{R}) \leq \text{card}(\mathbb{R}^{\mathbb{N}}) = (2^\omega)^\omega = 2^{\omega \cdot \omega} = 2^\omega,$$

so by the Cantor theorem all these cardinal numbers are equal. The cardinal number  $2^\omega$  is called the **cardinality continuum** and sometimes is also denoted by the symbol  $\mathfrak{c}$ .

Let  $C(\mathbb{R}, \mathbb{R})$  denote the set of all continuous functions from  $\mathbb{R}$  into  $\mathbb{R}$ . If  $f, g \in C(\mathbb{R}, \mathbb{R})$  and  $f|_{\mathbb{Q}} = g|_{\mathbb{Q}}$ , then, obviously,  $f = g$ . Hence, we obtain

$$2^\omega = \text{card}(\mathbb{R}) \leq \text{card}(C(\mathbb{R}, \mathbb{R})) \leq \text{card}(\mathbb{R}^{\mathbb{Q}}) = (2^\omega)^\omega = 2^\omega,$$

so we see that  $\text{card}(C(\mathbb{R}, \mathbb{R})) = 2^\omega$ .

For any infinite set  $X$  we frequently use the following three symbols:

$$\begin{aligned}[X]^\kappa &= \{Y \subseteq X : \text{card}(Y) = \kappa\}, \\ [X]^{<\kappa} &= \{Y \subseteq X : \text{card}(Y) < \kappa\}, \\ [X]^{\leq \kappa} &= \{Y \subseteq X : \text{card}(Y) \leq \kappa\}.\end{aligned}$$

It is easy to check that if  $\omega \leq \lambda \leq \kappa$ , then  $\text{card}([\kappa]^\lambda) = \kappa^\lambda$ . Hence, we see, for example, that  $\text{card}([\mathbb{R}]^\omega) = 2^\omega$ .

Now we define addition and multiplication for an arbitrary family of cardinal numbers. Let  $(\kappa_i)_{i \in I}$  be a family of cardinals. We put

$$\begin{aligned}\sum_{i \in I} \kappa_i &= \text{card}\left(\bigcup_{i \in I} (X_i \times \{i\})\right), \\ \prod_{i \in I} \kappa_i &= \text{card}\left(\prod_{i \in I} X_i\right),\end{aligned}$$

where  $(X_i)_{i \in I}$  is an arbitrary family of sets such that

$$\text{card}(X_i) = \kappa_i \quad (i \in I).$$

The next classical theorem establishes an important relation between addition and multiplication of cardinal numbers.

**Theorem 1.4 (König)** Let  $(\kappa_i)_{i \in I}$  and  $(\lambda_i)_{i \in I}$  be two families of cardinals such that for each  $i \in I$  we have  $\kappa_i < \lambda_i$ . Then the inequality

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$$

holds.

**Proof.** At first let us define a mapping

$$\phi : \bigcup_{i \in I} (\kappa_i \times \{i\}) \rightarrow \prod_{j \in I} \lambda_j$$

by the formula

$$\phi((\alpha, i))(j) = \begin{cases} \alpha & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It is easy to see that  $\phi$  is an injective mapping. Hence, the inequality

$$\sum_{i \in I} \kappa_i \leq \prod_{i \in I} \lambda_i$$

holds. We want to exclude the equality. To get a contradiction suppose that there exists a family  $(X_i)_{i \in I}$  of pairwise disjoint sets such that  $\text{card}(X_i) = \kappa_i$  for every  $i \in I$  and

$$\bigcup_{i \in I} X_i = \prod_{i \in I} \lambda_i.$$

Let us put

$$Y_i = \{f(i) : f \in X_i\}$$

for each  $i \in I$ . Obviously, we have  $\text{card}(Y_i) \leq \text{card}(X_i)$ . Now, take any  $g \in \prod_{i \in I} \lambda_i$  such that

$$g(i) \in \lambda_i \setminus Y_i \quad (i \in I).$$

Then  $g \notin \bigcup_{i \in I} X_i$ , so we obtain a contradiction.

The König theorem proved above is a generalization of the classical **Cantor inequality**  $\kappa < 2^\kappa$ , which, in fact, easily follows from the König inequality (put in the König theorem  $\kappa_i = 1$  and  $\lambda_i = 2$  for all  $i \in \kappa$ ).

Let  $\kappa$  and  $\lambda$  be two infinite cardinal numbers. We say that  $\lambda$  is cofinal with  $\kappa$  if there exists a transfinite sequence  $(\alpha_\zeta)_{\zeta \in \lambda}$  of ordinals such that

$$\alpha_\zeta < \kappa \quad (\zeta < \lambda),$$

$$\sup_{\zeta} \alpha_\zeta = \kappa.$$

We say that a cardinal number  $\lambda$  is the **cofinality** of  $\kappa$  (and write  $\lambda = \text{cf}(\kappa)$ ) if  $\lambda$  is the least cardinal cofinal with  $\kappa$ . Of course, the inequality  $\text{cf}(\kappa) \leq \kappa$  holds.

It is easy to show that  $\lambda = \text{cf}(\kappa)$  if and only if  $\lambda$  is the least cardinal number such that there exists a family of sets  $(X_i)_{i \in \lambda}$  satisfying the following relations:

$$\bigcup_{i \in \lambda} X_i = \kappa,$$

$$(\forall i)(i \in \lambda \rightarrow \text{card}(X_i) < \kappa).$$

**Theorem 1.5** For any infinite cardinal  $\kappa$  we have  $\text{cf}(2^\kappa) > \kappa$ .



**Proof.** Suppose that  $cf(2^\kappa) \leq \kappa$ . Let  $\lambda = cf(2^\kappa)$  and let  $(X_\alpha)_{\alpha \in \lambda}$  be a family of subsets of  $2^\kappa$  such that

$$\bigcup_{\alpha \in \lambda} X_\alpha = 2^\kappa$$

and  $card(X_\alpha) < 2^\kappa$  for each  $\alpha \in \lambda$ . Then, by the König inequality, we have

$$2^\kappa \leq \sum_{\alpha \in \lambda} card(X_\alpha) < \prod_{\alpha \in \lambda} 2^\kappa \leq (2^\kappa)^\kappa = 2^\kappa,$$

so we get a contradiction.

Similar arguments show that  $\lambda < \lambda^{cf(\lambda)}$ , for each infinite cardinal  $\lambda$ . Moreover,  $\lambda < cf(\kappa^\lambda)$  for any  $\kappa \geq 2$ . If we apply Theorem 5 to the cardinal  $\omega$ , then we get  $cf(2^\omega) > \omega$ , so the real line  $\mathbb{R}$  cannot be covered by a countable union of sets of cardinality less than  $2^\omega$ .

Notice that  $cf(\omega) = \omega$  and  $cf(\omega_\omega) = \omega$ , too. On the other hand, from the equality  $\kappa \cdot \kappa = \kappa$ , for infinite cardinals  $\kappa$ , we can easily deduce that  $cf(\kappa^+) = \kappa^+$ . Hence, for example,  $cf(\omega_1) = \omega_1$  and  $cf(\omega_2) = \omega_2$ . Those cardinal numbers  $\kappa$  for which we have  $cf(\kappa) = \kappa$  are called **regular** cardinals, and those for which we have  $cf(\kappa) < \kappa$  are called **singular**.

One of the simplest and the most natural questions which appear during the discussion of infinite cardinal numbers is the question about the size of  $2^\omega$ . How large is this cardinal? Since we know that  $2^\omega > \omega$ , the first possible candidate for  $2^\omega$  is  $\omega_1$ . The statement

$$2^\omega = \omega_1$$

is called the **Continuum Hypothesis** and is usually denoted by **CH**. Gödel showed that theory **(ZFC) & (CH)** is consistent, since it is valid in the Constructible Universe. On the other hand, Cohen showed that theory **(ZFC) & (¬CH)** is consistent, too. Hence, the sentence **CH** is independent of theory **ZFC**.

A stronger form of the Continuum Hypothesis is the **Generalized Continuum Hypothesis**, denoted by **GCH**, which says that

$$(\forall \kappa \geq \omega)(2^\kappa = \kappa^+).$$

Gödel also showed that theory **(ZFC) & (GCH)** is consistent, because it is valid in the Constructible Universe. The Generalized Continuum Hypothesis essentially simplifies cardinal arithmetic. Namely, if **GCH** holds, then for any infinite cardinals  $\kappa$  and  $\lambda$  we have

$$\kappa^\lambda = \begin{cases} \kappa & \text{if } \lambda < cf(\kappa), \\ \kappa^+ & \text{if } cf(\kappa) \leq \lambda < \kappa, \\ \lambda^+ & \text{if } \kappa \leq \lambda. \end{cases}$$

Now we change the subject of our discussion. We pay attention to the *Boolean algebras, ideals and filters*.

A **Boolean algebra** is an algebraic system

$$(B, \wedge, \vee, ', 0, 1),$$

where  $B$  is a basic set,  $\wedge$  and  $\vee$  are binary operations on  $B$ ,  $'$  is a unary operation on  $B$  and  $0, 1$  are two distinguished elements of  $B$ . These operations satisfy the following conditions:

$$1) \quad x \wedge y = y \wedge x; \quad x \vee y = y \vee x;$$

- 2)  $x \wedge (y \wedge z) = (x \wedge y) \wedge z; \quad x \vee (y \vee z) = (x \vee y) \vee z;$
- 3)  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z); \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z);$
- 4)  $x \wedge 1 = x; \quad x \vee 0 = x;$
- 5)  $x \wedge x' = 0; \quad x \vee x' = 1.$

These conditions imply, for example, that

$$x \wedge x = x, \quad x \vee x = x, \quad (x')' = x,$$

$$(x \vee y)' = x' \wedge y', \quad (x \wedge y)' = x' \vee y'$$

and so on.

The simplest example of a Boolean algebra is an algebra of the form

$$(P(E), \cap, \cup, ', \emptyset, E).$$

Here  $E$  is any basic set and  $'$  denotes the complement operation on the family of all subsets of  $E$ , i.e.

$$X' = E \setminus X \quad (X \subseteq E).$$

Such Boolean algebras are called **power set Boolean algebras**.

This example can be generalized in the following way. A non-empty family  $\mathcal{A}$  of subsets of a basic set  $E$  is an **algebra of subsets of  $E$**  if

- 1)  $\{X, Y\} \subseteq \mathcal{A} \rightarrow X \cap Y \in \mathcal{A};$
- 2)  $X \in \mathcal{A} \rightarrow E \setminus X \in \mathcal{A}.$

Note that if  $\mathcal{A}$  is an algebra of subsets of  $E$  and  $X, Y \in \mathcal{A}$ , then  $X \cup Y \in \mathcal{A}$ . In other words, we can say that algebras are *closed under finite unions, finite intersections and complement*.

If  $\mathcal{A}$  is any algebra of subsets of  $E$ , then the structure

$$(\mathcal{A}, \cap, \cup, ', \emptyset, E)$$

is a Boolean algebra. We shall show below that any Boolean algebra is isomorphic with such an algebra of sets.

Let  $\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$  be a Boolean algebra. We define a partial ordering  $\preceq$  on  $B$  by the formula

$$x \preceq y \iff x \wedge y = x.$$

It is easy to see that

$$x \preceq y \iff x \vee y = y.$$

The element 0 is the least element of  $B$  and 1 is the largest element of  $B$  in the just defined order. It is worth remarking here that in power set Boolean algebras the relation  $\preceq$  is the standard inclusion of sets.

An element  $b \in B$  is called an **atom** if  $0 \prec b$  and for every  $c \prec b$  we have  $c = 0$ . If  $P(E)$  is a power set algebra, then elements of the form  $\{x\}$ , where  $x \in E$ , are atoms. The Boolean algebra  $\mathcal{B}$  is **atomless** if it has no atoms at all.

A non-empty set  $F \subseteq B$  is called a **filter** in  $\mathcal{B}$  if

- 1)  $(a \in F \ \& \ a \preceq b) \rightarrow (b \in F);$
- 2)  $(a \in F \ \& \ b \in F) \rightarrow (a \wedge b \in F).$

A dual notion to filter is the notion of an **ideal**. A non-empty set  $I \subseteq B$  is an ideal in  $\mathcal{B}$  if

- 1)  $(a \in I \ \& \ a \succeq b) \rightarrow (b \in I)$ ;
- 2)  $(a \in I \ \& \ b \in I) \rightarrow (a \vee b \in I)$ .

If  $I$  is an ideal in  $\mathcal{B}$ , then the family

$$I' = \{a' : a \in I\}$$

is a filter in  $\mathcal{B}$ , called the **dual filter** to the ideal  $I$ . Conversely, if  $F$  is a filter in  $\mathcal{B}$ , then the family

$$F' = \{a' : a \in F\}$$

is an ideal in  $\mathcal{B}$ , called the **dual ideal** to the filter  $F$ .

A filter  $F \subseteq B$  is called **principal** if there exists  $b \in B$  such that  $F = F_b$ , where

$$F_b = \{x \in B : b \preceq x\}.$$

Similarly, we say that the ideal  $I \subseteq B$  is **principal** if the dual filter  $I'$  is principal.

A filter  $F \subseteq B$  is **proper** if  $F \neq B$  (equivalently, if  $0 \notin F$ ), and  $F$  is an **ultrafilter** if  $F$  is maximal (with respect to inclusion) among proper filters in  $\mathcal{B}$ .

A standard application of the Zorn lemma shows us that any proper filter in a Boolean algebra  $\mathcal{B}$  can be extended to an ultrafilter. Moreover, let  $a$  and  $b$  be two elements of  $B$  such that  $\neg(a \preceq b)$ . Then it is not difficult to prove that there exists an ultrafilter  $F \subseteq B$  satisfying the relations:

$$a \in F, \quad b \notin F.$$

From this fact we immediately obtain that any two distinct elements of a Boolean algebra can be separated by an ultrafilter in this algebra.

In the next theorem we give a simple characterization of ultrafilters.

**Theorem 1.6** *Let  $F$  be a proper filter in a Boolean algebra*

$$\mathcal{B} = (B, \wedge, \vee, ', 0, 1).$$

*Then the following three conditions are equivalent:*

- 1)  $F$  is an ultrafilter;
- 2)  $(\forall a, b \in B)(\text{if } a \vee b \in F \text{ then } a \in F \text{ or } b \in F)$ ;
- 3)  $(\forall a \in B)(a \in F \text{ or } a' \in F)$ .

**Proof.** Implications 2)  $\rightarrow$  3) and 3)  $\rightarrow$  1) are trivial. Let us prove implication 1)  $\rightarrow$  2). Assume that 1) holds and suppose that 2) is false. Hence, there are elements  $a, b \in B$  such that

$$a \vee b \in F, \ a \notin F, \ b \notin F.$$

Let us consider the set

$$S = \{x \in B : (\exists y \in F)(a \wedge y \preceq x)\}.$$

Then  $S$  is a filter in  $\mathcal{B}$  such that  $F \subseteq S$  and  $a \in S$ . Hence,  $S$  is not proper, so  $0 \in S$ . Therefore, for some  $y \in F$  we get  $a \wedge y = 0$ . But then we have

$$b \wedge y = ((a \vee b) \wedge y) \in F,$$

so  $b \in F$ . Hence, we obtain a contradiction, and the theorem is proved.

For any Boolean algebra  $\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$  we define the Stone set for  $\mathcal{B}$  by the formula

$$St(\mathcal{B}) = \{F \subseteq B : F \text{ is an ultrafilter in } \mathcal{B}\}$$

and the Stone mapping

$$st_{\mathcal{B}} : B \rightarrow P(St(\mathcal{B}))$$

by the formula

$$st_{\mathcal{B}}(b) = \{F \in St(\mathcal{B}) : b \in F\}.$$

Finally, let us put

$$C(\mathcal{B}) = \{st_{\mathcal{B}}(b) : b \in \mathcal{B}\}.$$

**Theorem 1.7 (Stone)** *Every Boolean algebra  $\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$  is isomorphic with an algebra of subsets of some set.*

**Proof.** From Theorem 6 we easily deduce that for any two elements  $a \in B$  and  $b \in B$  the following equalities hold :

$$st_{\mathcal{B}}(a') = St(\mathcal{B}) \setminus st_{\mathcal{B}}(a),$$

$$st_{\mathcal{B}}(a \wedge b) = st_{\mathcal{B}}(a) \cap st_{\mathcal{B}}(b),$$

$$st_{\mathcal{B}}(a \vee b) = st_{\mathcal{B}}(a) \cup st_{\mathcal{B}}(b).$$

These equalities imply that  $C(\mathcal{B})$  is an algebra of subsets of the set  $St(\mathcal{B})$  and that the mapping  $st_{\mathcal{B}}$  is an isomorphism between  $\mathcal{B}$  and the Boolean algebra

$$(C(\mathcal{B}), \cap, \cup, ', \emptyset, St(\mathcal{B})).$$

**Example 1.2** Let  $\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$  be any finite Boolean algebra. Since every filter in  $\mathcal{B}$  is closed under finite multiplication, we see that the product of all elements from the filter belongs to this filter, too. So, every filter in a finite Boolean algebra is principal. Let  $F$  be an ultrafilter in  $\mathcal{B}$  and let  $b \in B$  be such that  $F = F_b$ . Then  $b$  must be an atom of the algebra  $\mathcal{B}$ , since otherwise  $F$  would not be maximal. Let

$$At(\mathcal{B}) = \{b \in B : b \text{ is an atom in } \mathcal{B}\}.$$

Then we see that

$$card(St(\mathcal{B})) = card(At(\mathcal{B}))$$

and after identification of  $St(\mathcal{B})$  with  $At(\mathcal{B})$  we have

$$C(\mathcal{B}) = P(At(\mathcal{B})).$$

This gives us the complete description of all finite Boolean algebras. Namely, any finite Boolean algebra is isomorphic with the power set Boolean algebra of some finite set.

**Example 1.3** Let  $F$  be a proper filter in the power set Boolean algebra  $P(\omega)$  and let  $F$  extend the filter dual to the ideal  $[\omega]^{<\omega}$ . Then  $F$  is non-principal. Hence, there are non-principal ultrafilters in this case. In fact, there are non-principal ultrafilters in any infinite Boolean algebra (see exercises after this Chapter).

Let  $\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$  be a Boolean algebra. The binary operation

$$(a, b) \rightarrow (a \wedge b') \vee (a' \wedge b)$$

defined on  $B$  is a generalization of the standard operation of the symmetric difference between two sets and has the same properties as the symmetric difference.

Let  $I$  be an ideal in a Boolean algebra  $\mathcal{B}$ . We define a binary relation  $\sim$  on  $B$  by the formula

$$a \sim b \iff (a \wedge b') \vee (a' \wedge b) \in I.$$

It is easy to see that the relation  $\sim$  is an equivalence relation on the set  $B$ . It is also routine to check that the following definitions of operations on the quotient set  $B/\sim$  are correct :

$$[a]_{\sim} \vee [b]_{\sim} = [a \vee b]_{\sim};$$

$$[a]_{\sim} \wedge [b]_{\sim} = [a \wedge b]_{\sim};$$

$$([a]_{\sim})' = [a']_{\sim}.$$

Note that  $[0]_{\sim} = I$  and  $[1]_{\sim} = I'$ . The structure

$$(B/\sim, \wedge, \vee, ', [0]_{\sim}, [1]_{\sim})$$

is a Boolean algebra, too. It is denoted by  $\mathcal{B}/I$  and is called the **quotient Boolean algebra** (with respect to the given ideal  $I$ ). The canonical surjection  $\phi : \mathcal{B} \rightarrow \mathcal{B}/I$  defined by the formula

$$\phi(b) = [b]_{\sim} \quad (b \in B)$$

is a homomorphism of Boolean algebras and

$$\text{Ker}(\phi) = \{b \in B : \phi(b) = [0]_{\sim}\} = I.$$

**Example 1.4** Let us consider the quotient Boolean algebra

$$C = P(\omega)/([\omega]^{<\omega}).$$

This is an atomless Boolean algebra. It is interesting to note that this algebra has completely different properties than the power set algebra  $P(\omega)$ . For example, any countable decreasing sequence of non-zero elements of  $C$  has a non-zero lower bound in  $C$ . Indeed, suppose that

$$c_0 \succ c_1 \succ \dots \succ c_n \succ \dots$$

are non-zero elements of  $C$ . Let  $c_n = [C_n]$ . We may assume that

$$C_0 \supseteq C_1 \supseteq \dots \supseteq C_n \supseteq \dots$$

since otherwise we may consider the family

$$\{C_0, C_0 \cap C_1, C_0 \cap C_1 \cap C_2, \dots\}.$$

From  $c_n \neq c_{n+1}$  we get that  $C_n \setminus C_{n+1}$  is non-empty (in fact, infinite). Let  $Z \subseteq \omega$  be a set such that

$$\text{card}(Z \cap (C_n \setminus C_{n+1})) = 1$$

for every  $n \in \omega$ . Then  $Z$  is infinite and  $\text{card}(Z \setminus C_n) < \omega$  for any  $n \in \omega$ . This shows us that  $[Z] \succ 0$  and  $[Z] \prec c_n$  for every  $n \in \omega$ .

Let  $E$  be a non-empty basic set and let  $I$  be a proper ideal in the power set Boolean algebra  $P(E)$ . In this case we simply say that  $I$  is an ideal on  $E$ , or that  $I$  is an **ideal of subsets of  $E$** . Suppose that  $[E]^{<\omega} \subseteq I$ .

We define four cardinal numbers connected with the ideal  $I$ . These cardinal numbers describe important properties of  $I$ :

- 1)  $add(I) = \min\{card(\mathcal{D}) : \mathcal{D} \subseteq I \text{ \& } \bigcup \mathcal{D} \notin I\}$ ;
- 2)  $cov(I) = \min\{card(\mathcal{D}) : \mathcal{D} \subseteq I \text{ \& } \bigcup \mathcal{D} = E\}$ ;
- 3)  $non(I) = \min\{card(X) : X \subseteq E \text{ \& } X \notin I\}$ ;
- 4)  $cof(I) = \min\{card(\mathcal{D}) : \mathcal{D} \subseteq I \text{ \& } (\forall X \in I)(\exists Y \in \mathcal{D})(X \subseteq Y)\}$ .

The cardinal numbers defined above are respectively called **additivity** (or **completeness**), **covering** number, **nonuniformity** number and **cofinality** of the ideal  $I$ . It is easy to prove that for any ideal  $I$  on  $E$  such that  $[E]^{<\omega} \subseteq I$  the following diagram holds :

$$\omega \rightarrow add(I) \begin{array}{l} \nearrow cov(I) \\ \searrow non(I) \end{array} \begin{array}{l} \searrow \\ \nearrow \end{array} cof(I) \rightarrow 2^{card(E)},$$

where an arrow  $\kappa \rightarrow \lambda$  denotes the inequality  $\kappa \leq \lambda$  between cardinal numbers  $\kappa$  and  $\lambda$ . It can also easily be shown that

$$\begin{aligned} add(I) &\leq cf(non(I)), \\ add(I) &\leq cf(cof(I)). \end{aligned}$$

An ideal  $I$  is called a  $\sigma$ -**ideal**, or  $\sigma$ -**complete** ideal if  $add(I) > \omega$ .

**Example 1.5** Let  $\kappa$  be an infinite cardinal number and let  $I = [\kappa]^{<\omega}$ . Then we have

$$\begin{aligned} add(I) &= non(I) = \omega, \\ cov(I) &= cof(I) = \kappa. \end{aligned}$$

Let  $J = [\mathbb{R}]^{\leq\omega}$ . Then we have

$$\begin{aligned} add(J) &= non(J) = \omega_1, \\ cov(J) &= cof(J) = 2^\omega. \end{aligned}$$

Let  $\gamma$  be a limit ordinal. We say that a set  $A \subseteq \gamma$  is **unbounded** in  $\gamma$  if for each  $\xi < \gamma$  there is  $\eta \in A$  such that  $\xi < \eta$ . We say that a set  $A \subseteq \gamma$  is **closed** in  $\gamma$  if for each limit ordinal  $\delta < \gamma$  we have

$$sup(A \cap \delta) = \delta \rightarrow \delta \in A.$$

Let  $\kappa$  be an infinite cardinal, for which  $cf(\kappa) > \omega$ . We define

$$CUB_\kappa = \{X \subseteq \kappa : (\exists Y \subseteq \kappa)(Y \text{ is closed and unbounded in } \kappa \text{ \& } Y \subseteq X)\}.$$

It is easy to check that  $CUB_\kappa$  is a filter of subsets of  $\kappa$  and

$$add((CUB_\kappa)') = cf(\kappa).$$

We say that a set  $A \subseteq \kappa$  is a **stationary** subset of  $\kappa$  if

$$(\forall X)(X \in CUB_\kappa \rightarrow A \cap X \neq \emptyset).$$

The filters of closed unbounded subsets of regular cardinal numbers will play an important role in our further considerations.

The following simple proposition is true.

**Theorem 1.8** *Suppose that  $\kappa$  is an infinite regular cardinal number. Let  $f : [\kappa]^{<\omega} \rightarrow [\kappa]^{<\omega}$ . Then the set*

$$\{\xi < \kappa : (\forall a \in [\xi]^{<\omega})(f(a) \subseteq \xi)\}$$

*is a closed and unbounded subset of  $\kappa$ .*

We leave the easy proof of this theorem to the reader.

Let  $E$  be a basic set. A family  $S \subseteq P(E)$  is called  **$\sigma$ -algebra** in  $E$  if  $S$  is an algebra of subsets of  $E$  and for any countable family  $(X_n)_{n \in \omega} \subseteq S$  the union  $\bigcup_{n \in \omega} X_n$  belongs to  $S$ . We say in this case that  $S$  is **closed under countable unions**. It is easy to see that  $\sigma$ -algebra  $S$  is closed under countable intersections, too.

If  $I$  is an ideal of subsets of  $E$ , then  $I \cup I'$  is an algebra of subsets of  $E$ . Moreover, if  $I$  is a  **$\sigma$ -ideal** (i.e. ideal closed under countable unions) then  $I \cup I'$  is a  $\sigma$ -algebra. It is obvious that the intersection of any family of algebras ( $\sigma$ -algebras) in  $E$  is an algebra ( $\sigma$ -algebra) in  $E$ . Hence, if  $\mathcal{A}$  is any family of subsets of  $E$ , then

$$\bigcap \{S \subseteq P(E) : \mathcal{A} \subseteq S \text{ \& } S \text{ is an algebra } (\sigma\text{-algebra})\}$$

is the smallest algebra ( $\sigma$ -algebra) of subsets of  $E$  which contains the family  $\mathcal{A}$ . This algebra ( $\sigma$ -algebra) is called the algebra ( $\sigma$ -algebra) generated by the family  $\mathcal{A}$ . The  $\sigma$ -algebra generated by the family  $\mathcal{A}$  sometimes is denoted by the symbol  $\sigma(\mathcal{A})$ .

**Theorem 1.9** *If  $\mathcal{A} \subseteq P(E)$  and  $\text{card}(\mathcal{A}) > 1$ , then we have*

$$\text{card}(\sigma(\mathcal{A})) \leq (\text{card}(\mathcal{A}))^\omega.$$

**Proof.** For any family  $B \subseteq P(E)$  let us put

$$B_\sigma = \left\{ \bigcup D : D \in [B]^{\leq \omega} \right\},$$

$$B_c = \{E \setminus X : X \in B\} \cup B.$$

Note that  $\text{card}(B_\sigma) \leq (\text{card}(B))^\omega$  and  $\text{card}(B_c) \leq 2 \cdot \text{card}(B)$ . We define by transfinite recursion of length  $\omega_1$  a sequence of subsets of  $P(E)$  in the following way:

- 1)  $S_0 = \mathcal{A}$ ,
- 2)  $S_\alpha = ((\bigcup \{S_\beta : \beta < \alpha\})_\sigma)_c$  if  $1 \leq \alpha < \omega_1$ .

Then it is clear that

$$\beta < \alpha \rightarrow S_\beta \subseteq S_\alpha.$$

Let us put

$$S = \bigcup_{\alpha < \omega_1} S_\alpha.$$

By induction we can prove that

$$\text{card}(S_\alpha) \leq (\text{card}(\mathcal{A}))^\omega \quad (\alpha < \omega_1).$$

Hence, we have

$$\text{card}(S) \leq \omega_1 \cdot (\text{card}(\mathcal{A}))^\omega = (\text{card}(\mathcal{A}))^\omega.$$

It is clear that  $S$  is closed under complements. Suppose now that  $T \subseteq S$  and  $\text{card}(T) \leq \omega$ . Using the regularity of  $\omega_1$  we can find an ordinal  $\alpha < \omega_1$  such that  $T \subseteq S_\alpha$ . Then  $\bigcup T \in S_{\alpha+1}$ , so  $S$  is closed under countable unions. Hence, we see that  $S = \sigma(\mathcal{A})$  and the theorem is proved.

We would like to remark here that in further considerations we shall often deal with various  $\sigma$ -algebras of *measurable* sets (with respect to some *measures*) and various  $\sigma$ -algebras of sets having the *Baire property* (in some *topological spaces*).

In many modern mathematical constructions, especially in measure theory and general topology, a specific kind of partial ordering, called a *tree*, is used. So let us recall the definition of this important notion.

A partial ordering  $(T, \preceq)$  is called a **tree** if  $T$  has the least element and for each  $y \in T$  the set  $\{x \in T : x \preceq y\}$  is well-ordered by  $\preceq$ . The least element of  $T$  is called the **root** of  $T$ . For any ordinal number  $\alpha$  the  $\alpha$ -th level of  $T$  is the set

$$T_\alpha = \{y : \{x \in T : x \prec y\} \text{ has order type } \alpha\}.$$

The height of a tree  $T$  is the least ordinal  $\alpha$  such that the  $\alpha$ -th level of  $T$  is empty.

Let  $A$  be a non-empty set and let  $\alpha$  be an ordinal. The complete  $A$ -ary tree of height  $\alpha$ , which consists of all functions from  $\bigcup_{\beta < \alpha} A^\beta$  and is ordered by inclusion, is denoted by  $A^{<\alpha}$ . If  $A = \{0, 1\}$ , then the complete  $A$ -ary trees are called **binary trees**.

Any linearly ordered subset of a tree  $(T, \preceq)$  is called a **branch** in  $T$ . A subset  $P$  of  $T$  is called a **path** through the tree  $T$  if  $P$  is a branch and contains exactly one element from each non-empty level of  $T$ .

**Example 1.6** Any tree of the form  $A^{<\alpha}$  has a path. Similarly, any tree of a successor height has a path. But there are trees with no paths at all. Let us consider the tree  $(T, \subseteq)$ , where

$$T = \{f \in \omega^{<\omega} : f \text{ is strictly decreasing}\}.$$

Then  $T$  has height  $\omega$  but it has no paths, since there are no infinite strictly decreasing sequences of natural numbers.

**Example 1.7** Let  $(\{0, 1\}^{<\omega}, \subseteq)$  be the complete binary tree of height  $\omega$ . Note that  $\text{card}(\{0, 1\}^{<\omega}) = \omega$ . For every  $f \in \{0, 1\}^\omega$  let us put

$$A_f = \{f|n : n \in \omega\}.$$

Then  $A_f$  is a path through the tree  $\{0, 1\}^{<\omega}$ . If  $f, g \in \{0, 1\}^\omega$  and  $f \neq g$ , then

$$\text{card}(A_f \cap A_g) < \omega.$$

This shows us that in the algebra  $P(\omega)/([\omega]^{<\omega})$  there exists a family of cardinality  $2^\omega$  of non-zero elements such that the product of any two of these elements is zero. Obviously,  $\omega$  cannot be partitioned into more than  $\omega$  pairwise disjoint non-empty sets. So this example gives us the second essential difference between the Boolean algebras  $P(\omega)$  and  $P(\omega)/([\omega]^{<\omega})$ .



Next theorem, although quite simple, is very important and has a lot of applications in many branches of mathematics. It is called the **König Lemma**. Notice that it has stimulated the development of set theory and infinitary combinatorics (some useful facts from infinitary combinatorics are discussed in Appendix A of this book).

**Theorem 1.10 (König)** *Suppose that  $(T, \preceq)$  is a tree of height  $\omega$  such that all levels of  $T$  are finite. Then there exists a path through  $T$ .*

**Proof.** Let  $x_0$  be the root of  $T$ . For each  $n \in \omega \setminus \{0\}$  we can recursively pick an element  $x_n \in T_n$  such that  $x_n \succ x_{n-1}$  and the set

$$\{y \in T : x_n \prec y\}$$

is infinite. This is possible, since every level  $T_n$  of  $T$  is finite. Then  $(x_n)_{n \in \omega}$  is a path through  $T$ .

It is worth remarking that the obvious generalization of the König lemma to the trees of height  $\omega_1$  is false. It is possible to construct a tree  $(T, \preceq)$  of height  $\omega_1$  such that all levels  $T_\alpha$  ( $\alpha < \omega_1$ ) are countable, but there is no path through  $T$  (see exercises to this Chapter). Such a tree is called an  $\omega_1$ -**Aronszajn** tree.

Several kinds of trees are particularly used in constructions of different models of set theory, which show the consistency or independence of various set-theoretic statements.

From the moment when it turned out that the Continuum Hypothesis was independent of **ZFC**, the issue of adding new axioms became really unbalanced. While the Continuum Hypothesis is a very powerful assertion, perhaps even too strong, its negation is rather weak. Hence, there appeared a natural necessity to find an appropriate axiom which even in the absence of the Continuum Hypothesis could give tools efficient enough for mathematical constructions. *Martin's Axiom* turned out a very good candidate to fill up this place.

In fact, Martin's Axiom has many equivalent forms. It can be expressed in the partial order form, the Boolean algebra form, the topological form etc.

The partial order form is favored by set-theorists and seems to be most useful. To formulate it we need some notations and definitions.

Let  $(P, \preceq)$  be a partially ordered set. We say that a set  $D \subseteq P$  is **coinitial** in  $P$  if for each  $p \in P$  there is  $q \in D$  with  $q \preceq p$ . In other words,  $D \subseteq P$  is coinitial in  $P$  if it is cofinal in the partially ordered set  $(P, \succeq)$ .

A non-empty set  $G \subseteq P$  is called a **filter** in  $(P, \preceq)$  if

$$(\forall p \in G)(\forall q \in P)(p \preceq q \rightarrow q \in G),$$

$$(\forall p \in G)(\forall q \in G)(\exists r \in G)(r \preceq p \ \& \ r \preceq q).$$

Notice that this definition resembles the definition of a filter in a Boolean algebra.

Two elements  $p$  and  $q$  of  $P$  are called **inconsistent** if there is no  $r \in P$  such that  $r \preceq p$  and  $r \preceq q$ . We say that a set  $A \subseteq P$  is **totally inconsistent** if any two distinct elements of  $A$  are inconsistent. Finally, we say that  $(P, \preceq)$  satisfies the **countable chain condition** (or simply **c.c.c.**) if each totally inconsistent subset of  $P$  is at most countable.

**Martin's Axiom**, denoted usually by **MA**, is the following statement:

if  $(P, \preceq)$  is a partially ordered set satisfying c.c.c. and  $\mathcal{D}$  is a family of coinitial subsets of  $P$  with  $\text{card}(\mathcal{D}) < 2^\omega$ , then there exists a filter  $G \subseteq P$  which intersects every element of  $\mathcal{D}$ , i.e.

$$(\forall A \in \mathcal{D})(A \cap G \neq \emptyset).$$

The Continuum Hypothesis easily implies Martin's Axiom. Indeed, let us assume **CH**. Let  $(P, \preceq)$  be any partially ordered set and let  $(D_n)_{n \in \omega}$  be any sequence of coinital subsets of  $P$ . Then we can recursively construct a decreasing sequence  $(p_n)_{n \in \omega}$  of elements of  $P$  such that  $p_n \in D_n$  for each  $n \in \omega$ . Now, put

$$G = \{p \in P : (\exists n \in \omega)(p_n \preceq p)\}.$$

Evidently,  $G$  is a filter in  $P$  which intersects every  $D_n$ .

Martin and Solovay proved that the statement **(MA)** &  $(\neg\text{CH})$  is consistent with **ZFC**. The size of  $2^\omega$  is not determined precisely by **MA**. For example, each of the statements

$$(\mathbf{MA}) \& (2^\omega = \omega_2),$$

$$(\mathbf{MA}) \& (2^\omega = \omega_3)$$

is consistent with **ZFC**.

In the formulation of Martin's Axiom the restriction to family  $\mathcal{D}$  of coinital subsets with  $\text{card}(\mathcal{D}) < 2^\omega$  is not accidental. To see this let us consider the complete binary tree  $P = \{0, 1\}^{<\omega}$  ordered by the reverse inclusion. Let  $\mathcal{D}$  be the family consisting of all sets of the form

$$A_n = \{p \in P : n \in \text{dom}(p)\},$$

$$D_f = \{p \in P : \neg(p \subseteq f)\}$$

for  $n < \omega$  and  $f \in 2^\omega$ . It is easy to see that  $\text{card}(\mathcal{D}) = 2^\omega$  and each set from  $\mathcal{D}$  is coinital in  $P$ . Suppose that  $G$  is a filter in  $P$  which intersects every set  $D \in \mathcal{D}$ . Then

$$g = \bigcup G$$

is a function. Since  $G \cap A_n \neq \emptyset$  for each  $n < \omega$ , we see that  $\text{dom}(g) = \omega$ . Thus, we have

$$g : \omega \rightarrow \{0, 1\}.$$

But we also have  $G \cap D_f \neq \emptyset$  for any  $f \in 2^\omega$ . So  $g \neq f$  for every  $f \in 2^\omega$  and this is an absurd.

The restriction to c.c.c. partial ordering is also matured (see exercises after this Chapter).

Here we give only one example of applications of Martin's Axiom. Other applications will be given later. Here we examine the structure of the set of all functions from  $\omega$  into  $\omega$ . For this purpose we define the following relation  $\preceq$  on  $\omega^\omega$ :

$$f \preceq g \iff (\exists n \in \omega)(\forall m \geq n)(f(m) \leq g(m)).$$

If  $f \preceq g$ , then we say that function  $g$  **dominates** function  $f$ . Of course, the relation  $\preceq$  is not a partial ordering on  $\omega^\omega$  (in fact, it is a partial pre-ordering).

If we identify any two functions  $f, g \in \omega^\omega$  for which

$$\text{card}(\{n \in \omega : f(n) \neq g(n)\}) < \omega,$$

then we obtain a partial ordering on the set of equivalence classes. For this partial ordering we shall use the same symbol  $\preceq$ .

Let us define two important cardinal numbers:

$$\mathbf{b} = \min\{\text{card}(F) : F \subseteq \omega^\omega \ \& \ \neg(\exists g \in \omega^\omega)(\forall f \in F)(f \preceq g)\},$$

$$\mathbf{d} = \min\{\text{card}(F) : F \subseteq \omega^\omega \ \& \ (\forall g \in \omega^\omega)(\exists f \in F)(g \preceq f)\}.$$

Obviously, we have  $\mathbf{b} \leq \mathbf{d} \leq 2^\omega$ . It is easy to check that  $\omega < \mathbf{b}$ . Indeed, let  $(f_n)_{n \in \omega}$  be any sequence of elements of  $\omega^\omega$ . Put

$$f(n) = \max\{f_i(n) : i \leq n\} \quad (n \in \omega).$$

Then  $f$  is a function such that  $f_n \preceq f$  for every  $n \in \omega$ . So we see that any countable subset of  $\omega^\omega$  can be dominated by one function. Thus  $\omega < \mathbf{b}$ .

We also see that **CH** implies the equalities

$$\mathbf{b} = \mathbf{d} = 2^\omega.$$

On the other hand, it can be shown that the statement

$$(2^\omega = \omega_2) \ \& \ (\mathbf{b} = \mathbf{d} = \omega_1)$$

is consistent with **ZFC**. We shall discuss an appropriate axiom of set theory for showing this in the second Part of the book. However, Martin's Axiom decides about the value of these two cardinal numbers similarly to the Continuum Hypothesis.

**Theorem 1.11** *If **MA** holds, then  $\mathbf{b} = \mathbf{d} = 2^\omega$ .*

**Proof.** Let  $P = \omega \times \omega^\omega$ . We introduce a partial order  $\preceq$  on  $P$  putting

$$(n, f) \preceq (m, g) \iff (n \geq m) \ \& \ (f|_m = g|_m) \ \& \ (\forall k \geq m)(f(k) \geq g(k)).$$

It is not difficult to check that any uncountable subset of  $P$  has two distinct consistent elements. Hence,  $(P, \preceq)$  satisfies c.c.c., so in this case we may apply Martin's Axiom.

Suppose that  $F \subseteq \omega^\omega$  and  $\text{card}(F) < 2^\omega$ . For every  $f \in F$  and every  $n \in \omega$  define the set

$$D_{f,n} = \{(m, g) : n \leq m \ \& \ (\forall k \geq m)(g(k) \geq f(k))\}.$$

Notice that the set  $D_{f,n}$  is coinital in  $P$ . Using **MA** we see that there exists a filter  $G \subseteq P$  which intersects each set  $D_{f,n}$ . Let us put

$$h = \bigcup \{g|_m : (m, g) \in G\}.$$

Note that  $h \in \omega^\omega$ . Moreover, if  $f \in F$ , then  $f$  is dominated by  $h$ . This shows us that  $F$  is dominated by a single function from  $\omega^\omega$ , so  $\mathbf{b} = 2^\omega$ .

## Exercises

**Exercise 1.1** *Prove that for any ordinal number  $\alpha$  we have*

$$\alpha = \{\beta : \beta < \alpha\}.$$

**Exercise 1.2** *Define by transfinite recursion the exponentiation  $\alpha^\beta$  for arbitrary ordinal numbers  $\alpha$  and  $\beta$ . Prove some algebraic properties of this operation, for example, such as*

$$\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma, \quad \alpha^{\beta \cdot \gamma} = (\alpha^\beta)^\gamma.$$

*Show also that for the exponentiation  $\omega^\omega$  we have*

$$\text{card}(\omega^\omega) = \omega$$

**Exercise 1.3** Prove, in theory **ZF**, that any countable linearly ordered set is isomorphic to some subset of  $(\mathbb{Q}, \leq)$ , where  $\mathbb{Q}$  is the set of all rational numbers linearly ordered by the standard order. This is the classical Cantor theorem.

**Exercise 1.4** Let  $(P, \preceq)$  be a partially ordered set. We say that  $(P, \preceq)$  is **complete** if for any set  $X \subseteq P$  there exists  $\sup(X)$  in  $P$ . Prove that  $(P, \preceq)$  is complete if and only if for any set  $X \subseteq P$  there exists  $\inf(X)$  in  $P$ . Suppose that  $(P, \preceq)$  is complete and let  $f : P \rightarrow P$  be an increasing mapping, i.e.

$$x \preceq y \rightarrow f(x) \preceq f(y) \quad (x, y \in P).$$

Prove that there exists an invariant point for  $f$ , i.e.

$$(\exists x \in P)(f(x) = x).$$

This result is due to Tarski.

**Exercise 1.5** Let  $X, Y$  be any two sets and let

$$f : X \rightarrow Y, \quad g : Y \rightarrow X$$

be arbitrary mappings. Show, in theory **ZF**, that there exist sets  $X_1, X_2, Y_1$  and  $Y_2$  such that

$$\begin{aligned} X &= X_1 \cup X_2, & X_1 \cap X_2 &= \emptyset, & Y &= Y_1 \cup Y_2, & Y_1 \cap Y_2 &= \emptyset, \\ f(X_1) &= Y_1, & g(Y_2) &= X_2. \end{aligned}$$

The above result generalizes the Banach theorem presented in this Chapter.

**Exercise 1.6** Prove, in theory **ZF**, the equivalence between the Axiom of Choice and the Zorn Lemma.

**Exercise 1.7** Let  $(P, \preceq)$  be any partial ordering. Prove that there exists a linear ordering on  $P$ , extending the order  $\preceq$ . This result is due to Marczewski.

**Exercise 1.8** Let  $((P_i, \preceq_i))_{i \in I}$  be a family of partially ordered sets. The product of these sets is the partially ordered set  $(P, \preceq)$ , where  $P = \prod_{i \in I} P_i$  and  $\preceq$  is defined by the formula

$$x \preceq y \longleftrightarrow (\forall i \in I)(x_i \preceq_i y_i) \quad (x, y \in P).$$

Prove that any partially ordered set is isomorphic to a subset of some product of linearly ordered sets.

**Exercise 1.9** Let  $(X, \preceq)$  be an infinite set of real numbers linearly ordered by the standard order. Prove, in theory **ZF**, the existence of a partition  $\{X_0, X_1\}$  of  $X$ , such that the sets  $X_0$  and  $X_1$  are also infinite. Does the analogous result hold in **ZF** for any infinite linearly ordered set  $(P, \preceq)$ ?

**Exercise 1.10** Let  $\kappa$  and  $\lambda$  be arbitrary infinite cardinal numbers. Show that the following **Hausdorff formula** is true:

$$(\kappa^+)^{\lambda} = \kappa^+ \cdot \kappa^{\lambda}.$$

**Exercise 1.11** Let  $(B, \wedge, \vee, ', 0, 1)$  be a Boolean algebra. Let us consider the binary operation

$$a + b = (a \wedge b') \vee (a' \wedge b)$$

defined on  $B$ . Show that an algebraic system  $(B, +, \wedge, 0, 1)$  is an unitary ring in the usual algebraic sense. In this ring  $+$  is addition and  $\wedge$  is multiplication. For any  $a \in B$  we have

$$a + a = 0, \quad a \wedge a = a.$$

**Exercise 1.12** Show that any proper filter in a Boolean algebra can be extended to an ultrafilter.

**Exercise 1.13** Show that in an infinite Boolean algebra there exists a non-principal ultrafilter.

**Exercise 1.14** Prove that if  $X$  is an infinite set, then  $\text{card}(\text{St}(P(X))) = 2^{2^{\text{card}(X)}}$ .

**Exercise 1.15** Let  $\mathcal{A}$  be an arbitrary non-empty family of subsets of a basic set  $E$  and let  $\mathcal{B}$  be an algebra of subsets of  $E$ , generated by  $\mathcal{A}$ . Prove that the following inequality holds :

$$\text{card}(\mathcal{B}) \leq \omega \cdot \text{card}(\mathcal{A}).$$

**Exercise 1.16** Let  $I$  be an ideal on an infinite set  $X$  such that  $[X]^{<\omega} \subseteq I$ . Prove all inequalities contained in the diagram presented in this Chapter and describing relations between the cardinal coefficients

$$\text{add}(I), \text{cov}(I), \text{non}(I), \text{cof}(I)$$

. Show also that

$$\text{add}(I) \leq \text{cf}(\text{non}(I)), \quad \text{add}(I) \leq \text{cf}(\text{cof}(I)).$$

Moreover, show that no other inequality between these coefficients can be proved in general.

**Exercise 1.17** Prove that  $\text{add}((\text{CUB}_\kappa)') = \text{cf}(\kappa)$  for any infinite cardinal number  $\kappa$  with  $\text{cf}(\kappa) > \omega$ .

**Exercise 1.18** Prove Theorem 8 from this Chapter.

**Exercise 1.19** The **Countable form of the Axiom of Choice** is the following statement:

if  $(X_n)_{n \in \omega}$  is an arbitrary countable family of non-empty pairwise disjoint sets, then there exists a family  $(x_n)_{n \in \omega} \in \prod_{n \in \omega} X_n$ .

The **Axiom of Dependent Choices** is the following statement :

if  $S(x, y)$  is a binary relation on a non-empty set  $X$  such that

$$(\forall x \in X)(\exists y \in X)S(x, y),$$

then there exists a sequence  $(x_n)_{n \in \omega} \subseteq X$  such that

$$(\forall n \in \omega)S(x_n, x_{n+1}).$$

The Axiom of Dependent Choices is usually denoted by **DC**. It can be said that the Axiom **DC** is a safe form of **AC** which is completely sufficient for most branches of classical mathematics: geometry of the Euclidean spaces, mathematical analysis, real function theory, etc. On the other hand, it is worth noticing here that in a lot of domains of contemporary mathematics several uncountable forms of **AC** are frequently used as a necessary tool in proving interesting and strong results.

Show, in theory **ZF**, that

- 1) **DC** implies the countable form of **AC**;
- 2) the countable form of **AC** is enough to prove the equivalence of Cauchy and Heine definitions of continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ;

- 3) the countable form of **AC** implies that the union of a countable family of countable sets is also countable.
- 4) the countable form of **AC** implies that any infinite set contains a countable subset.

**Exercise 1.20** Prove, in theory **ZF**, the equivalence of the following two propositions:

- 1) the König theorem on  $\omega$ -trees ;
- 2) for any countable family  $(X_n)_{n \in \omega}$  of non-empty finite pairwise disjoint sets there exists a family  $(x_n)_{n \in \omega} \in \prod_{n \in \omega} X_n$ .

**Exercise 1.21** Prove the equivalence (in **ZFC**) of the following two propositions:

- 1) the Continuum Hypothesis;
- 2) for any partially ordered set  $(P, \preceq)$  and for any family  $\mathcal{D}$  of coinital subsets of  $P$  with  $\text{card}(\mathcal{D}) < 2^\omega$  there exists a filter  $G \subseteq P$  which intersects every element of  $\mathcal{D}$ .

This result shows us that in the formulation of Martin's Axiom c.c.c. is essential if we want to have an axiom strongly weaker than **CH**.

**Exercise 1.22** Using the method of the transfinite recursion construct an  $\omega_1$ -Aronszajn tree as a subtree of  $(\omega^{<\omega_1}, \subseteq)$ .

**Exercise 1.23** Let  $\{A, B\}$  be a partition of the real line  $\mathbb{R}$  such that there exists a bijection  $f : A \rightarrow B$ . Prove, in theory **ZF**, that there exists a bijection  $g : A \rightarrow \mathbb{R}$ .

**Exercise 1.24** Prove, in theory **ZF**, that there exists a partition  $\{X_\alpha : \alpha < \omega_1\}$  of the real line  $\mathbb{R}$ . Deduce from this fact (in **ZF**) that the set of all subsets of the real line cannot be represented as the union of a countable family of countable sets.

Notice in connection with this result that there are some models of theory **ZF** in which the real line  $\mathbb{R}$  can be represented as the union of a countable family of countable sets. On the other hand, show that  $\mathbb{R}$  cannot be represented as the union of a countable family of finite sets.

**Exercise 1.25** Prove, in theory **ZF**, that for every linearly ordered set  $X$  there exists a linearly ordered set  $Y$  such that  $\text{card}(X) < \text{card}(Y)$ . Moreover, if  $X$  is a well ordered set, then  $Y$  can also be taken as a well ordered set.

In particular, there exists (in **ZF**) a linearly ordered set  $E$  such that  $\text{card}(E) > \text{card}(\mathbb{R})$ . On the other hand, the existence of a well ordered set of the cardinality continuum cannot be proved in **ZF**



## Chapter 2

# Elements of General Topology

Let  $E$  be a basic set (we assume in general that  $E$  is infinite) and let  $T$  be a topological structure on  $E$ , i.e. a topology on  $E$ . As a rule, the pair  $(E, T)$  is called a topological space. If the topology  $T = T(E)$  is clear in the context, then we shall simply say that  $E$  is a topological space. In this Chapter we consider some facts from general topology which we need in the sequel. We assume, of course, that the reader knows some elementary facts and notions from this area of mathematics, for instance such notions as: *continuous mapping*, a *nowhere dense set*, *separation axioms* and the corresponding to them classes of topological spaces (*Hausdorff spaces*, *regular spaces*, *completely regular spaces*, *normal spaces* etc.), *quasi-compactness*, *metrizability* of topological space, *completeness* for *metric spaces* and others.

For our further purposes it is quite enough to consider only such topological spaces, in which any singleton is a closed set. We shall assume (as a rule) that every topological space under our considerations satisfies this condition.

The notion of quasi-compactness is one of the most important of the topological notions listed above. Let us recall that the topological space  $E$  is **quasi-compact** if every open covering of  $E$  contains a finite subcovering of  $E$ . The space  $E$  is called **compact** if it is Hausdorff and quasi-compact at the same time. There are a lot of remarkable theorems connected with the notion of quasi-compactness. The main one is, of course, the classical Tychonoff theorem.

**Theorem 2.1 (Tychonoff)** *The topological product of an arbitrary family of quasicompact spaces is a quasi-compact space. Conversely, if the topological product of the family of non-empty spaces is quasi-compact, then each of these spaces is quasi-compact.*

**Proof.** The simplest proof of this theorem can be done with the use of ultrafilters. Let us recall that a topological space is quasi-compact if and only if every ultrafilter in this space is convergent. Let  $(E_i)_{i \in I}$  be any family of non-empty quasi-compact topological spaces and let  $E = \prod_{i \in I} E_i$  be the topological product of these spaces. Let us consider any ultrafilter  $\Phi$  in the space  $E$ . For each  $i \in I$  denote by

$$pr_i : E \rightarrow E_i$$

the canonical projection which corresponds to the index  $i$ . Since  $pr_i$  is a surjection the family of sets  $pr_i(\Phi)$  is an ultrafilter in the space  $E_i$ . Hence, from the quasi-compactness of  $E_i$ , the ultrafilter  $pr_i(\Phi)$  converges to some point  $e_i \in E_i$ . It is



easy to prove, directly starting from the definition of the product topology, that the ultrafilter  $\Phi$  converges to the point  $(e_i)_{i \in I}$ .

The proof of the second part of the theorem follows immediately from quasi-compactness of continuous images of quasi-compact spaces.

The Tychonoff theorem has a lot of applications in many branches of mathematics (some of such applications are given in exercises to this Chapter).

Another important theorem for all mathematics also connected with the notion of quasi-compactness is the classical Baire theorem. This theorem is usually formulated in the following way.

**Theorem 2.2 (Baire)** *Let  $E$  be a non-empty topological space which satisfies one of the following two conditions:*

- 1)  *$E$  is locally compact, i.e.  $E$  is Hausdorff and every point of  $E$  has a compact neighbourhood;*
- 2)  *$E$  is metrizable by a complete metric.*

*Then the complement of the union of any countable family of nowhere dense subsets of  $E$  is dense in  $E$ . In particular,  $E$  is not the union of such a family of subsets (in other words,  $E$  is not a first category space).*

**Proof.** The proof in each of cases 1) and 2) is standard and classical. Let  $(F_n)_{n \in \omega}$  be any countable family of nowhere dense subsets of  $E$  and let  $V \subseteq E$  be any non-empty open set. By recursion we construct a sequence

$$V_0 \supseteq cl(V_1) \supseteq \dots \supseteq cl(V_n) \supseteq \dots,$$

where, as usual,  $cl(V_n)$  denotes the closure of the set  $V_n$  and

- a)  $V_0 = V$ ,
- b)  $V_n$  is non-empty and open in  $E$ ,
- c)  $F_n \cap cl(V_m) = \emptyset$  if  $m > n$ ,
- d) if condition 2) holds for  $E$ , then  $diam(V_n) < \frac{1}{n}$  for  $n \geq 1$ .

Let  $e$  be any point of the non-empty intersection of the family  $(cl(V_n))_{n \in \omega}$ . Then it is clear that  $e \in V \setminus \bigcup_{n \in \omega} F_n$ . Hence, the Baire theorem is proved.

Now we introduce an important notion. A topological space is a **Baire topological space** if each of its non-empty open subspaces is not a first category space. From the proof of the Baire theorem it follows that any locally compact topological space and any complete metric space are the Baire spaces.

It is worth remarking here that in every topological space there exists a biggest (with respect to the inclusion) open first category subspace. In order to show this we need to prove the famous Banach theorem about open first category sets.

**Theorem 2.3 (Banach)** *Let  $E$  be any topological space and let  $(V_i)_{i \in I}$  be any family of open first category subsets of  $E$ . Then the union  $V = \bigcup_{i \in I} V_i$  is an open first category set, too.*

**Proof.** Let  $(P_j)_{j \in J}$  be a maximal (with respect to inclusion) disjoint family of non-empty open subsets of  $E$  such that

$$(\forall j \in J)(\exists i \in I)(P_j \subseteq V_i).$$

The existence of such a family follows from Zorn's lemma. Now, from the maximality of  $(P_j)_{j \in J}$  we deduce that the closed set  $cl(V) \setminus \bigcup_{j \in J} P_j$  is nowhere dense in  $E$ . Hence, it is sufficient to show that the union  $\bigcup_{j \in J} P_j$  is a first category set. Indeed, we have

$$P_j = \bigcup_{n \in \omega} X_{j,n},$$

where  $X_{j,n}$  are nowhere dense subsets of  $E$ . For each  $n \in \omega$  we put

$$X_n = \bigcup_{j \in J} X_{j,n}.$$

Using the disjointness of the family  $(P_j)_{j \in J}$  we can deduce that for any  $n \in \omega$  the set  $X_n$  is also nowhere dense in  $E$ . But

$$\bigcup_{j \in J} P_j = \bigcup_{n \in \omega} X_n,$$

hence, the Baire theorem is proved.

From this theorem we immediately obtain the following result.

**Theorem 2.4** *Any topological space  $E$  can be represented as the union*

$$E = E_1 \cup E_2,$$

where the set  $E_1$  is an open first category subspace of  $E$ , the set  $E_2$  is a closed Baire subspace of  $E$  and  $E_1 \cap E_2 = \emptyset$ . Similarly, any topological space  $E$  can be represented as the union

$$E = E_1^* \cup E_2^*,$$

where the set  $E_1^*$  is a closed first category subspace of  $E$ , the set  $E_2^*$  is an open Baire subspace of  $E$  and  $E_1^* \cap E_2^* = \emptyset$ .

In order to avoid misunderstandings let us notice that the closed set  $E_1^*$  may not be nowhere dense in  $E$ .

The result formulated in Theorem 4 gives us in many cases the possibility to restrict our considerations only to the class of Baire topological spaces.

Now we shall introduce a definition which plays one of the main roles in this book.

Let  $E$  be any topological space and let  $X$  be a subset of  $E$ . We say that the set  $X$  has the **Baire property** (in  $E$ ) if  $X$  can be represented in the form

$$X = (V \cup Y) \setminus Z,$$

where  $V$  is an open subset of  $E$ , and  $Y$  and  $Z$  are first category subsets of  $E$ . It is easy to see that we obtain an equivalent definition if we change the above representation by the following one:

$$X = (V \setminus Y) \cup Z,$$

with the same  $V$ ,  $Y$  and  $Z$ . It is also easy to see that a set  $X \subseteq E$  has the Baire property if and only if there exists an open set  $V \subseteq E$  and a first category set  $P \subseteq E$  such that

$$X = V \Delta P.$$

The class of first category subsets of  $E$  will be denoted by the symbol  $K(E)$  and the class of all sets which have the Baire property in  $E$  will be denoted by the symbol  $\bar{B}(E)$ . We shall also call members of the class  $K(E)$  **meager** subsets of  $E$  and their complements to  $E$  **comeager** (or **residual**) subsets of  $E$ . It is easy to prove the following theorem.

**Theorem 2.5** *The class  $\bar{B}(E)$  is a  $\sigma$ -algebra of subsets of  $E$ . This  $\sigma$ -algebra is generated by the union  $T(E) \cup K(E)$ , where  $T(E)$  is the topology of the space  $E$ .*

Let us recall that the Borel  $\sigma$ -algebra of a topological space  $E$  is the  $\sigma$ -algebra generated by the topology  $T(E)$ . The Borel  $\sigma$ -algebra of a topological space  $E$  will be denoted by the symbol  $B(E)$ . It is clear that the inclusion

$$B(E) \subseteq \bar{B}(E)$$

always holds. Normally, the above inclusion is proper. But there are very interesting examples of such uncountable topological spaces  $E$  for which the equality  $B(E) = \bar{B}(E)$  holds. It is worth noticing that this situation may occur with some uncountable subsets  $E$  of the real line  $\mathbb{R}$  (here, of course,  $E$  is equipped with the induced topology from  $\mathbb{R}$ ). We shall discuss such a situation later on, in Chapter 8 of the book.

Another nontrivial question: does there exist a non-first category topological space  $E$  dense in itself such that  $\bar{B}(E) = P(E)$ , where  $P(E)$  is the family of all subsets of  $E$ ? It turns out that such a situation is possible, too.

The next result shows the transitivity of the Baire property.

**Theorem 2.6** *Let  $E_2$  be a subspace of a topological space  $E_1$  and let  $E_3$  be a subspace of a topological space  $E_2$ . If the set  $E_2$  has the Baire property in the space  $E_1$  and the set  $E_3$  has the Baire property in the space  $E_2$ , then the set  $E_3$  has the Baire property in the space  $E_1$ .*

The proof of this theorem is not difficult and we leave it to the reader as a useful exercise.

Let  $E$  be any topological space and  $X$  be a subset of this space. We say that  $X$  has the **Baire property in the restricted sense** if for each subspace  $E_1$  of  $E$  the set  $X \cap E_1$  has the Baire property in the space  $E_1$ . It is clear that the class of all subsets of  $E$  with the restricted Baire property is a  $\sigma$ -algebra in  $E$  contained in the  $\sigma$ -algebra  $\bar{B}(E)$ .

The following fact holds.

**Theorem 2.7** *For every topological space  $E$  and every subset  $X$  of  $E$  the subsequent three conditions are equivalent:*

- 1) *the set  $X$  has the Baire property in the restricted sense;*
- 2) *for each closed set  $F \subseteq E$  the intersection  $X \cap F$  has the Baire property in  $F$ ;*
- 3) *for each perfect set  $P \subseteq E$  the intersection  $X \cap P$  has the Baire property in  $P$ .*

We leave the proof of this theorem to the reader, too.

**Example 2.1** Let  $\mathbb{R}$  be the real line and let  $F$  be any non-empty perfect nowhere dense subset of  $\mathbb{R}$  (for instance,  $F$  can be the classical Cantor discontinuum in  $\mathbb{R}$ ). Later we will see that there exists a set  $X \subseteq F$  which does not have the Baire property in  $F$ . On the other hand, the same set  $X$  is nowhere dense in  $\mathbb{R}$  and hence, it has the Baire property in  $\mathbb{R}$ . So, we see that the set  $X$  has the Baire property in  $\mathbb{R}$  and at the same time does not have the Baire property in the restricted sense.

**Example 2.2** Let  $E$  be an arbitrary topological space and let  $X$  be an arbitrary Borel subset of  $E$ . It is easy to check that for each subspace  $E_1$  of  $E$  the following equality holds:

$$B(E_1) = \{Y \cap E_1 : Y \in B(E)\}.$$

Hence, the intersection  $X \cap E_1$  is a Borel subset of the space  $E_1$ . Therefore, any Borel subset of the space  $E$  has the Baire property in the restricted sense.

Let  $E_1$  and  $E_2$  be two topological spaces and let  $f$  be a mapping from  $E_1$  into  $E_2$ . Let us recall that the mapping  $f$  is a **Borel mapping** if

$$(\forall Z)(Z \in B(E_2) \rightarrow f^{-1}(Z) \in B(E_1)).$$

It is clear that a sufficient condition for  $f$  to be a Borel mapping is the following:

$$(\forall Z)(Z \in T(E_2) \rightarrow f^{-1}(Z) \in B(E_1)).$$

We say that a mapping

$$f : E_1 \rightarrow E_2$$

has the **Baire property** if for each open set  $V \subseteq E_2$  the set  $f^{-1}(V)$  has the Baire property in the space  $E_1$ . It is easy to show that the following proposition is true.

**Theorem 2.8** *If  $E_1$  and  $E_2$  are topological spaces and if  $f$  is a mapping from  $E_1$  into  $E_2$  then the next three sentences are equivalent:*

- 1) *the mapping  $f$  has the Baire property;*
- 2) *for any closed set  $Z \subseteq E_2$  the set  $f^{-1}(Z)$  has the Baire property in the space  $E_1$ ;*
- 3) *for any Borel set  $Z \subseteq E_2$  the set  $f^{-1}(Z)$  has the Baire property in the space  $E_1$ .*

In particular, any Borel (hence, any continuous) mapping  $f : E_1 \rightarrow E_2$  has the Baire property.

There exists a certain analogy between mappings with the Baire property and mappings measurable with respect to various measures (for instance, with respect to the classical Lebesgue measure). In particular, it is well known that all Lebesgue measurable real functions have the so called Luzin C-property (see, for example, Chapter 4 of this book). An analogous result for mappings with the Baire property is the following proposition.

**Theorem 2.9** *Let  $E_1$  and  $E_2$  be two topological spaces such that  $E_2$  satisfies the second countability axiom (i.e. there exists a countable base for  $T(E_2)$ ). Let  $f$  be a mapping from  $E_1$  into  $E_2$ . Then the next two sentences are equivalent :*

- 1) *the mapping  $f$  has the Baire property;*
- 2) *there exists a first category set  $Z \subseteq E_1$  such that the restriction of the mapping  $f$  to the set  $E_1 \setminus Z$  is continuous.*

**Proof.** Let  $f : E_1 \rightarrow E_2$  be a mapping which has the Baire property and let  $(V_i)_{i \in I}$  be any countable base for the space  $E_2$ . Every set  $f^{-1}(V_i)$  has the Baire property in the space  $E_1$ , so we can write

$$f^{-1}(V_i) = (U_i \setminus X_i) \cup Y_i,$$

where  $U_i$  is an open set in the space  $E_1$  and  $X_i$  and  $Y_i$  are first category subsets of  $E_1$ . Let us put

$$Z = \bigcup_{i \in I} (X_i \cup Y_i).$$

It is clear that  $Z$  is a first category subset of  $E_1$ . Let  $A = E_1 \setminus Z$ . We show that the mapping  $g = f|_A$  is continuous. Let us take any open subset  $V$  of  $E_2$ . There exists  $J \subseteq I$  such that

$$V = \bigcup_{j \in J} V_j.$$

Using the definition of the set  $A$  we get the following equalities:

$$g^{-1}(V) = \bigcup_{j \in J} (f^{-1}(V_j) \cap A) = \left( \bigcup_{j \in J} U_j \right) \cap A,$$

which give us the continuity of the mapping  $g$ .

Conversely, let  $f$  be a mapping from  $E_1$  into  $E_2$  such that there exists a first category set  $Z \subseteq E_1$  for which a mapping  $g = f|(E_1 \setminus Z)$  is continuous. We put, as above,  $A = E_1 \setminus Z$ . Let  $V$  be any open subset of the space  $E_2$ . Then the set  $g^{-1}(V)$  is open in  $A$ , i.e. there exists an open set  $U \subseteq E_1$  such that

$$g^{-1}(V) = U \cap A = U \setminus (U \cap Z).$$

From this we immediately obtain

$$f^{-1}(V) = g^{-1}(V) \cup (f^{-1}(V) \cap Z) = (U \setminus (U \cap Z)) \cup (f^{-1}(V) \cap Z),$$

hence, the mapping  $f$  has the Baire property.

Let us remark that in the second part of the proof we did not use the assumption that the topological space  $E_2$  satisfies the second countability axiom. But in the first part of the proof this assumption is essential. This can be shown by the next example.

**Example 2.3** Let  $\mathbb{R}$  be the set of all real numbers equipped with the standard Euclidean topology. Let us denote by  $T$  the class of all sets  $Y \subseteq \mathbb{R}$  which can be represented in the form

$$Y = V \setminus D,$$

where  $V$  is an open subset of  $\mathbb{R}$  and  $D$  is at most countable subset of  $\mathbb{R}$ . It can be checked that the class  $T$  is a topology on  $\mathbb{R}$  which properly extends the Euclidean topology on  $\mathbb{R}$ . The set  $\mathbb{R}$  with the introduced topology  $T$  is denoted by the symbol  $\mathbb{R}^*$ . It is evident that the space  $\mathbb{R}^*$  is not separable (i.e. does not contain a countable dense subset). Hence, this space does not satisfy the second countability axiom. It is also easy to check that

$$B(\mathbb{R}) = B(\mathbb{R}^*),$$

i.e. that the  $\sigma$ -algebras of Borel subsets of these two topological spaces coincide. Let  $f$  be a mapping from the space  $\mathbb{R}$  into the space  $\mathbb{R}^*$  defined by the formula

$$f(x) = x \quad (x \in \mathbb{R}).$$

It is clear that  $f$  is a Borel mapping from  $\mathbb{R}$  into  $\mathbb{R}^*$  (moreover,  $f$  is a Borel isomorphism between these two spaces). Hence, we see that the mapping  $f$  has the Baire property. Finally, we leave to the reader the proof that there is no first category subset  $Z$  of  $\mathbb{R}$  such that  $f|(\mathbb{R} \setminus Z)$  is continuous.

We can introduce, in a natural way, the notion of the mapping with the *Baire property in the restricted sense*. Let  $E_1$  and  $E_2$  be two topological spaces and let  $f$  be a mapping from  $E_1$  into  $E_2$ . We say that the mapping  $f$  has the **Baire property in the restricted sense** if, for each open subset  $V$  of the space  $E_2$ , the set  $f^{-1}(V)$  has the Baire property in the restricted sense in the space  $E_1$ . The class of all mappings with the Baire property and the class of all mappings with the Baire property in the restricted sense are quite similar. We suggest that the reader should think for a while about possible analogies between these classes. Of course, there are also some essential differences between them.

We know that any Borel mapping has the Baire property in the restricted sense. Next example shows us that there exist mappings with the restricted Baire property which are very far from the Borel mappings.

**Example 2.4** Looking slightly forward and using in this example the notion of the classical Lebesgue measure on  $\mathbb{R}$  we will show (with the help of Martin's Axiom) that there exists a function

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

which is non-measurable in the Lebesgue sense and, at the same time, has the Baire property in the restricted sense. Let us assume Martin's Axiom. Then by means of the method of transfinite recursion one can construct a set  $X \subseteq \mathbb{R}$  which satisfies the following two conditions :

- 1) for any non-empty perfect subset  $F$  of  $\mathbb{R}$  the intersection  $X \cap F$  is a first category set in the topological space  $F$ ;
- 2) for any closed subset  $F$  of  $\mathbb{R}$  with strictly positive Lebesgue measure the intersection  $X \cap F$  is non-empty.

A detailed construction of such set  $X$  will be considered in Chapter 8 of this book. We assume here that such  $X$  really exists in  $\mathbb{R}$ . From condition 1) it follows that the set  $X$  has the Baire property in the restricted sense. From condition 2) we obtain that  $X$  has full outer Lebesgue measure on the real line  $\mathbb{R}$ . So both conditions 1) and 2) give us that  $X$  is not a Lebesgue measurable subset of  $\mathbb{R}$ . Let  $f$  be a characteristic function (indicator) of the set  $X$ . Then the function  $f$  has the Baire property in the restricted sense but at the same time is not Lebesgue measurable. So, of course,  $f$  is not a Borel mapping from  $\mathbb{R}$  into  $\mathbb{R}$ .

It is impossible to omit the famous Kuratowski-Ulam theorem if we discuss some analogies between the Baire property and measurability. This theorem is a direct topological analogue of the well known Fubini theorem from abstract measure theory.

**Theorem 2.10 (Kuratowski-Ulam)** *Let  $E_1$  and  $E_2$  be two topological spaces and let  $E_2$  have a countable base. Let  $E_1 \times E_2$  be a topological product of these spaces and let  $Z \subseteq E_1 \times E_2$ . The following sentences hold:*

- 1) *if  $Z$  is a first category set in the space  $E_1 \times E_2$  then for almost every (in the category sense)  $x \in E_1$  the set*

$$Z(x) = \{y \in E_2 : (x, y) \in Z\}$$

*is a first category subset of the space  $E_2$ ;*

- 2) *if  $Z$  has the Baire property in the space  $E_1 \times E_2$ , then for almost every  $x \in E_1$  the set  $Z(x)$  has the Baire property in the space  $E_2$ ;*
- 3) *if  $Z$  has the Baire property in the space  $E_1 \times E_2$  and if for almost every  $x \in E_1$  the set  $Z(x)$  is a first category subset of the space  $E_2$ , then the set  $Z$  is a first category subset of the space  $E_1 \times E_2$ .*

**Proof.** All sentences 1), 2) and 3) can be proved by one schema. We shall show here only the proof of 1) and the remaining parts we leave to the reader (only once the reader must apply the Banach theorem on a first category open sets). Notice now that it is sufficient to prove 1) only for closed nowhere dense subsets  $Z$  of the space  $E_1 \times E_2$ . So, let  $F$  be a closed nowhere dense set in the space  $E_1 \times E_2$ . We shall show that for almost every  $x \in E_1$  the set  $F(x)$  is closed and nowhere dense in  $E_2$ . Let us put

$$G = (E_1 \times E_2) \setminus F.$$

We see that  $G$  is an open dense subset of the space  $E_1 \times E_2$ . Let  $(V_i)_{i \in I}$  be any countable base for the space  $E_2$ . For each index  $i \in I$  let us denote the set

$$\{x : (\exists y)(y \in V_i \ \& \ (x, y) \in G)\}$$

by the symbol  $G_i$ . It is easy to check that the equality

$$G_i = pr_1(G \cap (E_1 \times V_i))$$

holds. Hence, we conclude that the set  $G_i$  is open and dense in the space  $E_1$ . Thus, the intersection of the family  $(G_i)_{i \in I}$  is the complement of a first category subset of the space  $E_1$ . It is clear from the above that for each element  $x \in \bigcap_{i \in I} G_i$  the set

$$F(x) = E_2 \setminus G(x)$$

is closed and nowhere dense in the space  $E_2$  (because the set  $G(x)$  is open and dense in the same space). Hence, we have proved sentence 1).

Let us remark that in the Kuratowski-Ulam theorem the assumption that the space  $E_2$  has a countable base can be weakened (see exercises after this Chapter).

Now we want to introduce some simple cardinal-valued functions describing various properties of topological spaces. Let  $E$  be any topological space. We put

$$\begin{aligned} w(E) &= \inf \{ \text{card}(B) : B \text{ is a base of } T(E) \} + \omega; \\ c(E) &= \sup \{ \text{card}(B) : B \text{ is a family of pairwise} \\ &\quad \text{disjoint non-empty open sets in } E \} + \omega; \\ d(E) &= \inf \{ \text{card}(X) : X \text{ is a dense subset of } E \} + \omega; \\ \pi w(E) &= \inf \{ \text{card}(B) : B \subseteq T(E) \setminus \{\emptyset\} \text{ and } B \text{ is coinital} \\ &\quad \text{in } (T(E) \setminus \{\emptyset\}, \subseteq) \} + \omega. \end{aligned}$$

These functions are called usually:

$$\begin{array}{lll} w(E) & - \text{ the } \mathbf{weight} & \text{ of a space } E; \\ c(E) & - \text{ the } \mathbf{Suslin number} & \text{ of a space } E; \\ d(E) & - \text{ the } \mathbf{density} & \text{ of a space } E; \\ \pi w(E) & - \text{ the } \mathbf{\pi-weight} & \text{ of a space } E. \end{array}$$

We say that a topological space  $E$  satisfies **the Suslin condition** (or **the countable chain condition - c.c.c**) if  $c(E) = \omega$ .

We say that a family  $B \subseteq T(E) \setminus \{\emptyset\}$  is a  **$\pi$ -base** of the topological space  $E$  if  $B$  is a coinital subset of  $(T(E) \setminus \{\emptyset\}, \subseteq)$ .

The following inequalities are obvious:

$$c(E) \leq d(E) \leq \pi w(E) \leq w(E).$$

It is not difficult to prove that for a metric space  $E$  all these inequalities become equalities.

It is also easy to prove that the inequality

$$\text{card}(E) \leq 2^{2^{d(E)}}$$

holds for any Hausdorff topological space  $E$ .

**Example 2.5** Let  $\kappa$  be any infinite cardinal. Let us consider the space  $\mathbb{R}^\kappa$  equipped with the product topology. It can be shown that this space satisfies the Suslin condition, i.e.  $c(\mathbb{R}^\kappa) = \omega$ . Moreover, if  $\kappa = 2^\omega$ , then it can be shown that the space  $\mathbb{R}^\kappa$  is **separable**, i.e.  $d(\mathbb{R}^\kappa) = \omega$ . Indeed, in this case the space  $\mathbb{R}^\kappa$  can be identified with the space  $\mathbb{R}^{[0,1]}$ , consisting of all real functions defined on  $[0, 1]$ , equipped with

the pointwise convergence topology. Using, for instance, the classical Weierstrass theorem on approximation we deduce that the countable set of all polynomials on  $[0, 1]$  with rational coefficients is dense in the space  $\mathbb{R}^{[0,1]}$ .

If  $\kappa > 2^\omega$ , then the topological space  $\mathbb{R}^\kappa$  is not separable but, as noticed above, this space always satisfies the Suslin condition.

We say that a topological space  $E$  satisfies the **first countability axiom** if every point in  $E$  has a countable local base at this point (i.e. there exists a countable fundamental system of neighbourhoods of this point). For instance, any metrizable topological space satisfies the first countability axiom.

A metric space is called **topologically complete** if its topology is metrizable by a complete metric. It is clear that any closed subset of a topologically complete metric space is also topologically complete. It is not so trivial that the same fact holds for any open subset of a topologically complete metric space, either. Indeed, we have the following result.

**Theorem 2.11** *Let  $E$  be a topologically complete metric space and let  $G$  be any open subset of  $E$ . Then  $G$  is topologically complete, too.*

**Proof.** Let us consider the topological product  $\mathbb{R} \times E$ , where  $\mathbb{R}$  is the real line. This product is topologically complete. Let us put

$$Z = \{(t, x) \in \mathbb{R} \times E : t \cdot \rho(x, E \setminus G) = 1\},$$

where  $\rho$  is a metric in  $E$ . It is clear that  $Z$  is a closed subset of  $\mathbb{R} \times E$ . So  $Z$  is a topologically complete metric space. Let us consider the canonical projection

$$pr_2 : \mathbb{R} \times E \rightarrow E,$$

which is a continuous and open mapping. It is clear also that the mapping

$$pr_2|Z : Z \rightarrow G$$

is a bijection. Hence, this mapping is a homeomorphism and  $G$  is topologically complete.

Now, we shall establish one result of metamathematical character which will be used in the sequel.

**Theorem 2.12** *Suppose that we have some notion of completeness for the class of all Hausdorff topological spaces and suppose that the following conditions hold:*

- 1) *completeness is preserved under homeomorphisms;*
- 2) *completeness is preserved under taking closed subspaces;*
- 3) *for some fixed cardinal number  $\kappa$  the topological product of any family of cardinality  $\kappa$  of complete spaces is complete, too.*

*Then the completeness is preserved under taking intersections of any families of cardinality  $\kappa$  of complete subspaces of a given Hausdorff topological space.*

**Proof.** Let  $(E_i)_{i \in I}$  be a family of complete topological spaces such that

$$\text{card}(I) \leq \kappa, \quad (\forall i)(i \in I \rightarrow E_i \subseteq E),$$

where  $E$  is a Hausdorff topological space. Then the topological product  $\prod_{i \in I} E_i$  is complete, too. Let us consider the intersection

$$X = \left( \prod_{i \in I} E_i \right) \cap \text{diag}(E^I),$$



where  $\text{diag}(E^I)$  denotes the diagonal in the topological product  $E^I$ . Since  $\text{diag}(E^I)$  is closed in the space  $E^I$ , the set  $X$  is closed in the space  $\prod_{i \in I} E_i$ , and hence  $X$  is complete. But it is evident that  $X$  is homeomorphic to the intersection of the family  $(E_i)_{i \in I}$ . So this intersection is complete, too.

The next classical result can be obtained as an immediate application of Theorems 11 and 12.

**Theorem 2.13 (Alexandrov)** *In every complete metric space any  $G_\delta$ -subset (i.e. an intersection of a countable family of open sets) is topologically complete.*

Notice that there is a result which, in a certain sense, is a converse version of Theorem 13 (see exercises after this Chapter).

Now, we shall introduce one notion which is important for our purposes and plays a remarkable role in the classical descriptive set theory.

A topological space  $E$  is a **Polish space** if  $E$  is homeomorphic to a complete separable metric space.

It follows from Theorem 13 that any  $G_\delta$ -subset of a Polish space is a Polish space, too. It is also clear that any compact metric space is a Polish space. In particular, the **Cantor discontinuum**  $\{0, 1\}^\omega$  (where the set  $\{0, 1\}$  is equipped with discrete topology) is a Polish space. The space  $\omega^\omega = \mathbb{N}^\omega$ , where  $\mathbb{N}$  is equipped with the discrete topology, is another standard example of a Polish space. The space  $\mathbb{N}^\omega$  is usually called the **canonical Baire space**. It is not difficult to prove that  $\mathbb{N}^\omega$  is homeomorphic to the set of all irrational numbers in  $\mathbb{R}$ .

**Theorem 2.14** *The following three sentences hold:*

- 1) *any non-empty Polish space is a continuous image of the space  $\mathbb{N}^\omega$ ;*
- 2) *any non-empty compact metric space is a continuous image of the Cantor discontinuum  $\{0, 1\}^\omega$ ;*
- 3) *any separable metric space is topologically contained in the Hilbert cube  $[0, 1]^\omega$ .*

**Proof.** Let us remark that proofs of 1) and 2) can be done by one schema in which countable systems of countable (or finite) closed coverings of given spaces are used. These constructions are not difficult. So, we shall prove here only sentence 3).

Let  $(E, \rho)$  be any separable metric space. We may assume that the range of the metric  $\rho$  is contained in the segment  $[0, 1]$  (because, if we need, we may replace the original metric by a topologically equivalent metric bounded from above by 1). Let  $(e_n)_{n < \omega}$  be any sequence of points of  $E$  which is dense everywhere in  $E$ . We define a mapping

$$f : E \rightarrow [0, 1]^\omega$$

by the formula

$$f(e) = (\rho(e, e_n))_{n < \omega} \quad (e \in E).$$

It is easy to see that  $f$  is injective and continuous and that the converse mapping

$$f^{-1} : f(E) \rightarrow E$$

is continuous, too. Hence, the topological space  $E$  can be homeomorphically embedded into the Hilbert cube  $[0, 1]^\omega$ .

Let  $E$  be any Hausdorff topological space. By  $\text{Comp}(E)$  we denote the space of all non-empty compact subsets of  $E$ , i.e.

$$\text{Comp}(E) = \{X \in P(E) \setminus \{\emptyset\} : X \text{ is compact}\}$$

equipped with the **Viectoris topology**, generated by the family of sets of the form

$$\{F \in \text{Comp}(E) : F \cap U \neq \emptyset\}, \quad \{F \in \text{Comp}(E) : F \subseteq U\},$$

where  $U$  are open subsets of the space  $E$ . So the basic open sets in  $\text{Comp}(E)$  are of the form

$$\{F \in \text{Comp}(E) : F \subseteq U_1 \ \& \ F \cap U_2 \neq \emptyset \ \& \ \cdots \ \& \ F \cap U_n \neq \emptyset\},$$

where  $U_1, U_2, \dots, U_n$  are open sets in  $E$ .

If  $E$  is a metric space, then the space  $\text{Comp}(E)$  is metrizable, too. One of its metrics is the so called **Hausdorff metric**  $\rho$  defined by the formula

$$\rho(A, B) = \inf\{\epsilon > 0 : (\forall x \in A)(\rho(x, B) \leq \epsilon) \ \& \ (\forall y \in B)(\rho(y, A) \leq \epsilon)\}.$$

If  $E$  is a complete metric space, then  $\text{Comp}(E)$  is also a complete metric space. If  $E$  is a compact space, then  $\text{Comp}(E)$  is a compact space, too.

Notice that the set of all finite subsets of the space  $E$  is dense everywhere in  $\text{Comp}(E)$ . Notice also that the set of all finite subsets of an arbitrary countable dense subset of the space  $E$  is a countable dense subset of  $\text{Comp}(E)$ . Hence,  $\text{Comp}(E)$  is a Polish topological space, whenever  $E$  is a Polish topological space.

**Theorem 2.15** *Let  $E$  and  $F$  be any two Polish topological spaces. Then the following sentences hold:*

1) *the family of all perfect subsets of  $E$  is a  $G_\delta$ -set in  $\text{Comp}(E)$ ;*

2) *the function*

$$\Phi : \text{Comp}(\text{Comp}(E)) \rightarrow \text{Comp}(E),$$

*given by the formula  $\Phi(L) = \bigcup L$ , is continuous;*

3) *the function*

$$\Phi : \text{Comp}(E) \times \text{Comp}(E) \rightarrow \text{Comp}(E),$$

*given by the formula  $\Phi(A, B) = A \cup B$ , is continuous;*

4) *if the function  $f : E \rightarrow F$  is continuous, then the function*

$$f^* : \text{Comp}(E) \rightarrow \text{Comp}(F),$$

*given by the formula  $f^*(A) = f(A)$ , is continuous, too.*

We leave the proof of this theorem to the reader. In connection with Theorem 15 we only remark here that the function

$$g : \text{Comp}(E) \times \text{Comp}(E) \rightarrow \text{Comp}(E),$$

defined by the formula

$$g(A, B) = A \cap B,$$

is not necessarily continuous. However, if  $A$  is a closed and open subset of  $E$ , then the mapping

$$h : \text{Comp}(E) \rightarrow \text{Comp}(E),$$

given by the formula

$$h(B) = A \cap B,$$

is continuous.

At this point we shall finish our short review of topological notions and facts which will be used further in the text of this book.

## Exercises

**Exercise 2.1** Let  $X$  and  $Y$  be any two sets and let  $f : X \rightarrow Y$  be an arbitrary mapping. For every set  $Z \subseteq X$  let us put

$$cl(Z) = f^{-1}(f(Z)).$$

Prove that the operator  $cl$  is the closure operator in the sense of Kuratowski and hence, it defines a certain topology  $T$  on the set  $X$ . Investigate the properties of this topology. In particular, prove that  $T$  is discrete if and only if the mapping  $f$  is an injection.

**Exercise 2.2** Prove, in theory **ZF**, that the Tychonoff theorem about products of quasi-compact topological spaces is equivalent to the Axiom of Choice. Moreover, show in **ZF** that the following very weak version of the Tychonoff product theorem is also equivalent to the Axiom of Choice : the topological product of any family of spaces  $(E_i)_{i \in I}$ , where

$$(\forall i)(i \in I \rightarrow \text{card}(T(E_i)) \leq 3),$$

is quasi-compact.

**Exercise 2.3** Let  $I$  be any set of indices, let  $(X_i)_{i \in I}$  be a family of finite sets and let  $\Phi$  be a family of partial functions from  $I$  into  $\bigcup_{i \in I} X_i$ . Assume that the following conditions hold:

- a) for any  $\phi \in \Phi$  and  $i \in \text{dom}(\phi)$  we have  $\phi(i) \in X_i$ ;
- b) for any finite set  $J \subseteq I$  there exists a function  $\phi \in \Phi$  such that  $J \subseteq \text{dom}(\phi)$  (in particular, all sets  $X_i$  ( $i \in I$ ) are non-empty).

Prove that there exists a function

$$f : I \rightarrow \bigcup_{i \in I} X_i$$

which satisfies the following relations:

- c)  $(\forall i \in I)(f(i) \in X_i)$ ;
- d) for each finite set  $J \subseteq I$  one can find  $\phi \in \Phi$  such that  $\phi|_J = f|_J$ .

The formulated result is due to Rado. Its special case, when  $\text{card}(I) = \omega$ , is equivalent to the König theorem on  $\omega$ -trees.

**Exercise 2.4** Let  $I$  be any set of indices, again, and let  $(X_i)_{i \in I}$  be a family of finite sets. Show that the following two sentences are equivalent:

- a) there exists an injective family  $(x_i)_{i \in I}$  such that

$$(\forall i \in I)(x_i \in X_i);$$

- b) for every finite set  $J \subseteq I$  the inequality

$$\text{card}\left(\bigcup_{i \in J} X_i\right) \geq \text{card}(J)$$

holds.

This result is due to Hall and is usually called "the theorem about systems of pairwise different representatives".

**Exercise 2.5** Let  $X$  be a quasi-compact space. Show that for every topological space  $Y$  the mapping

$$pr_1 : Y \times X \rightarrow Y$$

is closed (i.e. the images of closed sets are closed). Does this property characterize quasi-compact spaces in the class of all topological spaces ?

**Exercise 2.6** Let  $C[0, 1]$  be the set of all continuous real functions defined on the segment  $[0, 1]$ . Let us consider in  $C[0, 1]$  the topology of uniform convergence on this segment. Then  $C[0, 1]$  becomes a separable Banach space. Let us put

$$\Phi = \{\phi \in C[0, 1] : \phi \text{ is nowhere differentiable on } [0, 1]\}.$$

Prove that the set  $\Phi$  is a comeager subset of the space  $C[0, 1]$ .

This classical result is due to Banach.

**Exercise 2.7** Let  $\Gamma$  be any topological group. Show that one of the following two sentences holds:

- a)  $\Gamma$  is a first category topological space;
- b)  $\Gamma$  is a Baire topological space.

**Exercise 2.8** Let  $E$  be a topological space. For every set  $Z \subseteq E$  let  $\text{int}(Z)$  be the set of all interior points of  $Z$ . A set  $X \subseteq E$  is called **regular open** (respectively, **regular closed**), if there exists a closed set  $F \subseteq E$  (respectively, an open set  $V \subseteq E$ ), for which  $X = \text{int}(F)$  (respectively,  $X = \text{cl}(V)$ ). Prove that

- a) a set  $X \subseteq E$  is regular open if and only if  $X = \text{int}(\text{cl}(X))$ ;
- b) a set  $X \subseteq E$  is regular closed if and only if  $X = \text{cl}(\text{int}(X))$ ;
- c) the intersection of two regular open sets is regular open, too;
- d) the union of two regular closed sets is regular closed, too;
- e) a set  $Y \subseteq E$  has the Baire property if and only if there exists a regular open set  $V \subseteq E$  and a first category set  $P \subseteq E$ , such that

$$Y = V \Delta P.$$

Moreover, prove that if  $E$  is a Baire space, then for any set  $Y \in \bar{B}(E)$  there exists exactly one pair  $(V, P)$  of sets satisfying the above conditions.

**Exercise 2.9** Let  $E$  be an arbitrary topological space. Prove that  $E$  can be represented in the form

$$E = X \cup Y,$$

where the sets  $X$  and  $Y$  satisfy the following relations:

- a)  $X \cap Y = \emptyset$ ;
- b)  $X$  is a perfect subset of  $E$ ;
- c)  $Y$  does not contain a non-empty dense in itself subset.

This is the classical Cantor-Bendixson theorem. In particular, if  $E$  has a countable base, then the set  $Y$  is at most countable.

**Exercise 2.10** Let  $E$  be a topological space and suppose that a set  $Y \subseteq E$  does not contain a non-empty dense in itself subset. Prove that the boundary of the set  $Y$  is a nowhere dense subset of  $E$ . Conclude from this fact that the set  $Y$  has the Baire property in  $E$ .

**Exercise 2.11** Let  $E$  be a topological space and let  $X$  and  $Y$  be some subsets of  $E$ . Suppose that  $X \subseteq \text{cl}(Y)$  and  $X$  has the Baire property in  $\text{cl}(Y)$ . Show that the set  $X \cap Y$  has the Baire property in  $Y$ .

**Exercise 2.12** Prove Theorem 6 from this Chapter.

**Exercise 2.13** Prove Theorem 7 from this Chapter.

**Exercise 2.14** Show that the Kuratowski-Ulam theorem remains true under a weaker assumption about the space  $E_2$ , namely if  $E_2$  has a countable  $\pi$ -base (i.e.  $\pi w(E_2) = \omega$ ). Show also that the Kuratowski-Ulam theorem, in general, is not true without some assumptions on the space  $E_2$ .

**Exercise 2.15** Prove that if a topological space  $E$  is Hausdorff, then the following inequality holds:

$$\text{card}(E) \leq 2^{2^{d(E)}}.$$

Check that this inequality is not true for the class of all topological spaces. Finally, for any infinite cardinal number  $\kappa$  construct a compact topological space  $E$  such that

$$d(E) = \kappa, \quad \text{card}(E) = 2^{2^\kappa}.$$

**Exercise 2.16** Let  $\kappa$  be any infinite cardinal number and let  $(E_i)_{i \in I}$  be a family of topological spaces such that

$$d(E_i) \leq \kappa \quad (i \in I).$$

Show that

$$c\left(\prod_{i \in I} E_i\right) \leq \kappa.$$

In particular, the topological product of any family of separable spaces satisfies the Suslin condition. This classical result is due to Marczewski.

**Exercise 2.17** Let  $(E_i)_{i \in I}$  be a family of non-empty topological spaces. Prove that the following two sentences are equivalent:

- a) the topological product  $\prod_{i \in I} E_i$  satisfies the Suslin condition;
- b) for each finite set  $J \subseteq I$  the topological product  $\prod_{i \in J} E_i$  satisfies the Suslin condition.

**Exercise 2.18** Let  $(E, T)$  be a topological space and let  $RO(T)$  be a family of all regular open sets in  $E$ . Prove that the inequality

$$\text{card}(RO(T)) \leq 2^{d(E)}$$

holds. Conclude from this fact, that for every regular topological space  $E$  the inequality

$$w(E) \leq 2^{d(E)}$$

holds. Deduce also from the last inequality that the topological space  $R^\kappa$  is non-separable whenever  $\kappa > 2^\omega$ .

**Exercise 2.19** Let  $\kappa$  be an infinite cardinal number and let  $E$  be a topological space such that  $c(E) \leq \kappa$ . Let  $(X_i)_{i \in I}$  be an arbitrary family of pairwise disjoint subsets of  $E$  which have the Baire property and are not first category sets. Prove that the inequality

$$\text{card}(I) \leq \kappa$$

holds.

**Exercise 2.20** Let  $E$  be a metric space and let  $X$  be a topologically complete subspace of  $E$ . Show that  $X$  is a  $G_\delta$ -set in  $E$ .

**Exercise 2.21** A topological space  $E$  is called an **isodyne** if for each non-empty open set  $V \subseteq E$  we have the equality

$$\text{card}(V) = \text{card}(E).$$

Show that in any topological space the class of all non-empty open isodyne subsets is a  $\pi$ -base.

**Exercise 2.22** Let  $\kappa$  be an infinite cardinal number. We say that a topological space  $E$  is  $\kappa$ -**inexhaustible** if  $E$  cannot be represented in the form

$$E = \bigcup_{i \in I} X_i,$$

where  $\text{card}(I) \leq \kappa$  and every  $X_i$  ( $i \in I$ ) is a first category subset of  $E$ . In particular, a space  $E$  is  $\omega$ -inexhaustible if and only if  $E$  is not a first category space.

Suppose that the Generalized Continuum Hypothesis holds and let  $E$  be any  $\kappa$ -inexhaustible isodyne Hausdorff topological space. Prove that

$$(\text{card}(E))^\kappa = \text{card}(E).$$

**Exercise 2.23** Show that the definition of Hausdorff metric is correct. Show also that if  $E$  is a Polish topological space, then the space  $\text{Comp}(E)$  is a Polish topological space, too.

**Exercise 2.24** Prove Theorem 15 from this Chapter.

**Exercise 2.25** Let  $(E, \rho)$  be a metric space. Let us define the Hausdorff metric  $\rho_1$  on the class  $E_1$  of all non-empty closed bounded subsets of the space  $E$ . Show that if  $(E, \rho)$  is complete then  $(E_1, \rho_1)$  is complete, too.

**Exercise 2.26** We say that a topological space  $E$  is **totally disconnected** if each point in  $E$  has a local base consisting of closed and open (at the same time) subsets of  $E$ .

Prove that any Boolean algebra is isomorphic to the algebra of all closed and open (at the same time) subsets of some compact, totally disconnected topological space (this is a topological version of the Stone theorem on the representations of Boolean algebras).

**Exercise 2.27** Prove that Martin's Axiom is equivalent to the following statement:

if  $E$  is a compact topological space with  $c(E) \leq \omega$ , then  $E$  is  $\kappa$ -inexhaustible for any  $\kappa < 2^\omega$ .

**Exercise 2.28** Let  $E_1$  be a topological space satisfying the first countability axiom such that every closed subset of  $E_1$  is  $G_\delta$ -set in  $E_1$ . Let  $E_2$  be a complete metric space. Finally, let  $X$  be a subset of  $E_1$  and let  $f : X \rightarrow E_2$  be a continuous mapping. Prove that there exist a set  $X^* \subseteq E_1$  and a continuous mapping  $f^* : X^* \rightarrow E_2$  with the following properties:

- a)  $X \subseteq X^* \subseteq cl(X)$ ;
- b)  $X^*$  is a  $G_\delta$ -set in  $E_1$ ;
- c)  $f^*$  is an extension of  $f$ .

**Exercise 2.29** Let  $E_1$  and  $E_2$  be two complete metric spaces. Let  $X \subseteq E_1$ ,  $Y \subseteq E_2$  and let

$$f : X \rightarrow Y$$

be an arbitrary homeomorphism between  $X$  and  $Y$ . Show that there exist some sets  $X^* \subseteq E_1$ ,  $Y^* \subseteq E_2$  and a mapping

$$f^* : X^* \rightarrow Y^*$$

with the following properties:

- a)  $X \subseteq X^*$  and  $X^*$  is a  $G_\delta$ -set in  $E_1$ ;
- b)  $Y \subseteq Y^*$  and  $Y^*$  is a  $G_\delta$ -set in  $E_2$ ;
- c)  $f^*$  is a homeomorphism between  $X^*$  and  $Y^*$ ;
- d)  $f^*$  is an extension of  $f$ .

This classical result is due to Lavrent'ev and is known as the Lavrent'ev theorem on extensions of homeomorphisms.

**Exercise 2.30** Show that there exist a subset  $X$  of the segment  $[0, 1]$  and an injective continuous mapping  $f : X \rightarrow [0, 1]$  such that  $f$  cannot be extended to an injective continuous mapping acting from a Borel subset of  $[0, 1]$  into  $[0, 1]$ .

**Exercise 2.31** Let  $E$  be an arbitrary locally compact topological space. Prove that there exists a compact topological space  $E^*$  such that for some point  $e \in E^*$  the space  $E^* \setminus \{e\}$  is homeomorphic with the space  $E$ . Prove also that the space  $E^*$  is unique with exactness to homeomorphism.

The mentioned space  $E^*$  is called the **Aleksandrov compactification** of the original space  $E$ .

**Exercise 2.32** Let us equip the first uncountable ordinal  $\omega_1$  with its order topology (generated by the natural ordering in  $\omega_1$ ). Prove that

- a) the space  $\omega_1$  is locally compact and locally separable;
- b)  $\omega_1$  satisfies the first countability axiom;
- c) there exists a closed subset of  $\omega_1$  which is not a  $G_\delta$ -set in  $\omega_1$ .
- d)  $c(\omega_1) = w(\omega_1) = \omega_1$ .

**Exercise 2.33** Using the Tychonoff theorem on products of quasi-compact topological spaces prove the following generalization of the König lemma:

Let  $(T, \preceq)$  be any tree all levels of which are finite. Then there exists a path through  $T$ .

## Chapter 3

# Elements of Descriptive Set Theory

In this Chapter we discuss some fundamental notions of the classical descriptive set theory in Polish topological spaces. We confine ourselves to *Borel* and *analytic* sets. A more general notion of a *projective* set will be thoroughly considered in Part 2 of the book.

Let  $E$  be an arbitrary topological space. As we know, the *Borel*  $\sigma$ -algebra  $B(E)$  is the  $\sigma$ -algebra of subsets of  $E$  generated by the family of all open subsets of  $E$ . The elements of the  $\sigma$ -algebra  $B(E)$  are called *Borel* sets in  $E$ .

In some particular but important for applications cases it is possible to give a description of the  $\sigma$ -algebra  $B(E)$  only in terms of countable unions and countable intersections.

A topological space  $E$  is called **perfect** if any closed subset of  $E$  is a  $G_\delta$ -set in  $E$  (or, equivalently, if any open subset of  $E$  is an  $F_\sigma$ -set in  $E$ ). It is not difficult to prove that for perfect topological spaces the following result holds.

**Theorem 3.1** *Let  $E$  be a topological space and let  $B^*(E)$  be the smallest (with respect to inclusion) class of subsets of  $E$  containing all open subsets of  $E$  and closed under the operations of countable unions and countable intersections. If  $E$  is a perfect topological space then the equality*

$$B(E) = B^*(E)$$

*holds.*

For any topological space  $E$  we can describe the class  $B^*(E)$  in a more concrete way. Indeed, we can write

$$B^*(E) = B_0^*(E) \cup \dots \cup B_\xi^*(E) \dots (\xi < \omega_1),$$

where the classes  $B_\xi^*(E)$  ( $\xi < \omega_1$ ) are defined by transfinite recursion:

1.  $B_0^*(E)$  is the class of all open subsets of  $E$ ;
2.  $B_\xi^*(E)$  is the class of all countable intersections of arbitrary elements of the class  $\bigcup_{\zeta < \xi} B_\zeta^*(E)$ , if ordinal  $\xi$  is odd;
3.  $B_\xi^*(E)$  is the class of all countable unions of arbitrary elements of the class  $\bigcup_{\zeta < \xi} B_\zeta^*(E)$ , if ordinal  $\xi > 0$  is even.



It is evident that for any topological space  $E$  we have

$$B^*(E) \subseteq B(E).$$

There are examples of topological spaces  $E$  such that

$$B^*(E) \neq B(E).$$

But for perfect topological spaces  $E$ , by the preceding theorem, the equality  $B^*(E) = B(E)$  holds and in many cases makes it easier to prove the properties of Borel subsets of  $E$  by the method of transfinite induction. We note here that the class of perfect topological spaces is wide enough for a lot of applications. In particular, this class contains in itself the class of all metrizable topological spaces. In what follows we mainly restrict our considerations to the class of perfect topological spaces.

The following auxiliary proposition is often applied in the theory of Borel sets.

**Lemma 3.1** *Let  $E$  be an arbitrary perfect topological space. Then the Borel  $\sigma$ -algebra  $B(E)$  of this space is the smallest (with respect to inclusion) class  $K$  such that*

- 1) *all open subsets of  $E$  belong to  $K$ ;*
- 2)  *$K$  is closed under countable intersections;*
- 3)  *$K$  is closed under countable unions of its pairwise disjoint elements.*

**Proof.** One can prove this lemma by the method of transfinite induction. But here we give another, simpler argument. At first it is clear that  $K \subseteq B(E)$ . Let us put

$$H = \{X \subseteq E : X \in K \text{ \& } E \setminus X \in K\}.$$

Then, for the class  $H$ , it is not difficult to check that

- 1) all open subsets of  $E$  belong to  $H$ ;
- 2)  $H$  is closed under countable intersections;
- 3)  $H$  is closed under countable unions of its pairwise disjoint elements.

By the definition of the class  $K$ , we have  $K = H$ . Thus, we see that  $K$  is a  $\sigma$ -algebra of subsets of  $E$  containing all open subsets of  $E$ . So  $B(E) \subseteq K$ , and finally,  $B(E) = K$ , as required.

A more general operation than that of countable unions and countable intersections is the so called  $(A)$ -operation which was introduced by Alexandrov and Hausdorff and which is defined in the following way.

Let  $E$  be a basic set (in general, not equipped with any topology) and let  $\Phi$  be a class of subsets of  $E$ . Let us consider an arbitrary system of sets

$$(X_s)_{s \in \mathbb{N}^{<\omega}},$$

where  $\mathbb{N}^{<\omega}$  denotes the complete  $\mathbb{N}$ -ary tree of height  $\omega$ , i.e. the set of all finite sequences of elements of  $\mathbb{N}$ , and every set  $X_s$  belongs to the class  $\Phi$ . The result of  $(A)$ -operation applied to the system of sets mentioned above is the set of the form

$$X = \bigcup_{f \in \mathbb{N}^\omega} \bigcap_k X_{f|k}.$$

This set is also sometimes denoted by

$$X = (A)((X_s)_{s \in \mathbb{N}^{<\omega}}).$$

The set  $X$  is also called an **analytic** set (over the original class  $\Phi$ ) corresponding to the system  $(X_s)_{s \in \mathbb{N}^{<\omega}}$ . The family of analytic sets over the class  $\Phi$ , corresponding to all systems  $(X_s)_{s \in \mathbb{N}^{<\omega}} \subseteq \Phi$ , is denoted by  $(A)(\Phi)$  and is called the **analytic class** over  $\Phi$ . It is clear that always  $\Phi \subseteq (A)(\Phi)$ .

If  $s = (s_1, \dots, s_n)$  and  $t = (t_1, \dots, t_k)$  are arbitrary sequences from  $\mathbb{N}^{<\omega}$  then by  $s * t$  we denote the concatenation of sequences  $s$  and  $t$ , i.e. a sequence from  $\mathbb{N}^{<\omega}$  defined by the equality

$$s * t = (s_1, \dots, s_n, t_1, \dots, t_k).$$

If  $s \in \mathbb{N}^{<\omega}$  and  $k \in \mathbb{N}$  then we put

$$s * k = s * (k).$$

We denote the empty sequence by the symbol  $()$ . For any  $s \in \mathbb{N}^{<\omega}$  let  $lh(s)$  be the length of  $s$ .

Let  $(Y_k)_{k \in \mathbb{N}}$  be an arbitrary countable family of sets belonging to the original class  $\Phi$ . Let us consider the system  $(X_s)_{s \in \mathbb{N}^{<\omega}}$  where

$$X_s = Y_{lh(s)}.$$

Then, evidently, we have

$$(A)((X_s)_{s \in \mathbb{N}^{<\omega}}) = \bigcap_k Y_k.$$

Now let us consider the system  $(X_s)_{s \in \mathbb{N}^{<\omega}}$  where

$$X_s = Y_{s_1}.$$

Then we obviously get

$$(A)((X_s)_{s \in \mathbb{N}^{<\omega}}) = \bigcup_k Y_k.$$

Thus, we see that  $(A)$ -operation is a generalization of usual operations of countable unions and countable intersections. The following theorem shows us that  $(A)$ -operation is idempotent.

**Theorem 3.2** *For any basic set  $E$  and for any class  $\Phi$  of subsets of  $E$  the equality*

$$(A)((A)(\Phi)) = (A)(\Phi)$$

*holds. In particular, the class  $(A)(\Phi)$  is closed under countable unions and countable intersections.*

We leave a non-trivial proof of this result to the reader as a very useful exercise. Notice that from this theorem we can easily conclude that if  $E$  is a perfect topological space and  $\Phi(E)$  is either the class of all open subsets of  $E$  or the class of all closed subsets of  $E$ , then

$$B(E) \subseteq (A)(\Phi(E)).$$

Now, let  $E$  be an arbitrary topological space. We define **analytic sets** in  $E$  to be the sets from the class  $(A)(F(E))$ , where  $F(E)$  is the class of all closed subsets of  $E$ . The class  $(A)(F(E))$  will be denoted simply by  $A(E)$ . By the above remark in any perfect topological space  $E$  every Borel set is also an analytic one.

The following easy proposition quite often appears to be useful in technical aspect.

**Theorem 3.3** *Let  $E$  be a basic set and let  $\Phi$  be a class of subsets of  $E$  closed under finite intersections. Then, the analytic class  $(A)(\Phi)$  coincides with the class of sets of the form*

$$X = \bigcup_{f \in \mathbb{N}^\omega} \bigcap_k X_{f|k},$$

where system  $(X_s)_{s \in \mathbb{N}^{<\omega}}$  is regular, i.e.

$$X_{s*i} \subseteq X_s$$

for every  $s \in \mathbb{N}^{<\omega}$  and every  $i \in \mathbb{N}$ .

In particular, when we deal with analytic subsets of a topological space  $E$  then without loss of generality we may consider only such analytic sets which are generated by regular systems of closed subsets of  $E$ .

Another purely technical proposition is also very useful in many situations.

**Theorem 3.4** *Let  $E$  be a basic set with a given system*

$$(X_s)_{s \in \mathbb{N}^{<\omega}}$$

of its subsets. If this system is regular and additionally satisfies the condition

$$(lh(s) = lh(t) \ \& \ s \neq t) \rightarrow X_s \cap X_t = \emptyset \quad (s, t \in \mathbb{N}^{<\omega})$$

then the following equality holds:

$$(A)((X_s)_{s \in \mathbb{N}^{<\omega}}) = \bigcap_k \bigcup_{lh(s)=k} X_s.$$

In other words, in this case the  $(A)$ -operation reduces itself to countable unions and countable intersections.

The proof of Theorem 4 relies on direct checking of the equality above, and therefore is left to the reader.

Let  $E$  be a basic set, let  $\Phi$  be a class of subsets of  $E$  and let  $(X_s)_{s \in \mathbb{N}^{<\omega}}$  be a system of sets from the class  $\Phi$ . Let us define by transfinite recursion over  $\xi < \omega_1$  the sets

- 1)  $X_s^0 = X_s$ ,
- 2)  $X_s^{\xi+1} = X_s^\xi \cap (\bigcup_{n \in \mathbb{N}} X_{s*n}^\xi)$ ,
- 3)  $X_s^\xi = \bigcap_{\zeta < \xi} X_s^\zeta$  for limit ordinal  $\xi < \omega_1$ .

It is easy to show by transfinite induction that for  $\zeta \leq \xi$  we have

$$X_s^\xi \subseteq X_s^\zeta.$$

Now, for  $\xi < \omega_1$  let us put

- 1)  $Y_\xi = \bigcup_n X_n^\xi$ ,
- 2)  $T_\xi = \bigcup_s (X_s^\xi \setminus X_s^{\xi+1})$ ,
- 3)  $Z_\xi = Y_\xi \setminus T_\xi$ .

Let us notice here that all the sets defined above:

$$X_s^\xi, \quad Y_\xi, \quad T_\xi, \quad Z_\xi$$

belong to the  $\sigma$ -algebra (and more precisely to the  $\sigma$ -ring) of sets generated by the original class  $\Phi$ . The following important result was obtained by Sierpiński.

**Theorem 3.5** For any system of sets

$$(X_s)_{s \in \mathbb{N}^{<\omega}}$$

the following equalities hold:

$$(A)((X_s)_{s \in \mathbb{N}^{<\omega}}) = \bigcap_{\xi < \omega_1} Y_\xi = \bigcup_{\xi < \omega_1} Z_\xi.$$

**Proof.** At first we prove the inclusion

$$\bigcup_{\xi < \omega_1} Z_\xi \subseteq (A)((X_s)_{s \in \mathbb{N}^{<\omega}}). \quad (3.1)$$

Let  $z \in \bigcup_{\xi < \omega_1} Z_\xi$ . Then for some  $\xi < \omega_1$  we have

$$z \in Y_\xi, \quad z \notin T_\xi.$$

Thus for some  $n_0$  we have  $z \in X_{n_0}^\xi$  and at the same time

$$z \notin X_{n_0}^\xi \setminus X_{n_0}^{\xi+1},$$

whence we obtain that  $z \in X_{n_0}^{\xi+1}$ . But then, remembering that

$$X_{n_0}^{\xi+1} \subseteq \bigcup_n X_{n_0 n}^\xi,$$

we obtain that for some  $n_1$  we have  $z \in X_{n_0 n_1}^\xi$ . Continuing this process by recursion we see that for some sequence

$$f = (n_0, n_1, n_2, \dots) \in \mathbb{N}^\omega,$$

the element  $z$  belongs to the intersection

$$\bigcap_k X_{f|k}^\xi \subseteq (A)((X_s)_{s \in \mathbb{N}^{<\omega}}).$$

Thus, the required inclusion (3.1) is established. Now we verify that the inclusion

$$(A)((X_s)_{s \in \mathbb{N}^{<\omega}}) \subseteq \bigcap_{\xi < \omega_1} Y_\xi \quad (3.2)$$

is true. For this purpose we notice at first that for every  $\xi < \omega_1$ , every  $r \in \mathbb{N}$  and all  $f \in \mathbb{N}^\omega$  we have the inclusion

$$\bigcap_k X_{f|k}^\xi \subseteq X_{f|r}^\xi,$$

which can easily be obtained by transfinite induction on  $\xi$ . Thus, if  $\xi < \omega_1$ , the following relations hold:

$$\bigcap_k X_{f|k}^\xi \subseteq X_{f(0)}^\xi \subseteq \bigcup_n X_n^\xi = Y_\xi,$$

from which we immediately obtain (3.2). Now we verify the equality

$$\bigcap_{\xi < \omega_1} T_\xi = \emptyset. \quad (3.3)$$

Suppose that  $x \in \bigcap_{\xi < \omega_1} T_\xi$ . Then for any ordinal  $\xi < \omega_1$  we can find a sequence  $s \in \mathbb{N}^{<\omega}$ , depending on  $\xi$ , such that

$$x \in X_s^\xi \setminus X_s^{\xi+1}.$$

Thus, there are two distinct indices  $\xi$  and  $\zeta$ , for instance  $\xi < \zeta$ , with the same corresponding  $s \in \mathbb{N}^{<\omega}$ . But in this case we obtain that

$$x \in X_s^\zeta \subseteq X_s^{\xi+1},$$

which is impossible. Finally, taking into account all three formulas (3.1), (3.2) and (3.3) we come to the required result.

As a trivial consequence of the theorem proved above we obtain the following proposition.

**Theorem 3.6** *Let  $E$  be a topological space and let  $X$  be a subset of  $E$  which is either an analytic set or the complement of an analytic set. Then we have*

$$X = \bigcup_{\xi < \omega_1} X_\xi,$$

where all the sets  $X_\xi$  ( $\xi < \omega_1$ ) are pairwise disjoint and Borel in the space  $E$ .

We notice here that the above representation of the set  $X$  as the union of  $\omega_1$ -sequence of pairwise disjoint Borel sets was obtained effectively, i.e. without the help of the Axiom of Choice.

So far we have considered the  $(A)$ -operation in a basic set  $E$ , which, in general, is not equipped with any topology. But it should be noticed that the most content theory of this operation may be developed when  $E$  is a topological space and the  $(A)$ -operation is applied to the class  $\Phi = F(E)$  of all closed subsets of  $E$ . From the point of view of applications to other domains of mathematics the most important case is when  $E$  is a Polish topological space. As we have said above, we denote the class of all analytic subsets of a Polish space  $E$  by the symbol  $A(E)$ . Since every Polish space  $E$  is a perfect topological space, we have the inclusion

$$B(E) \subseteq A(E).$$

We want to remark here that if  $E$  is an uncountable space, then this inclusion is proper (see exercises after this Chapter).

**Theorem 3.7** *Let  $E$  be a Polish space. Then the class  $A(E) \setminus \emptyset$  can be described as the class of all subsets of the space  $E$  which are continuous images of the canonical Baire space  $\mathbb{N}^\omega$ .*

**Proof.** Let  $s$  be an arbitrary finite sequence of natural numbers. Let us put

$$\mathbb{N}^\omega(s) = \{z \in \mathbb{N}^\omega : (\forall i)(i < lh(s) \rightarrow z(i) = s(i))\}.$$

It is obvious that the system of sets

$$(\mathbb{N}^\omega(s))_{s \in \mathbb{N}^{<\omega}}$$

forms a base of the space  $\mathbb{N}^\omega$  such that all elements of this base are non-empty, open and closed subsets of  $\mathbb{N}^\omega$ .

Now, let  $E$  be an arbitrary non-empty Polish topological space. Then it is not difficult to construct a regular system

$$(F_s)_{s \in \mathbb{N}^{<\omega}}$$

of closed subsets of  $E$  such that

- 1)  $\lim_k(\text{diam}F_{(n_1, \dots, n_k)}) = 0$ ,
- 2)  $E = (A)((F_s)_{s \in \mathbb{N}^{<\omega}})$ .

From the existence of such a system it follows among other things that a given space  $E$  is a continuous image of the Baire space  $\mathbb{N}^\omega$ . Now, let us suppose that for some set  $X \subseteq E$  we have

$$X = (A)((X_s)_{s \in \mathbb{N}^{<\omega}}),$$

where the sets  $(X_s)_{s \in \mathbb{N}^{<\omega}}$  are closed in  $E$  and form a regular system. Taking into account what we have just said about the space  $E$  and by the existence of the canonical isomorphism between  $\mathbb{N}^\omega$  and  $\mathbb{N}^\omega \times \mathbb{N}^\omega$  we may assume without the loss of generality that

$$\lim_k \text{diam}(X_{(n_0, \dots, n_k)}) = 0.$$

Now let us put

$$Z = \{f \in \mathbb{N}^\omega : \bigcap_k X_{f|k} \neq \emptyset\}.$$

It is not difficult to check that the set  $Z$  is closed in the space  $\mathbb{N}^\omega$ . So  $Z$  is a Polish space. Let us define a mapping

$$\Phi : Z \rightarrow E,$$

putting at  $f \in Z$  the value  $\Phi(f)$  equal to the unique point which belongs to the intersection  $\bigcap_k X_{f|k}$ . Then it is easy to see that the mapping  $\Phi$  is continuous and  $\Phi(Z) = X$ . Further, if  $X \neq \emptyset$ , then  $Z \neq \emptyset$ , too. Therefore, in this case there exists a continuous mapping from the space  $\mathbb{N}^\omega$  onto  $Z$ . Thus, if the set  $X$  is non-empty, then it is a continuous image of the space  $\mathbb{N}^\omega$ . In this way we have shown that any non-empty set from the class  $A(E)$  is a continuous image of the Baire space  $\mathbb{N}^\omega$ .

Conversely, let  $Y \subseteq E$  be such that there exists a continuous surjection

$$\Phi : \mathbb{N}^\omega \rightarrow Y.$$

Then for every element  $f \in \mathbb{N}^\omega$  we can write

$$\{\Phi(f)\} \subseteq \bigcap_k \Phi(\mathbb{N}^\omega(f|k)) \subseteq \bigcap_k \text{cl}(\Phi(\mathbb{N}^\omega(f|k))).$$

By the virtue of the continuity of the mapping  $\Phi$  we have

$$\lim_k \text{diam}(\text{cl}(\Phi(\mathbb{N}^\omega(f|k)))) = 0.$$

Thus, we obtain that

$$\{\Phi(f)\} = \bigcap_k \text{cl}(\Phi(\mathbb{N}^\omega(f|k))).$$

Whence it could immediately be seen that

$$Y = (A)((X_s)_{s \in \mathbb{N}^{<\omega}}),$$

where the system of sets  $(X_s)_{s \in \mathbb{N}^{<\omega}}$  is defined by the formula

$$X_s = \text{cl}(\Phi(\mathbb{N}^\omega(s))),$$

i.e.  $Y \in A(E)$ . This completes the proof of Theorem 7.

Since for any Polish topological space  $E$  we have the inclusion

$$B(E) \subseteq A(E),$$

from Theorem 7 it follows that in the space  $E$  every non-empty Borel subset is a continuous image of the Baire space  $\mathbb{N}^\omega$ . The next result strengthens Theorem 7 in the case of Borel subsets of a Polish space.

**Theorem 3.8** . *Let  $E$  be a Polish topological space. Then every Borel subset of  $E$  is an injective continuous image of some Polish space.*

**Proof.** Evidently, the assertion of the theorem is true for any open subset of  $E$ , since open subsets of  $E$  are Polish spaces. Now let  $(X_i)_{i \in I}$  be an arbitrary countable family of pairwise disjoint subsets of the space  $E$  such that there exist a countable family  $(P_i)_{i \in I}$  of Polish topological spaces and a countable family

$$\Phi_i : P_i \rightarrow X_i \quad (i \in I)$$

of bijective continuous mappings. Then it is obvious that there exists a bijective continuous mapping

$$\Phi : P \rightarrow \bigcup_{i \in I} X_i,$$

where  $P$  denotes the topological sum of the family of the spaces  $(P_i)_{i \in I}$ . Of course,  $P$  is a Polish topological space, too.

Now, let  $(Y_i)_{i \in I}$  be any countable family of subsets of  $E$  such that there exist a countable family  $(Q_i)_{i \in I}$  of Polish topological spaces and a countable family

$$\Psi_i : Q_i \rightarrow Y_i \quad (i \in I)$$

of bijective continuous mappings. It is clear that the topological product  $\prod_{i \in I} Q_i$  is a Polish topological space. Analogically, the topological product  $E^I$  may also be considered as a Polish space. Let us take the continuous mapping

$$\Psi : \prod_{i \in I} Q_i \rightarrow E^I$$

defined by the formula

$$\Psi(q) = (\Psi_i(q_i))_{i \in I} \quad (q \in \prod_{i \in I} Q_i).$$

Obviously, the mapping  $\Psi$  is injective and continuous. Let us put

$$Q = \Psi^{-1}(\text{diag}(E^I)),$$

where  $\text{diag}(E^I)$  is the diagonal of the space  $E^I$ . Clearly, the set  $Q$  is closed in the Polish space  $\prod_{i \in I} Q_i$ , so  $Q$  is a Polish space, too. Besides, it is evident that the mapping

$$\Psi|_Q : Q \rightarrow \left(\prod_{i \in I} Y_i\right) \cap \text{diag}(E^I)$$

is bijective. Now it remains to notice that the set  $(\prod_{i \in I} Y_i) \cap \text{diag}(E^I)$  is homeomorphic to the intersection of the family of sets  $(Y_i)_{i \in I}$ , thus there exists a continuous bijection from the space  $Q$  onto the set  $\bigcap_{i \in I} Y_i$ . Resuming the above ideas and applying Lemma 1 we immediately obtain the assertion of the present theorem.

Taking into account that there exists the canonical continuous bijection from the Baire space  $\mathbb{N}^\omega$  onto the Hilbert cube  $[0, 1]^\omega$  and using the fact that any Polish space is topologically contained (as a  $G_\delta$ -subset) in the Hilbert cube  $[0, 1]^\omega$  we can deduce from Theorem 8 that every Borel subset of a Polish topological space is an injective continuous image of some Polish subspace of the Baire space  $\mathbb{N}^\omega$ .

**Lemma 3.2** *Let  $E$  be an arbitrary non-empty complete metric space without isolated points. Then there exists a subset of  $E$  which is homeomorphic to the Cantor discontinuum  $\{0, 1\}^\omega$ .*

The proof of this well-known result does not present any difficulties. It is carried out by a standard method of constructing a dyadic system of non-empty closed subsets of a given space  $E$ . Combining this simple result with Theorem 8, we immediately obtain that every uncountable Borel subset of a Polish topological space contains in itself some set homeomorphic to the Cantor discontinuum (this classical result is due to Alexandrov and Hausdorff). So, we see that Borel subsets of Polish topological spaces realize in a sense the Continuum Hypothesis: they are either countable or they are of the cardinality continuum. An analogous result for analytic subsets of Polish spaces will be established below.

**Lemma 3.3** *Let  $E$  be an arbitrary separable metric space and let  $g$  be a continuous mapping from the space  $E$  onto some uncountable Hausdorff topological space. Then there exist non-empty open sets  $U \subseteq E$  and  $V \subseteq E$  such that*

- 1) both sets  $g(\text{cl}(U))$  and  $g(\text{cl}(V))$  are uncountable;
- 2)  $g(\text{cl}(U)) \cap g(\text{cl}(V)) = \emptyset$ .

The proof of Lemma 3 is also quite simple and we leave it to the reader.

**Theorem 3.9** *Let  $E$  be a Polish topological space, let  $Y$  be an uncountable analytic subset of some (possibly different from  $E$ ) Polish space and let a mapping*

$$g : E \rightarrow Y$$

*be surjective and continuous. Then there exists a set  $X \subseteq E$  homeomorphic to the Cantor discontinuum  $\{0, 1\}^\omega$  and such that the mapping*

$$g|X : X \rightarrow g(X)$$

*is a homeomorphism.*

**Proof.** In the view of the result of Lemma 3 by the method of mathematical recursion it is easy to construct a dyadic system

$$(F_s)_{s \in \{0, 1\}^{<\omega}}$$

of closed balls in the space  $E$ , so that for any  $s \in \{0, 1\}^{<\omega}$  they satisfy the following relations:

- 1)  $F_{s*0} \cup F_{s*1} \subseteq F_s$ ;
- 2)  $g(F_{s*0}) \cap g(F_{s*1}) = \emptyset$ ;
- 3)  $g(F_s)$  is an uncountable set;
- 4)  $\text{diam}(F_s) < \frac{1}{lh(s)+1}$ .

Having constructed this system of balls we put

$$X = \bigcap_k \bigcup_{lh(s)=k} F_s.$$

Then a direct checking shows us that the set  $X$  is homeomorphic to the Cantor discontinuum and that the mapping  $g|X$  is injective. Hence, from the compactness of the set  $X$  it follows that the mapping  $g|X$  is a homeomorphism between  $X$  and  $g(X)$ .

As an immediate consequence of the theorem proved above we obtain that every analytic subset of a Polish topological space is either countable or contains in itself



a set homeomorphic to the Cantor discontinuum (in the latter case, of course, the cardinality of the analytic subset will be equal to the cardinality continuum).

Let  $E$  be an arbitrary topological space and let  $X$  and  $Y$  be some subsets of  $E$ . We say that the sets  $X$  and  $Y$  can be separated by Borel sets if in the space  $E$  there exist Borel sets  $X_1$  and  $Y_1$  such that

$$X \subseteq X_1, \quad Y \subseteq Y_1, \quad X_1 \cap Y_1 = \emptyset.$$

The following auxiliary proposition may be easily proved using the closedness of the class of all Borel sets in  $E$  under countable unions and countable intersections.

**Lemma 3.4** *Let  $E$  be a topological space, let  $X$  and  $Y$  be subsets of  $E$  and let*

$$X = \bigcup_{n \in \mathbb{N}} X_n, \quad Y = \bigcup_{n \in \mathbb{N}} Y_n.$$

*If the sets  $X$  and  $Y$  cannot be separated by Borel sets, then there exists a pair of indices  $(n, m)$  such that the sets  $X_n$  and  $Y_m$  cannot be separated by Borel sets, either.*

The classical result given below is due to Luzin and is called the principle of separation for analytic sets.

**Theorem 3.10** *Let  $E$  be a Polish topological space and let  $X$  and  $Y$  be two disjoint analytic sets in  $E$ . Then  $X$  and  $Y$  can be separated by Borel sets.*

**Proof.** Assume otherwise that  $X$  and  $Y$  cannot be separated by Borel sets. Let

$$\varphi : \mathbb{N}^\omega \rightarrow X, \quad \psi : \mathbb{N}^\omega \rightarrow Y$$

be surjective continuous mappings which define the sets  $X$  and  $Y$ . Using the notation from the proof of Theorem 7, we can write

$$X = \bigcup_n \varphi(\mathbb{N}^\omega(n)), \quad Y = \bigcup_n \psi(\mathbb{N}^\omega(n)).$$

Whence, by the preceding lemma, we conclude that there exists a pair of indices  $(n_0, m_0)$  for which the sets  $\varphi(\mathbb{N}^\omega(n_0))$  and  $\psi(\mathbb{N}^\omega(m_0))$  cannot be separated by Borel sets. Repeating this process recursively, we construct two sequences of natural numbers

$$x = (n_0, \dots, n_k, \dots), \quad y = (m_0, \dots, m_k, \dots)$$

such that for every  $k$  the sets

$$\varphi(\mathbb{N}^\omega(x|k)), \quad \psi(\mathbb{N}^\omega(y|k))$$

cannot be separated by Borel sets. But since the mappings  $\varphi$  and  $\psi$  are continuous this leads immediately to contradiction, as with a sufficiently large  $k$  the sets  $\varphi(\mathbb{N}^\omega(x|k))$  and  $\psi(\mathbb{N}^\omega(y|k))$  lie in disjoint open neighbourhoods of the points

$$\varphi(x) \in X, \quad \psi(y) \in Y.$$

The obtained contradiction completes the proof of this theorem.

From the theorem proved above we immediately get that if  $X$  is an analytic subset of a Polish space  $E$ , such that its complement  $E \setminus X$  is also analytic, then the set  $X$  is a Borel subset of  $E$ . This classical result is due to Suslin.

**Theorem 3.11** *Suppose that  $(X_n)_{n \in \omega}$  is an arbitrary family of pairwise disjoint analytic subsets of some Polish topological space  $E$ . Then there exists a family  $(Y_n)_{n \in \omega}$  of pairwise disjoint Borel subsets of  $E$  such that*

$$X_n \subseteq Y_n \quad (n \in \omega).$$

**Proof.** From the previous theorem we deduce that for any pair  $(n, m)$  of distinct indices there exists a Borel set  $P_{nm} \subseteq E$  such that

$$X_n \subseteq P_{nm} \subseteq E \setminus X_m.$$

We put by recursion

$$Y_0 = \bigcap_{m \geq 1} P_{0m},$$

$$Y_n = \left( \bigcap_{m \in \omega \setminus \{n\}} P_{nm} \right) \setminus (Y_0 \cup \dots \cup Y_{n-1}) \quad (n \geq 1).$$

It is easy to check that  $(Y_n)_{n \in \omega}$  is the required family.

For future needs we shall give one abstract topological characterization of the canonical Baire space  $\mathbb{N}^\omega$ . Evidently, this space is homeomorphic with the set  $\mathbb{Z}$  of all irrational numbers of the real line  $\mathbb{R}$ . But obviously the set  $\mathbb{Z}$  has the following properties : it is non-empty, zero-dimensional (i.e. any of its points has a fundamental system consisting of closed and open neighbourhoods), it is a dense  $G_\delta$ -subset of  $\mathbb{R}$  and its complement is dense in  $\mathbb{R}$ , too. It is easy to see that these properties topologically characterize the space  $\mathbb{N}^\omega$ . In other words, the following easy lemma holds.

**Lemma 3.5** *Let  $E$  be any zero-dimensional Polish topological space and let  $Z$  be a dense  $G_\delta$ -subset of  $E$  such that its complement is dense in  $E$ , too. Then  $Z$  is homeomorphic with the space  $\mathbb{N}^\omega$ .*

We leave details of the proof of this lemma to the reader. He must only check that the properties of  $Z$  mentioned above imply that this set can be obtained by (A)-operation applied to some regular system

$$(F_s)_{s \in \mathbb{N}}^{<\omega}$$

consisting of non-empty, closed and open sets which are pairwise disjoint (when  $lh(s)$  is fixed) and the diameters of which converge to zero when  $lh(s)$  converges to infinity.

The following auxiliary proposition is an easy corollary from Lemma 5.

**Lemma 3.6** *Any uncountable zero-dimensional Polish topological space  $E$  can be represented in the form*

$$E = Z \cup D,$$

where  $Z \cap D = \emptyset$ , the set  $Z$  is homeomorphic to the Baire space  $\mathbb{N}^\omega$  and the set  $D$  is at most countable.

**Proof.** Let  $P$  be the set of condensation points of the space  $E$ . Then  $P$  is non-empty, perfect and  $E \setminus P$  is at most countable. Let  $C$  be any countable subset of  $P$  dense in  $P$ . Then  $Z = P \setminus C$  is a dense  $G_\delta$ -subset of  $P$  such that its complement to  $P$  is dense, too. Hence, the previous lemma implies that the set  $Z$  is homeomorphic to the space  $\mathbb{N}^\omega$ . It suffices to put  $D = (E \setminus P) \cup C$ .

**Theorem 3.12** *Let  $E_1$  and  $E_2$  be two Polish spaces, let  $X$  be a Borel subset of  $E_1$  and let a mapping*

$$g : X \rightarrow E_2$$

*be injective and continuous. Then the set  $g(X)$  is a Borel subset of  $E_2$ .*

**Proof.** The last lemma states that each uncountable zero-dimensional Polish topological space after removing from it at most countable subset is homeomorphic to the space  $\mathbb{N}^\omega$ . The remark made after the proof of Theorem 8 says that any Borel subset of a Polish topological space is an injective continuous image of some Polish subspace of the zero-dimensional Baire space  $\mathbb{N}^\omega$ . So it is sufficient to prove that for each injective continuous mapping  $g$  from  $\mathbb{N}^\omega$  into a Polish topological space  $E$  the set  $g(\mathbb{N}^\omega)$  is a Borel subset of  $E$ . Let us prove this assertion.

If a natural index  $k$  is fixed then the analytic sets

$$g(\mathbb{N}^\omega(s)) \quad (s \in \mathbb{N}^k)$$

are pairwise disjoint since the mapping  $g$  is injective. Hence, it follows from the separation principle for analytic sets that there are pairwise disjoint Borel sets

$$B_s \quad (s \in \mathbb{N}^k)$$

in the space  $E$  such that

$$g(\mathbb{N}^\omega(s)) \subseteq B_s.$$

We define by recursion the following sets:

$$B_{(n_0)}^* = B_{(n_0)} \cap cl(g(\mathbb{N}^\omega((n_0))))),$$

$$B_{s*n}^* = B_{s*n} \cap cl(g(\mathbb{N}^\omega(s*n))) \cap B_s^*.$$

From this definitions, using the induction on length of  $s$ , we can immediately check the inclusion

$$g(\mathbb{N}^\omega(s)) \subseteq B_s^* \subseteq cl(g(\mathbb{N}^\omega(s))).$$

Hence, for any  $z \in \mathbb{N}^\omega$  we get

$$g(z) = \bigcap_k B_{z|k}^*.$$

Therefore, we have

$$g(\mathbb{N}^\omega) = \bigcup_{z \in \mathbb{N}^\omega} \bigcap_k B_{z|k}^*,$$

where the sets  $B_{(n_0, \dots, n_k)}^*$  are Borel and pairwise disjoint when  $k$  is fixed. Finally, we apply Theorem 4 to the system of sets

$$(B_s^*)_{s \in \mathbb{N}^{<\omega}}$$

and get that the set  $g(\mathbb{N}^\omega)$  is a Borel subset of the space  $E$ .

Comparing the theorem proved above to Theorem 8 we see that Borel subsets of Polish topological spaces can be characterized as injective continuous images of  $G_\delta$ -subsets of the canonical Baire space  $\mathbb{N}^\omega$ .

The following result generalizes the preceding theorem.

**Theorem 3.13** *Let  $E_1$  and  $E_2$  be any two Polish topological spaces, let  $X$  be a Borel subset of  $E_1$  and let*

$$g : X \rightarrow E_2$$

*be some injective Borel mapping. Then the set  $g(X)$  is a Borel subset of the space  $E_2$ .*

**Proof.** In the topological product  $X \times E_2$  we consider a graph of the mapping  $g$ , i.e. the set

$$\Gamma(g) = \{(x, y) \in X \times E_2 : y = g(x)\}.$$

First, let us show that this set is a Borel subset of the product  $X \times E_2$ . For this purpose let us consider the mapping

$$\varphi : X \times E_2 \rightarrow \mathbb{R},$$

defined by the formula

$$\varphi(x, y) = \rho(g(x), y),$$

where  $\rho$  is any metric in the space  $E_2$ . The mapping  $\varphi$  is a composition of Borel mappings. Hence, it is a Borel mapping, too. Thus, the set

$$\Gamma(g) = \varphi^{-1}(\{0\})$$

is a Borel subset of  $X \times E_2$ . Finally, let us consider the mapping

$$pr_2|_{\Gamma(g)} : \Gamma(g) \rightarrow E_2,$$

which is the restriction of the canonical projection  $pr_2$  to the set  $\Gamma(g)$ . Then we have  $pr_2(\Gamma(g)) = g(X)$  and it is clear that  $pr_2|_{\Gamma(g)}$  is an injective continuous mapping. Hence, using the preceding theorem, we get the required result.

In such a way we see that if  $X$  and  $Y$  are some Borel subsets of Polish topological spaces and

$$g : X \rightarrow Y$$

is a bijective Borel mapping then the converse mapping

$$g^{-1} : Y \rightarrow X$$

is a Borel mapping, too. In other words, in such a case  $g$  is a Borel isomorphism between the sets  $X$  and  $Y$ . Using this result let us prove the following theorem.

**Theorem 3.14** *Let  $X$  and  $Y$  be any Borel subsets of Polish topological spaces. If  $card(X) = card(Y)$ , then these sets are Borel isomorphic.*

**Proof.** A Borel isomorphism between the Cantor space  $\{0, 1\}^\omega$  and the unit closed interval  $[0, 1]$  can be constructed directly. Hence, it follows that

$$\{0, 1\}^\omega = (\{0, 1\}^\omega)^\omega$$

is Borel isomorphic to the Hilbert cube  $[0, 1]^\omega$ . From this fact and the fact that every separable metric space is topologically contained in the Hilbert cube  $[0, 1]^\omega$  we deduce that any separable metric space is Borel isomorphic with some subset of the Cantor space  $\{0, 1\}^\omega$ . Now, let  $X$  and  $Y$  be given Borel subsets of Polish topological spaces. Without loss of generality we may assume that both sets  $X$  and  $Y$  are uncountable. Hence, any of these sets contains a subset which is homeomorphic to the Cantor space  $\{0, 1\}^\omega$ . Thus, we see that  $X$  is Borel isomorphic to some Borel

subset of  $Y$  and, conversely, that  $Y$  is Borel isomorphic to some Borel subset of  $X$ . Now we can apply the Banach theorem, which was proved in Chapter 1, and immediately get the required result.

The theorem proved above is of primary importance because it allows us in many cases to consider a concrete uncountable Borel set and automatically extend obtained results onto any other uncountable Borel set. In particular, we have that all Polish topological spaces of the same cardinality are Borel isomorphic to each other. We add to this result the following useful theorem.

**Theorem 3.15** *Let  $E_1$  and  $E_2$  be arbitrary uncountable Polish topological spaces without isolated points. Then there exists a Borel isomorphism*

$$g : E_1 \rightarrow E_2,$$

*which preserves the first category sets, i.e. for any  $X \subseteq E_1$  we have*  
 $X$  *is first category set*  $\iff$   $g(X)$  *is first category set.*

*In particular,  $g$  preserves the Baire property.*

**Proof.** After removing from both spaces  $E_1$  and  $E_2$  nowhere dense boundaries of all balls from two countable families we get zero-dimensional Polish spaces  $E_1^*$  and  $E_2^*$ . Now, let  $X_1$  and  $X_2$  be any two dense  $G_\delta$ -subsets of  $E_1^*$  and  $E_2^*$ , respectively, such that their complements to  $E_1^*$  and  $E_2^*$ , respectively, are dense, too. As we know, the sets  $X_1$  and  $X_2$  are homeomorphic to the canonical Baire space  $\mathbb{N}^\omega$  and so they are homeomorphic to each other. Let

$$\varphi : X_1 \rightarrow X_2$$

be any homeomorphism between  $X_1$  and  $X_2$ . Remark that the sets  $E_1 \setminus X_1$  and  $E_2 \setminus X_2$  are first category Borel subsets of  $E_1$  and  $E_2$ , respectively. Let us notice that without loss of generality we may assume that they have the same cardinality. Finally, let

$$\psi : E_1 \setminus X_1 \rightarrow E_2 \setminus X_2$$

be any Borel isomorphism. Then it is easy to see that the common extension of functions  $\varphi$  and  $\psi$  is the required Borel isomorphism  $g$ .

Let us remark also that an analogous result can be formulated in terms of measure theory. This fact will be discussed later on in the book.

Till now we have considered various properties of Borel and analytic sets. These classes of sets are important for applications, for instance in analysis, in measure theory, in probability theory and other branches of mathematics. But in the descriptive set theory more general classes of sets are investigated, too. An example of such a class of sets are the *projective* sets which were introduced by Luzin and Sierpiński. This notion has not so many applications as Borel and analytic sets. However, this notion occurs to be rather deep and connected with the foundations of mathematics. Moreover, it is worth remarking that from the point of view of projective sets many aspects and properties of Borel and analytic sets are more clearly seen. We discuss projective sets and some of their properties in Part 2 of this book. Here we shall only give a classical definition of these sets.

Let  $E$  be an arbitrary Polish topological space. We define the classes of sets

$$Pr_0(E), Pr_1(E), \dots, Pr_n(E), \dots$$

by recursion. Let us put

$$Pr_0(E) = B(E).$$

Suppose now that for natural number  $n \geq 1$  the classes  $Pr_k(E)$ , where  $k < n$ , are defined. If  $n$  is odd then  $Pr_n(E)$  is the class of all continuous images (in  $E$ ) of sets from the class  $Pr_{n-1}(E)$ . If  $n$  is even then  $Pr_n(E)$  is the class of all complements of the sets from the class  $Pr_{n-1}(E)$ . Now we put

$$Pr(E) = \bigcup_n Pr_n(E).$$

Projective subsets of the topological space  $E$  are members of the class  $Pr(E)$ . Hence, we see that the Borel subsets of  $E$  (i.e. sets from the class  $Pr_0(E)$ ) and the analytic subsets of  $E$  (i.e. sets from the class  $Pr_1(E)$ ) are only very particular cases of projective sets. Let us notice at this place that many natural questions can be solved for Borel and analytic subsets of Polish spaces (questions about cardinality, measurability, Baire property etc.). Analogous questions for projective sets of higher levels are, as a rule, undecidable by the contemporary set theory.

## Exercises

**Exercise 3.1** Prove Theorem 1 from this Chapter.

**Exercise 3.2** Prove Theorem 2 from this Chapter.

**Exercise 3.3** Let  $E$  be any non-empty topological space. Show that

$$\text{card}(B(E)) \leq \text{card}(T(E))^\omega,$$

where  $T(E)$  is the topology of the space  $E$ . Show also that if  $E$  is an infinite Polish topological space, then

$$\text{card}(Pr(E)) = 2^\omega.$$

Deduce from this fact that if  $E$  is an uncountable Polish space, then there are subsets of  $E$  which are not projective (in particular, are not Borel or analytic).

**Exercise 3.4** Prove Theorem 3 from this Chapter.

**Exercise 3.5** Prove Theorem 4 from this Chapter.

**Exercise 3.6** Show that in any Polish topological space  $E$  every uncountable set  $X$  from the class  $Pr_3(E)$  is the union of a family  $(X_\xi)_{\xi < \omega_1}$  of non-empty pairwise disjoint Borel subsets of  $E$ .

**Exercise 3.7** Let  $C$  be the standard Cantor set on the unit closed interval  $[0, 1]$  and let  $D$  be the set of all end-points of the connected components of  $[0, 1] \setminus C$ . Show that the set  $C \setminus D$  is homeomorphic to the set  $\mathbb{Z}$  of all irrational numbers. Starting from this fact construct two bijective, continuous mappings

$$g_1 : \mathbb{N}^\omega \rightarrow [0, 1],$$

$$g_2 : \mathbb{N}^\omega \rightarrow [0, 1]^\omega.$$

**Exercise 3.8** Prove Lemma 2 from this Chapter.

**Exercise 3.9** Prove Lemma 3 from this Chapter.

**Exercise 3.10** Prove Lemma 4 from this Chapter.

**Exercise 3.11** Deduce directly from the separation principle for analytic sets that if  $E_1$  and  $E_2$  are two Polish topological spaces and

$$g : E_1 \rightarrow E_2$$

is a bijective Borel mapping, then  $g$  is a Borel isomorphism between these spaces.

Let again  $E_1$  and  $E_2$  be two Polish topological spaces and let

$$g : E_1 \rightarrow E_2$$

be some mapping. Show that if the graph  $\Gamma(g)$  of the mapping  $g$  is an analytic subset of the topological product  $E_1 \times E_2$ , then  $g$  is a Borel mapping and hence, the graph  $\Gamma(g)$  is a Borel subset of  $E_1 \times E_2$ . Prove a more general version of this proposition when in the place of Polish topological spaces  $E_1$  and  $E_2$  we take two arbitrary Borel sets  $X_1 \subseteq E_1$  and  $X_2 \subseteq E_2$ .

**Exercise 3.12** Let  $X$  be any analytic subset of a Polish topological space, let  $E$  be any metrizable topological space and let

$$g : X \rightarrow E$$

be a Borel mapping. Prove that the set  $g(X) \subseteq E$  is separable.

**Exercise 3.13** Suppose that the inequality

$$2^{\omega_0} < 2^{\omega_1}$$

holds (this inequality follows, for instance, from the Continuum Hypothesis). Let  $E_1$  and  $E_2$  be any two metrical spaces such that  $E_1$  is separable and let

$$g : E_1 \rightarrow E_2$$

be any Borel mapping from  $E_1$  into  $E_2$ . Prove that the set  $g(E_1) \subseteq E_2$  is separable. Is the assumption of metrizability of the space  $E_2$  essential here?

**Exercise 3.14** Prove Lemma 5 from this Chapter.

**Exercise 3.15** Let  $E$  be an arbitrary metric space and let  $X$  be a subset of  $E$ . Let

$$g : X \rightarrow \mathbb{R}$$

be a Borel mapping. Show that there exist a Borel set  $X^* \subseteq E$  and a Borel mapping

$$g^* : X^* \rightarrow \mathbb{R}$$

such that  $X \subseteq X^*$  and  $g^*|_X = g$ .

**Exercise 3.16** Let  $E$  be an arbitrary Polish topological space of the cardinality continuum. Show that there exists a real function

$$g : E \rightarrow \mathbb{R}$$

such that for every set  $X \subseteq E$  of the cardinality continuum the restriction

$$g|_X : X \rightarrow \mathbb{R}$$

is not a Borel mapping.

**Exercise 3.17** Let  $K = \{0, 1\}^\omega \times [0, 1]$ . It is obvious that the topological space  $K$  is compact, so the space  $\text{Comp}(K)$  is compact, too. Using the fact that there exists a continuous surjection

$$g : \{0, 1\}^\omega \rightarrow \text{Comp}(K),$$

prove that the space  $\{0, 1\}^\omega$  contains an analytic set which is not a Borel subset of this space. Deduce also from this fact that any uncountable Polish space contains an analytic subset which is not Borel.

**Exercise 3.18** Give an example of two Polish topological spaces  $E_1$  and  $E_2$  without isolated points and of a Borel isomorphism

$$g : E_1 \rightarrow E_2$$

between these spaces such that  $g$  does not preserve the Baire property.





## Chapter 4

# Some Facts from Measure Theory

A measure of a set is a generalization of the notion of the length of an interval, the area of a plane figure and the volume of a three-dimensional body. The notion of a measurable set appeared in the real function theory during investigations and generalizations of the concept of integral. A classical example of a measure is the Lebesgue measure on the real line  $\mathbb{R}$ , defined by Lebesgue in 1902. It extends the notion of the length of an interval onto a much bigger class of subsets of  $\mathbb{R}$ . This class of sets contains all Borel and all analytic subsets of the real line and many other subsets of  $\mathbb{R}$ .

In this section we shall remind the reader of some well known basic facts from general measure theory and then we will discuss some theorems from this domain of mathematics which are deeper or more specific.

We suppose, of course, that the reader is familiar with some definitions and facts from measure theory (for instance, that he knows the first chapters of the popular Halmos book on this subject).

Let  $E$  be a non-empty basic set, let  $S$  be any algebra of subsets of  $E$  and let  $\mu$  be a function from  $S$  into the extended real line

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$$

such that

$$\text{card}(\text{ran}(\mu) \cap \{-\infty, +\infty\}) \leq 1.$$

We say that  $\mu$  is **finitely additive** (or **additive**) if for every finite family  $\{X_1, \dots, X_n\} \subseteq S$  of pairwise disjoint sets we have

$$\mu\left(\bigcup_i X_i\right) = \sum_{i=1}^n \mu(X_i).$$

Similarly, we say that  $\mu$  is **countably additive** (or  **$\sigma$ -additive**) if for every countable family  $(X_i)_{i \in \mathbb{N}} \subseteq S$  of pairwise disjoint sets such that  $\bigcup_i X_i \in S$  we have

$$\mu\left(\bigcup_i X_i\right) = \sum_{i=1}^{\infty} \mu(X_i).$$

Finally, we say that a function  $\mu : S \rightarrow \mathbb{R}^*$  is a **measure** (defined on the algebra  $S$ ) if the following conditions hold:

- a)  $\mu(\emptyset) = 0$ ;

b)  $(\forall X \in S)(\mu(X) \geq 0)$ ;

c)  $\mu$  is  $\sigma$ -additive.

A **measure space** is a triple  $(E, S, \mu)$ , where  $E$  is a non-empty basic set,  $S$  is an algebra of subsets of  $E$  and  $\mu$  is a measure on  $S$ .

A measure  $\mu$  is called  **$\sigma$ -finite** if there exists a countable family  $(X_n)_{n \in \mathbb{N}}$  of subsets of  $S$  such that

$$\bigcup_n X_n = E, \quad (\forall n \in \mathbb{N})(\mu(X_n) < +\infty).$$

We remark here that the measures met most frequently are  $\sigma$ -finite, and we shall mainly deal with  $\sigma$ -finite measures in the sequel.

A measure  $\mu$  is called **finite** if

$$(\forall X \in S)(\mu(X) < +\infty).$$

Since  $S$  is an algebra of subsets of  $E$ , the last relation is equivalent to the relation  $\mu(E) < +\infty$ .

We say that a measure  $\mu$  is a **probability measure** if  $\mu(E) = 1$ .

We say that a measure  $\mu$  is **complete** if for every  $X \in S$  with  $\mu(X) = 0$  we have

$$(\forall Y)(Y \subseteq X \rightarrow Y \in S).$$

A measure  $\mu$  is called **diffused** (or continuous) if

$$(\forall x \in E)(\{x\} \in S \ \& \ \mu(\{x\}) = 0).$$

For any non-zero measure  $\mu$  we denote

$$L(\mu) = \{X : (\exists Y)(X \subseteq Y \ \& \ Y \in S \ \& \ \mu(Y) = 0)\}.$$

It is clear that the class  $L(\mu)$  is a  $\sigma$ -ideal of subsets of  $E$ . Moreover, one can see that the measure  $\mu$  is complete if and only if  $L(\mu) \subseteq S$ .

The members of the class  $L(\mu)$  are called  **$\mu$ -measure zero sets** or  **$\mu$ -negligible sets**.

The following fact is fundamental for the whole measure theory.

**Theorem 4.1 (Caratheodory)** *Let  $\mu$  be a measure on an algebra  $S$  of subsets of a basic set  $E$ . Then there exists a measure extending the original measure  $\mu$  onto the  $\sigma$ -algebra generated by the algebra  $S$ . If the original measure  $\mu$  is  $\sigma$ -finite, then this extension is unique.*

So we see that in the class of  $\sigma$ -finite measures we can consider only such measures which are defined on  $\sigma$ -algebras.

We shall remind the reader of some details of the construction of the extension of a measure from an algebra to the  $\sigma$ -algebra generated by this algebra.

Suppose that  $\mu$  is a measure on an algebra  $S$  of subsets of  $E$ . We define a real function  $\mu^*$  on the class  $P(E)$  by the formula

$$\mu^*(X) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu(Y_n) : (Y_n)_{n \in \mathbb{N}} \subseteq S \ \& \ X \subseteq \bigcup_{n \in \mathbb{N}} Y_n \right\}.$$

This function is called the **outer measure** associated with  $\mu$ .

We say that a subset  $Z$  of  $E$  is  **$\mu^*$ -measurable** if for any set  $X \subseteq E$  the following **Caratheodory condition** holds :

$$\mu^*(X \cap Z) + \mu^*(X \cap (E \setminus Z)) = \mu^*(X).$$

It can be shown that the class of all  $\mu^*$ -measurable sets  $Z \subseteq E$  is a  $\sigma$ -algebra of subsets of  $E$  which contains the original algebra  $S$ . Moreover, the function  $\mu^*$  considered only on the class of all  $\mu^*$ -measurable sets is  $\sigma$ -additive, so it is a measure. It extends also the original measure  $\mu$  and is complete. These facts immediately imply Theorem 1.

It is sometimes useful to consider with the outer measure its dual notion - the **inner measure**. Recall here that if the original measure  $\mu$  is defined on a  $\sigma$ -algebra  $S$ , then the inner measure associated with  $\mu$  is a function  $\mu_*$  defined on the class  $P(E)$  by the formula

$$\mu_*(X) = \sup\{\mu(Y) : Y \in S \text{ \& } Y \subseteq X\}.$$

A set  $X \subseteq E$  is called  $\mu$ -**massive** (or a set with **full outer measure** with respect to  $\mu$ ) if the equality

$$\mu_*(E \setminus X) = 0$$

holds. It is easy to see that when we have a finite measure  $\mu$ , a set  $X \subseteq E$  is  $\mu$ -massive if and only if

$$\mu^*(X) = \mu(E).$$

Notice that the complements of  $\mu$ -massive subsets of  $E$  may be successfully applied to various problems connected with extensions of measures.

During the discussion about extensions of measures it is impossible to omit one construction frequently met.

Let  $S$  be any  $\sigma$ -algebra of subsets of a basic set  $E$  and let  $I$  be any  $\sigma$ -ideal of subsets of  $E$ . Let us put

$$S(I) = \sigma(S \cup I),$$

where  $\sigma(S \cup I)$ , as usual, denotes the  $\sigma$ -algebra generated by the class  $S \cup I$ . It is easy to check that

$$S(I) = \{X \Delta Y : X \in S \text{ \& } Y \in I\}.$$

**Theorem 4.2** *Let  $(E, S, \mu)$  be a measure space and let  $I$  be a  $\sigma$ -ideal of subsets of  $E$  such that*

$$(\forall Y)(Y \in I \rightarrow \mu_*(Y) = 0).$$

*Then the formula*

$$\nu(X \Delta Y) = \mu(X) \quad (X \in S, Y \in I)$$

*correctly defines a measure  $\nu$  on the  $\sigma$ -algebra  $S(I)$  which extends the original measure  $\mu$ .*

**Proof.** Let us check that the function  $\nu$  is well defined. Suppose hence that

$$X_1, X_2 \in S, \quad Y_1, Y_2 \in I, \quad X_1 \Delta Y_1 = X_2 \Delta Y_2.$$

Then we have

$$X_1 \Delta X_2 = Y_1 \Delta Y_2,$$

so  $X_1 \Delta X_2 \in I$  and  $\mu(X_1 \Delta X_2) = 0$ . Therefore,  $\mu(X_1) = \mu(X_2)$ . In the same way it may be checked that the function  $\nu$  is  $\sigma$ -additive. Finally, it is clear that  $\nu$  extends the original measure  $\mu$ .

If  $(E, S, \mu)$  is not a complete measure space, then we can apply Theorem 2 to the ideal  $L(\mu)$  and extend the measure  $\mu$  to a complete measure  $\bar{\mu}$  on  $S(L(\mu))$ . The obtained space

$$(E, S(L(\mu)), \bar{\mu})$$

is called the **measure completion** of  $(E, S, \mu)$ .

**Example 1.** Let us recall the construction of the classical **Lebesgue measure**  $\lambda$  on the real line  $\mathbb{R}$ . Consider first the class  $S$  of subsets of  $\mathbb{R}$  which consists of all finite unions of half-open intervals, i.e. the class of sets of the form

$$\bigcup_{1 \leq i \leq n} [a_i, b_i[,$$

where  $n \in \mathbb{N}$  and

$$-\infty < a_1 \leq b_1 \leq a_2 \leq b_2 \leq \dots \leq a_n \leq b_n < +\infty.$$

Note that the class  $\sigma(S)$  coincides with the  $\sigma$ -algebra  $B(\mathbb{R})$  of all Borel subsets of  $\mathbb{R}$ . We define a function  $\lambda$  for sets from the class  $S$  by the following formula:

$$\lambda \left( \bigcup_{1 \leq i \leq n} [a_i, b_i[ \right) = \sum_{i=1}^n (b_i - a_i).$$

The correctness of this definition and the additivity of  $\lambda$  can easily be checked. Moreover, the compactness of any closed and bounded interval in  $\mathbb{R}$  implies that the function  $\lambda$  is  $\sigma$ -additive. Hence, by Theorem 1  $\lambda$  can be extended to a measure defined on the class  $\sigma(S) = B(\mathbb{R})$ .

**Lebesgue measurable sets** on  $\mathbb{R}$  are the  $\lambda^*$ -measurable subsets of  $\mathbb{R}$  and the **Lebesgue measure**, denoted also by  $\lambda$ , is the restriction of the function  $\lambda^*$  to the class of all Lebesgue measurable sets.

The class of Lebesgue measurable subsets of  $\mathbb{R}$  is much wider than the Borel class  $B(\mathbb{R})$ . In order to show this let us recall the construction of the classical Cantor subset  $C$  of  $\mathbb{R}$ .

Let  $T = \{0, 1\}^{<\omega}$  be the complete binary tree of height  $\omega$ , i.e. the family of all finite sequences with ranges contained in  $\{0, 1\}$ . Let  $(J_s)_{s \in T}$  be a family of closed intervals on  $\mathbb{R}$  such that

$$J_{\emptyset} = [0, 1],$$

and for any  $s \in T$  we have

- a)  $J_{s*0} \cap J_{s*1} = \emptyset$ ,
- b)  $J_{s*0} \cup J_{s*1} \subseteq J_s$ ,
- c) the length of  $J_s$  is equal to  $(\frac{1}{3})^{lh(s)}$ ,
- d) the left end-point of  $J_{s*0}$  coincides with the left end-point of  $J_s$  and the right end-point of  $J_{s*1}$  coincides with the right end-point of  $J_s$ .

For any index  $n \in \mathbb{N}$  we put

$$F_n = \bigcup \{J_s : lh(s) = n\}.$$

The Cantor set  $C$  is defined by the formula

$$C = \bigcap_n F_n.$$

Note that for each  $n \in \mathbb{N}$  we have

$$\lambda(F_n) = \left(\frac{2}{3}\right)^n,$$

hence, the equality

$$\lim_n \lambda(F_n) = 0$$

holds. Therefore,  $\lambda(C) = 0$ . It is not difficult to prove that  $C$  is a nowhere dense perfect subset of  $\mathbb{R}$  with the cardinality continuum. It is also clear that the set  $C$  is homeomorphic to the Cantor discontinuum  $\{0, 1\}^\omega$ . Since the Lebesgue measure is complete we see that any subset of  $C$  is  $\lambda$ -measurable. Thus, there are  $2^{\mathfrak{c}}$  Lebesgue measurable subsets of  $\mathbb{R}$ . On the other hand, we know that  $\text{card}(B(\mathbb{R})) = \mathfrak{c}$ , so we see that there are Lebesgue measurable sets on  $\mathbb{R}$  which are not Borel subsets of  $\mathbb{R}$ .

As mentioned above, the family of all Lebesgue measurable subsets of  $\mathbb{R}$  has the same cardinality as the family of all subsets of  $\mathbb{R}$ . However, it can be proved (using some uncountable forms of the Axiom of Choice) that there exist subsets of  $\mathbb{R}$  which are not Lebesgue measurable. In connection with this fact it is necessary to note here that we can sometimes use Theorem 2 and obtain certain interesting extensions of the Lebesgue measure which are defined on wider classes of sets. Some problems of this type will be considered more deeply in Part 2 of the book.

Now we formulate and prove two simple classical facts, which are usually called the Borel-Cantelli lemmas.

Recall that if  $(X_n)_{n \in \mathbb{N}}$  is any sequence of sets, then its upper limit (in the set-theoretical sense) is defined by the formula

$$\limsup(X_n)_{n \in \mathbb{N}} = \bigcap_n \left( \bigcup_{m > n} X_m \right).$$

It is easy to see that  $\limsup(X_n)_{n \in \mathbb{N}}$  is the set of all elements which belong to infinitely many members of the sequence  $(X_n)_{n \in \mathbb{N}}$ .

**Theorem 4.3** *Let  $(E, S, \mu)$  be a measure space and let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of  $\mu$ -measurable subsets of  $E$ . If the series  $\sum_n \mu(X_n)$  is convergent, then*

$$\mu(\limsup(X_n)_{n \in \mathbb{N}}) = 0.$$

**Proof.** Let us fix  $i \in \mathbb{N}$ . Then we have

$$\mu(\limsup(X_n)_{n \in \mathbb{N}}) \leq \mu\left(\bigcup_{m > i} X_m\right) \leq \sum_{m > i} \mu(X_m).$$

But since the series  $\sum_n \mu(X_n)$  is convergent we have

$$\lim_i \sum_{m > i} \mu(X_m) = 0.$$

This immediately gives us the required result.

Now suppose that  $(E, S, \mu)$  is a probability measure space. A family  $T \subseteq S$  is called  $\mu$ -**independent** if for every finite family  $\{X_1, \dots, X_n\} \subseteq T$  of pairwise different sets the equality

$$\mu(X_1 \cap \dots \cap X_n) = \mu(X_1) \cdot \dots \cdot \mu(X_n)$$

holds. It is easy to see that if a family  $T \subseteq S$  is  $\mu$ -independent, then for every finite family  $\{X_1, \dots, X_n\} \subseteq T$  of pairwise different sets we have

$$\mu(Y_1 \cap \dots \cap Y_n) = \mu(Y_1) \cdot \dots \cdot \mu(Y_n),$$

where

$$(\forall i)(1 \leq i \leq n \rightarrow (Y_i = X_i \vee Y_i = E \setminus X_i)).$$

**Theorem 4.4** Let  $(E, S, \mu)$  be a probability measure space. If  $(X_n)_{n \in \mathbb{N}} \subseteq S$  is a sequence of  $\mu$ -independent sets and the series  $\sum_n \mu(X_n)$  is divergent, then

$$\mu(\limsup(X_n)_{n \in \mathbb{N}}) = 1.$$

**Proof.** It is sufficient to prove that

$$\mu(E \setminus (\limsup(X_n)_{n \in \mathbb{N}})) = 0,$$

i.e. to prove that

$$\mu\left(\bigcap_{i \geq n} (E \setminus X_i)\right) = 0$$

for every  $n \in \mathbb{N}$ . Let us fix  $n$  and  $m > n$ . Then we have

$$\begin{aligned} \mu\left(\bigcap_{n \leq i \leq m} (E \setminus X_i)\right) &= \prod_{i=n}^m (1 - \mu(X_i)) \leq \\ &\prod_{i=n}^m \exp(-\mu(X_i)) = \exp\left(-\sum_{i=n}^m \mu(X_i)\right). \end{aligned}$$

But since the series  $\sum_n \mu(X_n)$  is divergent we have

$$\lim_m \left(\exp\left(-\sum_{i=n}^m \mu(X_i)\right)\right) = 0.$$

Hence, we obtain

$$\mu\left(\bigcap_{i \geq n} (E \setminus X_i)\right) = \lim_m \mu\left(\bigcap_{n \leq i \leq m} (E \setminus X_i)\right) = 0$$

for every  $n \in \mathbb{N}$ . This ends the proof of the theorem.

Let  $E$  be an arbitrary basic set. By a real function on a set  $E$  we shall understand a function defined on  $E$  with the range contained in the extended real line

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}.$$

To avoid possible misunderstanding let us recall that  $\mathbb{R}^*$  is homeomorphic with the closed interval  $[-1, 1]$ . In all further considerations we shall use the following convention:

$$0 \cdot (-\infty) = 0 \cdot (+\infty) = 0.$$

Let  $E$  be a basic set again and let  $S$  be a  $\sigma$ -algebra of subsets of  $E$ . The pair  $(E, S)$  is usually called a **measurable space**.

A real function  $f$  on  $E$  is called  **$S$ -measurable** if for every Borel set  $X \subseteq \mathbb{R}^*$  the set  $f^{-1}(X)$  is in  $S$ . The simplest examples of  $S$ -measurable functions are the characteristic functions  $1_Y$  where  $Y \in S$ . Any linear combination of several characteristic functions is called a **step function**.

The following theorem shows that the class of all  $S$ -measurable functions is closed under all natural algebraic operations.

**Theorem 4.5** Let  $E$  be a basic set and let  $S$  be some  $\sigma$ -algebra of subsets of  $E$ . Suppose that

$$\Phi : \mathbb{R}^* \times \mathbb{R}^* \rightarrow \mathbb{R}^*$$

is a  $B(\mathbb{R}^* \times \mathbb{R}^*)$ -measurable function and suppose that  $f$  and  $g$  are two  $S$ -measurable functions. Then the function  $h : E \rightarrow \mathbb{R}^*$  defined by the formula

$$h(x) = \Phi(f(x), g(x)) \quad (x \in E)$$

is  $S$ -measurable, too.

The simple proof of this theorem is left to the reader.

Let  $f : E \rightarrow \mathbb{R}^*$  be a function. We put

$$f^+(x) = \max\{f(x), 0\} \quad (x \in E),$$

$$f^-(x) = \max\{-f(x), 0\} \quad (x \in E).$$

It is evident that

$$f^+ \geq 0, \quad f^- \geq 0, \quad f = f^+ - f^-.$$

From Theorem 5 we can deduce that the function  $f$  is  $S$ -measurable if and only if both functions  $f^+$  and  $f^-$  are  $S$ -measurable.

The class of  $S$ -measurable real functions is also closed under limit operations : if  $(f_n)_{n \in \mathbb{N}}$  is any sequence of  $S$ -measurable functions and

$$f = \lim_n f_n, \quad g = \inf_n f_n, \quad h = \sup_n f_n,$$

then  $f, g$  and  $h$  are also  $S$ -measurable functions.

Let  $(E, S, \mu)$  be a measure space, let  $X$  be a  $\mu$ -measurable subset of  $E$  and let  $f : E \rightarrow \mathbb{R}$  be an  $S$ -measurable and nonnegative function.

The  $\mu$ -**integral** of the function  $f$  on the set  $X$  is defined by the formula

$$\int_X f d\mu = \sup \left\{ \sum_n (\inf f|_{X_n}) \cdot \mu(X_n) : (X_n)_{n \in \mathbb{N}} \subseteq S \text{ is a partition of the set } X \right\}.$$

In many cases the real number  $\int_X f d\mu$  will be also denoted by the symbol

$$\int_X f(x) d\mu(x).$$

Suppose now that  $f : E \rightarrow \mathbb{R}$  is any  $S$ -measurable function. Then we put

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu,$$

if at least one of the integrals from the right side of this equality is finite. If both integrals from the right side of this equality are finite, then we say that the function  $f$  is **integrable** on the set  $X$  and the real number  $\int_X f d\mu$  is called the  $\mu$ -**integral** of  $f$  on the set  $X$ . We say that the function  $f$  is  $\mu$ -integrable if it is  $\mu$ -integrable on the whole basic set  $E$ .

The class of all  $\mu$ -integrable functions on  $E$  is a Banach space with respect to the norm

$$\|f\| = \int_E (f^+ + f^-) d\mu.$$

Of course, we identify here the functions which are equivalent with respect to the measure  $\mu$ , i.e. we identify the functions which coincide almost everywhere (with respect to  $\mu$ ) on the basic set  $E$ .

We assume that the reader knows some standard facts about integrable real functions such as the Lebesgue theorem on majorated convergence, the Fatou lemma, absolute continuity of integrals etc.

Let us take a look once more at the notion of a measure space with a  $\sigma$ -finite measure. Suppose that  $(E, S, \mu)$  is such a space and assume that  $\mu(E) = +\infty$ . Let  $(X_n)_{n \in \mathbb{N}} \subseteq S$  be a countable family of pairwise disjoint sets such that  $\bigcup_n X_n = E$  and

$$0 < \mu(X_n) < +\infty$$



for each  $n \in \mathbb{N}$ . Let us consider the measure  $\nu$  on the  $\sigma$ -algebra  $S$  defined by the formula

$$\nu(X) = \sum_n \frac{1}{2^{n+1}} \cdot \frac{\mu(X \cap X_n)}{\mu(X_n)} \quad (X \in S).$$

Observe that  $\nu$  is a probability measure on  $S$ . If  $X$  is an arbitrary set from  $S$ , then  $\nu(X) > 0$  if and only if  $\mu(X) > 0$ . In this case we say that the measures  $\mu$  and  $\nu$  are equivalent. Hence, we can conclude that if in some considerations we investigate only the notion of "being a strictly positive measure set", or "being a measure zero set," then we can restrict our attention from the class of  $\sigma$ -finite measures to the class of probability measures. Obviously, we cannot do this replacement of measure when we deal with the class of all integrable functions, or when we calculate the precise value of a measure of a given set.

Let  $I$  be a non-empty set of indices and suppose that  $((E_i, S_i, \mu_i))_{i \in I}$  is a family of probability spaces. We shall define the product

$$(E, S, \mu) = \otimes_{i \in I} (E_i, S_i, \mu_i)$$

of this family of spaces.

Let  $E = \prod_{i \in I} E_i$  be the Cartesian product of the family of basic sets  $(E_i)_{i \in I}$ . A subset  $X$  of  $E$  is called a **rectangular set** if it can be represented in the form

$$X = \prod_{i \in I} X_i,$$

where  $X_i \in S_i$  for every  $i \in I$  and the set

$$\{i \in I : X_i \neq E_i\}$$

is finite. The family of all rectangular subsets  $X$  of  $E$  is denoted by the symbol  $P_0$  and the family of all finite unions of rectangular sets is denoted by the symbol  $P$ . Obviously, the family  $P$  is an algebra of subsets of  $E$  generated by  $P_0$ . We define a function

$$\mu : P_0 \rightarrow \mathbb{R}$$

by the formula

$$\mu\left(\prod_{i \in I} X_i\right) = \prod_{i \in I} \mu_i(X_i).$$

It may be checked that the function  $\mu$  can be uniquely extended to a  $\sigma$ -additive function on the algebra  $P$ . We shall denote this extension by the same symbol  $\mu$ . Hence, by Theorem 1 the measure  $\mu$  can be extended to the uniquely determined measure on the  $\sigma$ -algebra  $S = \sigma(P)$ . The last  $\sigma$ -algebra is denoted by  $\otimes_{i \in I} S_i$ . The extended measure defined on the  $\sigma$ -algebra  $\otimes_{i \in I} S_i$  is denoted by  $\otimes_{i \in I} \mu_i$  and it is called the **product measure** of the family of measures  $(\mu_i)_{i \in I}$ . The product of the family  $((E_i, S_i, \mu_i))_{i \in I}$  is the measure space

$$\left(\prod_{i \in I} E_i, \otimes_{i \in I} S_i, \otimes_{i \in I} \mu_i\right).$$

Notice that quite frequently the measure space corresponding to the completion of the measure  $\otimes_{i \in I} \mu_i$  is also called the product of the mentioned family of measure spaces.

**Example 2.** Let  $I$  be a non-empty set. For every index  $i \in I$  we put  $E_i = \{0, 1\}$ ,  $S_i = P(E_i)$ . Then we define the measure  $\mu_i$  on  $S_i$  by the formula

$$\mu_i(X) = \frac{\text{card}(X)}{2}.$$

It is clear that  $((E_i, \mathcal{S}_i, \mu_i))_{i \in I}$  is a family of probability spaces. By the symbol

$$(\{0, 1\}^I, \otimes_{i \in I} \mathcal{S}_i, \otimes_{i \in I} \mu_i)$$

we shall denote the product probability space of this family. Of course, we can treat each set  $E_i$  as a topological space with the discrete topology, so  $\{0, 1\}^I$  can be treated as a product topological space, called the **generalized Cantor discontinuum**. It is worth remarking here that the  $\sigma$ -algebra  $B(\{0, 1\}^I)$  of all Borel subsets of the space  $\{0, 1\}^I$  is, in general, bigger than the  $\sigma$ -algebra generated by all rectangular subsets of the Cartesian product  $\{0, 1\}^I$ . These two  $\sigma$ -algebras coincide if  $\text{card}(I) \leq \omega$ .

In the previous example the special case when  $I = \mathbb{N}$  is rather important. Let  $f$  be a function from  $\{0, 1\}^{\mathbb{N}}$  into  $[0, 1]$  defined by the following formula:

$$f(x) = \sum_{n \in \mathbb{N}} \frac{x(n)}{2^{n+1}} \quad (x \in \{0, 1\}^{\mathbb{N}}).$$

This function is not a bijection but it is a surjection and its restriction to the set

$$D = \{x \in \{0, 1\}^{\mathbb{N}} : (\forall n)(\exists m > n)(x(m) = 1)\}$$

is a bijection onto  $]0, 1[$ . The complement of  $D$  to  $\{0, 1\}^{\mathbb{N}}$  is countable, hence there exists a bijection

$$g : \{0, 1\}^{\mathbb{N}} \rightarrow [0, 1]$$

such that

$$\text{card}(\{x \in \{0, 1\}^{\mathbb{N}} : f(x) \neq g(x)\}) = \omega.$$

This bijection gives us an isomorphism between the product measure space

$$(\{0, 1\}^{\mathbb{N}}, B(\{0, 1\}^{\mathbb{N}}), \mu)$$

and the probability space

$$([0, 1], B([0, 1]), \lambda),$$

where  $\lambda$  is the restriction of the Lebesgue measure to the class of all Borel subsets of the closed unit interval  $[0, 1]$ .

The construction of the product of probability measure spaces described above can be carried for  $\sigma$ -finite measure spaces in the case when the set of indices  $I$  is finite. In particular, the  $n$ -dimensional Lebesgue measure space can be defined as the product of  $n$  copies of the measure space  $(\mathbb{R}, \text{dom}(\lambda), \lambda)$ , where  $\lambda$  is the Lebesgue measure on  $\mathbb{R}$ .

Let us denote this product measure space by the symbol

$$(\mathbb{R}^n, \text{dom}(\lambda^n), \lambda^n).$$

Let us also sketch another, more direct, construction of the  $n$ -dimensional Lebesgue measure  $\lambda^n$ . An  $n$ -dimensional rectangular parallelepiped in the space  $\mathbb{R}^n$  is a set of the form

$$T = \prod_{1 \leq i \leq n} [a_i, b_i[$$

for some real numbers

$$-\infty < a_1 < b_1 < +\infty, \dots, -\infty < a_n < b_n < +\infty.$$

The  $n$ -dimensional volume of  $T$  is the real strictly positive number

$$\text{vol}(T) = \prod_{i=1}^n (b_i - a_i).$$

Let  $S_n$  denote the algebra generated by all  $n$ -dimensional rectangular parallelepipeds in  $\mathbb{R}^n$ . There exists a unique extension of the volume function  $\text{vol}$  to an additive function on  $S_n$ . This unique extension is a  $\sigma$ -additive function on  $S_n$ . Hence, we may use the Caratheodory theorem and extend this function to a measure on the  $\sigma$ -algebra  $\sigma(S_n) = B(\mathbb{R}^n)$ . Finally, the  $n$ -dimensional Lebesgue measure may be defined as the completion of thus obtained measure.

From this explicit construction of the Lebesgue measure we can easily deduce the following simple formulas for the outer measure  $(\lambda^n)^*$  and the inner measure  $(\lambda^n)_*$  canonically associated with the Lebesgue measure  $\lambda^n$ :

$$(\lambda^n)^*(X) = \inf\{\lambda^n(U) : U \text{ is open in } \mathbb{R}^n \text{ \& } X \subseteq U\},$$

$$(\lambda^n)_*(X) = \sup\{\lambda^n(F) : F \text{ is closed in } \mathbb{R}^n \text{ \& } F \subseteq X\},$$

where  $X$  is an arbitrary subset of the space  $\mathbb{R}^n$ .

From the last property of the inner  $n$ -dimensional Lebesgue measure  $(\lambda^n)_*$  we obtain

$$\lambda^n(X) = \sup\{\lambda^n(K) : K \text{ is compact in } \mathbb{R}^n \text{ \& } K \subseteq X\}$$

for every Lebesgue measurable subset  $X$  of  $\mathbb{R}^n$ . One of the direct applications of this observation is the Luzin characterization of  $\lambda^n$ -measurable real functions, which shows us some analogy between the notions of a measurable function and a function with the Baire property (see Chapter 2).

**Theorem 4.6 (Luzin)** *Suppose that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is given. Then the following two sentences are equivalent:*

- 1) *the function  $f$  is  $\lambda^n$ -measurable;*
- 2) *the function  $f$  has  $C$ -property, i.e. for every  $\epsilon > 0$  there exists a closed set  $F \subseteq \mathbb{R}^n$  such that  $f|_F$  is continuous on  $F$  and  $\lambda^n(\mathbb{R}^n \setminus F) < \epsilon$ .*

**Proof.** Let  $f$  satisfy condition 1). Since the extended real line  $\mathbb{R}^*$  is homeomorphic to the closed interval  $[-1, 1]$  we may assume that  $\text{ran}(f) \subseteq ]-1, 1[$ . Let us consider the sequence  $(f_n)_{n \geq 1}$  of step real functions defined by the formula

$$f_n = \sum_{i=-n}^{i=n-1} \frac{i}{n} \cdot 1_{f^{-1}([\frac{i}{n}, \frac{i+1}{n}[)}.$$

This sequence uniformly converges to the function  $f$ . For every  $n \geq 1$  we can find a closed set  $F_n \subseteq \mathbb{R}^n$  such that  $f_n|_{F_n}$  is continuous on  $F_n$  and

$$\lambda^n(\mathbb{R}^n \setminus F_n) < \frac{\epsilon}{2^n}.$$

Then the set  $F = \bigcap_n F_n$  is also closed in  $\mathbb{R}^n$ , the function  $f|_F$  is continuous on  $F$  and  $\lambda^n(\mathbb{R}^n \setminus F) < \epsilon$ . So, we see that the function  $f$  satisfies condition 2).

We leave the simple proof of the converse implication 2)  $\rightarrow$  1) to the reader.

The Luzin theorem together with the well known Urysohn theorem on extensions of continuous real functions gives us that for any  $\lambda^n$ -measurable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and for any  $\epsilon > 0$  there exists a continuous function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\lambda^n(\{x \in \mathbb{R}^n : f(x) \neq \varphi(x)\}) < \epsilon.$$

The Luzin theorem also implies that if  $f$  is an arbitrary  $\lambda^n$ -measurable real function, then there exist a subset  $X$  of  $\mathbb{R}^n$  and a Borel function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

- a)  $X$  is a  $\sigma$ -compact set in  $\mathbb{R}^n$ , i.e.  $X$  can be represented as a countable union of compact subsets of  $\mathbb{R}^n$ ;
- b)  $\lambda^n(\mathbb{R}^n \setminus X) = 0$ ;
- c)  $f|_X = g|_X$ .

We want to remind the reader of another classical fact from measure theory - the well known Fubini theorem, which reduces the integration of real functions defined on product measure space to integration on the factors.

**Theorem 4.7 (Fubini)** *Let  $(E_1, S_1, \mu_1)$  and  $(E_2, S_2, \mu_2)$  be two measure spaces with  $\sigma$ -finite measures and let*

$$(E, S, \mu) = (E_1, S_1, \mu_1) \otimes (E_2, S_2, \mu_2)$$

. Suppose that  $f : E \rightarrow \mathbb{R}$  is a  $\mu$ -integrable function. Then

- 1) for  $\mu_1$ -almost every  $x \in E_1$  the function

$$y \rightarrow f(x, y) \quad (y \in E_2)$$

is  $\mu_2$ -integrable;

- 2) for  $\mu_2$ -almost every  $y \in E_2$  the function

$$x \rightarrow f(x, y) \quad (x \in E_1)$$

is  $\mu_1$ -integrable;

- 3) the function

$$x \rightarrow \int_{E_2} f(x, y) d\mu_2(y)$$

is  $\mu_1$ -integrable and the function

$$y \rightarrow \int_{E_1} f(x, y) d\mu_1(x)$$

is  $\mu_2$ -integrable ;

- 4) the equalities

$$\begin{aligned} \int_{E_1} \left( \int_{E_2} f(x, y) d\mu_2(y) \right) d\mu_1(x) &= \\ \int_{E_2} \left( \int_{E_1} f(x, y) d\mu_1(x) \right) d\mu_2(y) &= \\ \int \int_{E_1 \times E_2} f(x, y) d(\mu_1(x) \otimes \mu_2(y)) & \end{aligned}$$

hold.

The proof of the Fubini theorem is not connected with any principal difficulties. At the beginning we check the validity of the assertion of this theorem for  $\mu$ -measurable step functions defined on  $E$ , and then we apply the Lebesgue theorem on majorated convergence to get the thesis in the general case.

In an analogous way we can formulate and prove the Fubini theorem for a product of finitely many measure spaces with  $\sigma$ -finite measures.

Let  $S$  be a given  $\sigma$ -algebra of subsets of a basic set  $E$ . A function

$$\nu : S \rightarrow \mathbb{R}^*$$

is called a **signed measure** on  $S$  if

- a)  $\nu(\emptyset) = 0$ ;
- b)  $\text{card}(\text{ran}(\nu) \cap \{-\infty, +\infty\}) \leq 1$ ;
- c)  $\nu$  is  $\sigma$ -additive.

The next result, in fact, reduces the notion of a signed measure to the usual notion of measure.

**Theorem 4.8 (Hahn)** *Suppose that  $\nu$  is a signed measure on a  $\sigma$ -algebra  $S$  of subsets of a basic set  $E$ . Then there exist two sets  $A \subseteq E$  and  $B \subseteq E$  such that*

- 1)  $A \cap B = \emptyset, \quad A \cup B = E$ ;
- 2)  $A \in S, \quad B \in S$ ;
- 3) for every  $X \in S$  we have  $\nu(A \cap X) \geq 0$  and  $\nu(B \cap X) \leq 0$ .

**Proof.** Without loss of generality we may assume that

$$\text{ran}(\nu) \cap \{+\infty\} = \emptyset.$$

Let us put

$$S_0 = \{Y \in S : (\forall Z \subseteq Y)(Z \in S \rightarrow \nu(Z) \geq 0)\}.$$

Then it is clear that

$$\delta = \sup\{\nu(Y) : Y \in S_0\}$$

is a finite real number and there exists a set  $A \in S_0$  such that

$$\nu(A) = \delta.$$

Now let us put  $B = E \setminus A$ . Then it is not difficult to check that the partition  $\{A, B\}$  of the set  $E$  satisfies conditions 1), 2) and 3).

The decomposition  $\{A, B\}$  of the basic set  $E$ , corresponding to the given signed measure  $\nu$ , is called the **Hahn decomposition** of  $E$  with respect to  $\nu$ .

We define

$$\begin{aligned} \nu^+(X) &= \nu(X \cap A) & (X \in S), \\ \nu^-(X) &= -\nu(X \cap B) & (X \in S). \end{aligned}$$

It is obvious that  $\nu^+$  and  $\nu^-$  are ordinary measures on the  $\sigma$ -algebra  $S$ . Moreover, we have

$$\nu = \nu^+ - \nu^-.$$

So, we obtain that any signed measure  $\nu$  can be represented as a difference between two ordinary measures. This representation is called the **Jordan decomposition** of the signed measure  $\nu$ .

Let us notice also that the function  $|\nu|$  defined by the formula

$$|\nu| = \nu^+ + \nu^-$$

is a measure on the  $\sigma$ -algebra  $S$  and it is called the **total variation** of the given signed measure  $\nu$ .

We say that a signed measure  $\nu$  is  **$\sigma$ -finite** if there exists a family  $(X_n)_{n \in \mathbb{N}} \subseteq S$  such that  $\bigcup_n X_n = E$  and  $|\nu|(X_n) < +\infty$  for every  $n \in \mathbb{N}$ .

Suppose now that  $(E, S, \mu)$  is a measure space and that  $\nu$  is a signed measure on the same  $\sigma$ -algebra  $S$ . We say that  $\nu$  is **absolutely continuous with respect to  $\mu$**  if

$$(\forall X \in S)(\mu(X) = 0 \rightarrow \nu(X) = 0).$$

The next result which can be derived from Theorem 8 plays an important role in modern analysis and probability theory.

**Theorem 4.9 (Radon-Nikodym)** *Suppose that  $(E, S, \mu)$  is a measure space with a  $\sigma$ -finite measure and that  $\nu$  is a  $\sigma$ -finite signed measure on  $S$ , absolutely continuous with respect to  $\mu$ . Then there exists a  $\mu$ -measurable function  $f : E \rightarrow \mathbb{R}$  such that for every  $X \in S$  we have*

$$\nu(X) = \int_X f d\mu.$$

The above theorems (Fubini's, Hahn's and Radon-Nikodym's) are typical results of pure measure theory. On the other hand, in most of the situations which can be met in modern analysis a measure does not appear separately but it is tightly connected with other fundamental mathematical structures. This concerns primarily the topological structure, which we have already noticed above on the example of the classical Lebesgue measure (see the Luzin theorem above).

Let  $E$  be an arbitrary topological space and let  $B(E)$  be the Borel  $\sigma$ -algebra of this space. We say that a measure  $\mu$  is a **Borel measure** (on  $E$ ) if the equality

$$\text{dom}(\mu) = B(E)$$

holds. It is obvious that the concrete properties of the original topological space  $E$  often imply the corresponding properties of the Borel measures on  $E$ .

**Example 3.** In the sequel we shall see that there exists an uncountable subspace  $E$  of  $\mathbb{R}$  such that any  $\sigma$ -finite diffused Borel measure on  $E$  is identically equal to zero.

The following definition describes a very important class of Borel measures.

Let  $E$  be an arbitrary Hausdorff topological space and let  $\mu$  be a Borel measure on  $E$ . We say that the measure  $\mu$  is a **Radon measure** if for each set  $X \in B(E)$  we have

$$\mu(X) = \sup\{\mu(K) : K \text{ is compact in } E \text{ \& } K \subseteq X\}.$$

We say that a Hausdorff topological space  $E$  is a **Radon space** if every  $\sigma$ -finite Borel measure on  $E$  is a Radon measure. From the last definition it immediately follows that any Borel subset of a Radon topological space is also a Radon space.

We have already mentioned above that the classical  $n$ -dimensional Lebesgue measure (considered only on the Borel  $\sigma$ -algebra  $B(\mathbb{R}^n)$ ) is a Radon measure. It turns out that this fact is a rather particular case of the following proposition essentially due to Ulam.

**Theorem 4.10** *Any Polish topological space  $E$  is a Radon space.*

**Proof.** Let  $\mu$  be an arbitrary  $\sigma$ -finite Borel measure on  $E$ . Without loss of generality we may assume that  $\mu$  is a finite measure. Let  $X \in B(E)$ . It is easy to see that

$$\begin{aligned}\mu(X) &= \inf\{\mu(V) : V \text{ is open in } E \text{ \& } X \subseteq V\}, \\ \mu(X) &= \sup\{\mu(F) : F \text{ is closed in } E \text{ \& } F \subseteq X\}.\end{aligned}$$

Moreover, it is easy to check that these formulas are true for any perfect topological space  $E$  and for any Borel set  $X \subseteq E$ . Now let an arbitrary real number  $\epsilon > 0$  be given. First of all, we can find a closed subset  $F$  of a Polish space  $E$  such that

$$F \subseteq X, \quad \mu(X \setminus F) < \frac{\epsilon}{2}.$$

Furthermore, since  $F$  is separable, there exists a sequence

$$F_0, F_1, \dots, F_n, \dots$$

of subsets of  $F$  such that

- a) the set  $F_n$  is the union of a finite family of closed balls in  $F$ , the diameters of which do not exceed  $\frac{1}{n+1}$ ;
- b)  $\mu(F \setminus F_n) < \frac{\epsilon}{2^{n+2}}$ .

Now let us put  $K = \bigcap_n F_n$ . It is clear that  $K$  is compact in  $E$  (since  $K$  is totally bounded and closed in a complete space  $E$ ). Moreover, we have  $K \subseteq X$  and

$$\mu(X \setminus K) = \mu(X \setminus F) + \mu(F \setminus K) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, the formulated theorem is proved.

From this theorem it directly follows that any Borel subset of a Polish topological space is a Radon space. In the sequel we shall see that the analytic subsets of a Polish topological space and their complements are also Radon spaces. On the other hand, it is impossible to decide in theory **ZFC** for the projective sets from the class  $\text{Pr}_3(\mathbb{R})$  whether they are Radon spaces or not.

The next result shows us that in a separable metric space any  $\sigma$ -finite diffused Borel measure is concentrated on a set of the first category.

**Theorem 4.11** *Let  $E$  be a topological space and let  $X$  be a subset of  $E$  satisfying the following relations:*

- 1) *the set  $X$  is countable and everywhere dense in  $E$ ;*
- 2) *for every point  $x \in X$  the one-element set  $\{x\}$  is a  $G_\delta$ -subset of  $E$ .*

*Furthermore, let  $\mu$  be an arbitrary  $\sigma$ -finite Borel measure on  $E$  such that*

$$(\forall x \in X)(\mu(\{x\}) = 0).$$

*Then there exists a set  $Y \subseteq E$  satisfying the following relations:*

- a)  $X \subseteq Y$ ;
- b)  $Y$  is a  $G_\delta$ -subset of  $E$ ;
- c)  $\mu(Y) = 0$ .

In particular, the measure  $\mu$  is concentrated on the set  $E \setminus Y$  which is a first category  $F_\sigma$ -subset of  $E$ .

**Proof.** Without loss of generality we may assume that the measure  $\mu$  is finite. Let us put

$$X = \{x_0, x_1, \dots, x_n, \dots\}.$$

From the conditions of theorem it follows that for every pair  $(n, k)$  of natural numbers there exists an open neighbourhood  $V_k(x_n)$  of the point  $x_n$ , such that

$$\mu(V_k(x_n)) < \frac{1}{2^{n+k}}.$$

Now we may put

$$Y_k = \bigcup_n V_k(x_n),$$

$$Y = \bigcap_k Y_k.$$

It is not difficult to check that the set  $Y$  is a required one.

Applying this result to the classical  $n$ -dimensional Lebesgue measure  $\lambda^n$ , we see that  $\lambda^n$  is concentrated on a first category  $F_\sigma$ -subset of the space  $\mathbb{R}^n$ . In other words, there exists a partition  $\{A, B\}$  of the space  $\mathbb{R}^n$  such that

$$\lambda^n(A) = 0, \quad B \in K(\mathbb{R}^n).$$

This fact means also that the  $\sigma$ -ideals  $L(\lambda^n)$  and  $K(\mathbb{R}^n)$  are **orthogonal** to each other.

**Theorem 4.12** *Let  $E_1$  and  $E_2$  be any two Polish topological spaces. Let  $\mu_1$  be a probability diffused Borel measure on the space  $E_1$  and let  $\mu_2$  be a probability diffused Borel measure on the space  $E_2$ . Then there exists a Borel isomorphism*

$$\varphi : (E_1, B(E_1)) \rightarrow (E_2, B(E_2))$$

such that

$$(\forall X \in B(E_1))(\mu_1(X) = \mu_2(\varphi(X))).$$

In other words, the mapping

$$\varphi : (E_1, B(E_1), \mu_1) \rightarrow (E_2, B(E_2), \mu_2)$$

is also an isomorphism between the given measure spaces.

**Proof.** As we know, all uncountable Polish spaces are Borel isomorphic. Therefore, without loss of generality we may assume that

$$(E_1, B(E_1), \mu_1) = ([0, 1], B([0, 1]), \mu),$$

$$(E_2, B(E_2), \mu_2) = ([0, 1], B([0, 1]), \lambda),$$

where  $\mu$  is a probability diffused Borel measure on the unit segment  $[0, 1]$  and  $\lambda$  is the restriction of the classical Lebesgue measure to the Borel  $\sigma$ -algebra of this segment.

For any natural number  $n \geq 1$  there exists a finite sequence of closed intervals

$$I_{n,1} = [a_{n,1}, b_{n,1}], \quad I_{n,2} = [a_{n,2}, b_{n,2}], \dots, \quad I_{n,k(n)} = [a_{n,k(n)}, b_{n,k(n)}]$$

which satisfies the following relations:



- 1)  $a_{n,1} \geq 0, \quad b_{n,k(n)} \leq 1$ ;
- 2)  $(\forall j)(1 \leq j < k(n) \rightarrow b_{n,j} \leq a_{n,j+1})$ ;
- 3)  $(\forall j)(1 \leq j \leq k(n) \rightarrow 0 < b_{n,j} - a_{n,j} < \frac{1}{n})$ ;
- 4)  $(\forall j)(1 \leq j \leq k(n) \rightarrow 0 < \mu([a_{n,j}, b_{n,j}]) < \frac{1}{n})$ ;
- 5)  $\mu([0, 1] \setminus (I_{n,1} \cup \dots \cup I_{n,k(n)})) = 0$ .

The existence of such a sequence of closed intervals easily follows from the compactness of  $[0, 1]$  and the diffuseness of the measure  $\mu$ . The constructed sequence of intervals uniquely determines another sequence of closed intervals

$$I'_{n,1} = [a'_{n,1}, b'_{n,1}], \dots, I'_{n,k(n)} = [a'_{n,k(n)}, b'_{n,k(n)}]$$

which satisfies the following conditions:

$$\begin{aligned} a'_{n,1} &= 0, \quad b'_{n,k(n)} = 1; \\ (\forall j)(1 \leq j < k(n) \rightarrow b'_{n,j} &= a'_{n,j+1}); \\ (\forall j)(1 \leq j \leq k(n) \rightarrow \mu([a_{n,j}, b_{n,j}]) &= b'_{n,j} - a'_{n,j}). \end{aligned}$$

Moreover, without loss of generality we may assume that for any  $n \geq 1$  the family of closed intervals

$$\{I_{n+1,1}, \dots, I_{n+1,k(n+1)}\}$$

is inscribed into the family

$$\{I_{n,1}, \dots, I_{n,k(n)}\}$$

and similarly, the family of closed intervals

$$\{I'_{n+1,1}, \dots, I'_{n+1,k(n+1)}\}$$

is inscribed into the family

$$\{I'_{n,1}, \dots, I'_{n,k(n)}\}.$$

Now let us put

$$\begin{aligned} I_n &= I_{n,1} \cup \dots \cup I_{n,k(n)}, \\ I &= \bigcap_n I_n, \end{aligned}$$

$$D_1 = \{x : (\exists n)(\exists j)(x \text{ is an end-point of } I_{n,j})\},$$

$$D_2 = \{x : (\exists n)(\exists j)(x \text{ is an end-point of } I'_{n,j})\}.$$

It is clear that

$$\text{card}(D_1) \leq \omega, \quad \text{card}(D_2) \leq \omega.$$

Let  $x$  be any point from  $I \setminus D_1$ . Then this point uniquely determines the sequence

$$I_{1,j(1)}, I_{2,j(2)}, \dots, I_{n,j(n)}, \dots$$

of closed intervals, for which we have

$$x \in I_{1,j(1)} \cap I_{2,j(2)} \cap \dots \cap I_{n,j(n)} \cap \dots \quad .$$

It is obvious that there exists only one point  $y$  such that

$$y \in I'_{1,j(1)} \cap I'_{2,j(2)} \cap \dots \cap I'_{n,j(n)} \cap \dots \quad .$$

Taking into account these facts let us put

$$\psi(x) = y.$$

For the mapping  $\psi$  it is not difficult to check that

- a)  $\psi$  is a Borel mapping;
- b) for any Borel set  $X \subseteq I \setminus D_1$  we have  $\mu(X) = \lambda(\psi(X))$ ;
- c)  $[0, 1] \setminus D_2 \subseteq \text{ran}(\psi)$ ;
- d) the restriction of  $\psi$  to the set  $I \setminus (D_1 \cup \psi^{-1}(D_2))$  is an injection.

Starting with these properties and changing, if necessary, the mapping  $\psi$  on a  $\mu$ -measure zero Borel set, we shall obtain the required isomorphism  $\varphi$ .

The theorem proved above is principal and important. In particular, it shows us that any probability diffused Borel measure on a Polish topological space is isomorphic to the classical Lebesgue measure  $\lambda$  on  $[0, 1]$ . Moreover, and it is of major importance, this isomorphism between measures is also an isomorphism between Borel structures. From Theorem 12 we can also deduce a more general fact. Namely, let  $E_1$  and  $E_2$  be two Borel subsets of a Polish topological space and let  $\mu_1$  and  $\mu_2$  be two probability diffused Borel measures on  $E_1$  and  $E_2$ , respectively. Then there exists a Borel isomorphism

$$\varphi : (E_1, B(E_1)) \rightarrow (E_2, B(E_2)),$$

which is also an isomorphism between the measures  $\mu_1$  and  $\mu_2$ . Finally, let us consider a more general situation when two measure spaces

$$(E_1, B(E_1), \mu_1), \quad (E_2, B(E_2), \mu_2)$$

are given, where  $E_1$  and  $E_2$  are Borel subsets of a Polish topological space and  $\mu_1$  and  $\mu_2$  are non-zero  $\sigma$ -finite diffused Borel measures on  $E_1$  and  $E_2$ , respectively. Then there exists a Borel isomorphism

$$\varphi : (E_1, B(E_1)) \rightarrow (E_2, B(E_2)),$$

such that

$$\varphi(L(\mu_1)) = L(\mu_2).$$

In other words, the mapping  $\varphi$  transforms the  $\sigma$ -ideal of  $\mu_1$ -measure zero sets onto the  $\sigma$ -ideal of  $\mu_2$ -measure zero sets. Of course, in this case  $\varphi$  need not be an isomorphism between the given measure spaces. There can not be any isomorphisms between those spaces when  $\mu_1$  is finite and  $\mu_2$  is not.

Another important mathematical structure which is tightly connected with measures is a group structure.

Let  $E$  be a basic set and let  $\Gamma$  be some group of transformations of this set. Further, let  $\mathcal{D}$  be some class of subsets of  $E$ . We say that the class  $\mathcal{D}$  is  $\Gamma$ -**invariant** if

$$(\forall X \in \mathcal{D})(\forall g \in \Gamma)(g(X) \in \mathcal{D}).$$

Let  $S$  be some  $\sigma$ -algebra of subsets of  $E$  and let  $\mu$  be a measure defined on  $S$ . We say that the measure  $\mu$  is  $\Gamma$ -**quasi-invariant** if

- 1)  $S$  is a  $\Gamma$ -invariant class of subsets of  $E$ ;
- 2) the class  $L(\mu)$  of all  $\mu$ -measure zero sets is also  $\Gamma$ -invariant.

If instead of condition 2) the stronger condition holds

- 3)  $(\forall X \in S)(\forall g \in \Gamma)(\mu(X) = \mu(g(X)))$ ,

then we say that the measure  $\mu$  is a  $\Gamma$ -invariant measure.

**Example 4.** Let  $\Gamma$  be an arbitrary group of motions (i.e. isometric transformations) of the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . Then it is obvious that the Lebesgue measure  $\lambda^n$  is a  $\Gamma$ -invariant measure. If  $\Gamma$  is an arbitrary group of non-degenerated affine transformations of the space  $\mathbb{R}^n$ , then  $\lambda^n$  is a  $\Gamma$ -quasi-invariant measure but, of course, in this case  $\lambda^n$  need not be a  $\Gamma$ -invariant measure. Similarly, if  $\Gamma$  is the group of all such homeomorphisms  $f : \mathbb{R} \rightarrow \mathbb{R}$  that  $f$  and  $f^{-1}$  both satisfy the Lipschitz condition, then the Lebesgue measure  $\lambda$  on  $\mathbb{R}$  is  $\Gamma$ -quasi-invariant, but not  $\Gamma$ -invariant.

Let  $(\Gamma, \cdot)$  be an arbitrary, locally compact topological group. For such a group we have a well-known result, due to Haar, which states the existence (and in a certain sense the uniqueness) of a non-zero invariant Borel measure  $\mu$  on  $\Gamma$ . More precisely, the measure  $\mu$  satisfies the following relations:

- 1)  $\mu$  is a locally finite measure, i.e. for each point  $x \in \Gamma$  there exists an open neighbourhood  $V(x)$  of this point such that  $\mu(V(x)) < +\infty$ ;
- 2)  $\mu$  is a Radon measure;
- 3)  $\mu$  is invariant under all left translations of  $\Gamma$ , i.e.

$$(\forall X \in B(\Gamma))(\forall g \in \Gamma)(\mu(X) = \mu(g \cdot X)).$$

The measure  $\mu$  is called the (left) **Haar measure** on the group  $\Gamma$ .

If the group  $\Gamma$  is  $\sigma$ -compact, then the Haar measure  $\mu$  on  $\Gamma$  is a  $\sigma$ -finite measure. If the group  $\Gamma$  is compact, then the Haar measure  $\mu$  on  $\Gamma$  is a finite measure, and we can obviously assume that in this case  $\mu$  is a probability measure.

We remark here that the classical Lebesgue measure  $\lambda^n$  considered only on the Borel  $\sigma$ -algebra of the Euclidean space  $\mathbb{R}^n$  is a rather particular case of the Haar measure.

**Example 5.** Let us return to the generalized Cantor space  $\{0, 1\}^I$ , where  $I$  is an arbitrary set of indices. As we know  $\{0, 1\}^I$  is a compact topological space. If we consider in the set  $\{0, 1\}^I$  the operation of addition modulo 2, then we obtain a compact topological group. Therefore, for this group there exists a probability Haar measure  $\mu$ . Notice here that the completion of the measure  $\mu$  is an extension of the product measure defined for the space  $\{0, 1\}^I$  in Example 2 of this Chapter.

## Exercises

**Exercise 4.1** Let  $(E, S, \mu)$  be a measure space with a finite measure  $\mu$ . For any sets  $X \subseteq E$  and  $Y \subseteq E$  let us put

$$\rho_\mu(X, Y) = \mu^*(X \Delta Y)$$

and let us identify any two sets  $X \subseteq E$  and  $Y \subseteq E$  for which  $\rho_\mu(X, Y) = 0$ . Show that

- a)  $(P(E), \rho_\mu)$  is a complete metric space;
- b)  $(S, \rho_\mu)$  is a closed subset of  $(P(E), \rho_\mu)$  and thus is a complete metric space, too.

The space  $(S, \rho_\mu)$  is called the metric space canonically associated with the given measure  $\mu$ .

The measure  $\mu$  is called **separable** if the metric space  $(S, \rho_\mu)$  is separable. Otherwise, the measure  $\mu$  is called **non-separable**.

Show that if the  $\sigma$ -algebra  $S$  is countably generated, i.e. it is generated by a countable family of subsets of  $E$ , then the measure  $\mu$  is separable. Give an example of a separable measure  $\mu$ , for which the  $\sigma$ -algebra  $S$  is not countably generated. Check also that the product measure on the Cantor space  $\{0, 1\}^I$  considered in Example 2 from this Chapter is separable if and only if  $\text{card}(I) \leq \omega$ .

**Exercise 4.2** Let  $(E, S, \mu)$  be a measure space with a  $\sigma$ -finite measure  $\mu$  and let  $(X_n)_{n \in \mathbb{N}}$  be an arbitrary, increasing with respect to the inclusion, sequence of subsets of  $E$ . Prove that

$$\mu^*\left(\bigcup_n X_n\right) = \lim_n \mu^*(X_n).$$

On the other hand, prove that there exists a decreasing, with respect to the inclusion, sequence  $(Y_n)_{n \in \mathbb{N}}$  of subsets of the segment  $[0, 1]$  such that

$$a) \lambda^*(Y_n) = 1 \text{ for every } n \in \mathbb{N};$$

$$b) \bigcap_n Y_n = \emptyset.$$

**Exercise 4.3** Let  $(X_n)_{n \in \mathbb{N}}$  be an arbitrary sequence of sets. The **lower limit** (in the set-theoretical sense) is the set

$$\liminf(X_n)_{n \in \mathbb{N}} = \bigcup_n \bigcap_{m > n} X_m.$$

In other words,  $\liminf(X_n)_{n \in \mathbb{N}}$  is the set of those elements which belong to all, except for finitely many, of the sets  $(X_n)_{n \in \mathbb{N}}$ . If the equality

$$\liminf(X_n)_{n \in \mathbb{N}} = \limsup(X_n)_{n \in \mathbb{N}}$$

holds, then we say that the sequence  $(X_n)_{n \in \mathbb{N}}$  converges in the set-theoretical sense and write

$$\lim(X_n)_{n \in \mathbb{N}} = \liminf(X_n)_{n \in \mathbb{N}} = \limsup(X_n)_{n \in \mathbb{N}}.$$

In particular, it is easy to see that any monotonic (with respect to the inclusion) sequence of sets  $(X_n)_{n \in \mathbb{N}}$  converges. Namely, if  $(X_n)_{n \in \mathbb{N}}$  is an increasing sequence, then we have

$$\lim(X_n)_{n \in \mathbb{N}} = \bigcup_n X_n,$$

and if  $(X_n)_{n \in \mathbb{N}}$  is a decreasing sequence, then we have

$$\lim(X_n)_{n \in \mathbb{N}} = \bigcap_n X_n.$$

Now let  $(E, S, \mu)$  be a measure space with a finite measure  $\mu$  and let  $(X_n)_{n \in \mathbb{N}}$  be an arbitrary sequence of subsets of  $E$ . Prove that the inequality

$$\mu^*(\liminf(X_n)_{n \in \mathbb{N}}) \leq \liminf_n \mu^*(X_n)$$

holds. Prove also that if  $(X_n)_{n \in \mathbb{N}} \subseteq S$ , then the inequality

$$\mu(\limsup(X_n)_{n \in \mathbb{N}}) \geq \limsup_n \mu(X_n)$$

holds. Deduce from these facts that if the sequence of sets  $(X_n)_{n \in \mathbb{N}} \subseteq S$  is convergent, then

$$\mu(\lim(X_n)_{n \in \mathbb{N}}) = \lim_n \mu(X_n).$$

**Exercise 4.4** Let  $(E, S, \mu)$  be a measure space with a finite separable measure  $\mu$  and let  $(X_\xi)_{\xi < \omega_1}$  be an uncountable family of  $\mu$ -measurable subsets of  $E$  such that

$$(\forall \xi)(\xi < \omega_1 \rightarrow \mu(X_\xi) > 0).$$

Show that there exists an  $\omega$ -sequence of indices

$$\xi_1 < \xi_2 < \dots < \xi_n < \dots$$

such that

$$\mu(X_{\xi_1} \cap X_{\xi_2} \cap \dots \cap X_{\xi_n} \cap \dots) > 0.$$

**Exercise 4.5** Let  $(T_i)_{i \in I}$  be an arbitrary family of the closed triangles on the Euclidean plane  $\mathbb{R}^2$  (we suppose that they are non-degenerated, i.e.  $\text{int}(T_i) \neq \emptyset$  for each index  $i \in I$ ). Prove that the set  $\bigcup_{i \in I} T_i$  is measurable with respect to the classical two-dimensional Lebesgue measure  $\lambda^2$ .

**Exercise 4.6** Prove Theorem 5 from this Chapter.

**Exercise 4.7** Construct a Lebesgue measurable function  $f : [0, 1] \rightarrow \mathbb{R}$  such that for each  $\lambda$ -measurable set  $X \subseteq [0, 1]$  with  $\lambda(X) = 1$  the function  $f|_X$  is everywhere discontinuous on  $X$ .

**Exercise 4.8** Let  $V$  be a non-empty open convex set in the Euclidean space  $\mathbb{R}^n$  and let  $f : V \rightarrow \mathbb{R}$  be an arbitrary convex real function. Let us put

$$D = \{x \in V : \text{there exists the derivative } f'(x)\}.$$

Show that

- a) the set  $D$  is measurable with respect to  $\lambda^n$  and  $\lambda^n(V \setminus D) = 0$ ;
- b) the set  $V \setminus D$  is a first category set in  $V$  (and, therefore, in  $\mathbb{R}^n$ ).

**Exercise 4.9** Let  $(\{0, 1\}^\omega, B(\{0, 1\}^\omega), \mu)$  be the product measure space considered in Example 2 from this Chapter. Let us put

$$X = \{x \in \{0, 1\}^\omega : \lim_n \frac{x(0) + \dots + x(n)}{n+1} = \frac{1}{2}\}.$$

Show that

- a)  $X \in B(\{0, 1\}^\omega)$ ;
- b)  $\mu(X) = 1$ ;
- c)  $X$  is a first category subset of the Cantor space  $\{0, 1\}^\omega$ .

**Exercise 4.10** Deduce the Radon–Nikodym theorem from the Hahn theorem on a decomposition of a basic set  $E$  with respect to a given signed measure  $\nu$ .

**Exercise 4.11** Let  $(\mu_n)_{n \in \mathbb{N}}$  be a sequence of probability measures defined on a measurable space  $(E, S)$ . Suppose that for every set  $X \in S$  the limit  $\lim_n \mu_n(X)$  exists and denote it by the symbol  $\mu(X)$ . Prove that  $\mu$  is also a probability measure on  $S$ . This result is known as the Vitali–Hahn–Saks theorem.

**Exercise 4.12** Let  $E$  be a  $\sigma$ -compact locally compact topological space and let  $\mu$  be an arbitrary locally finite Borel measure on  $E$ . Show that  $\mu$  is a  $\sigma$ -finite measure.

Let  $E$  be a Hausdorff topological space and let  $\mu$  be a  $\sigma$ -finite Radon measure on  $E$ . Prove that there exists a countable family  $(K_i)_{i \in I}$  of compact sets in  $E$  such that

- a)  $(\forall i \in I)(\mu(K_i) < +\infty)$ ;  
 b)  $\mu(E \setminus \bigcup_i K_i) = 0$ .

**Exercise 4.13** Formulate and prove an analogue of Luzin's  $C$ -property for Radon measures.

**Exercise 4.14** Let  $E$  be an arbitrary Hausdorff topological space. Suppose that  $\mu$  is a finite, non-negative, finitely additive real function defined on the Borel  $\sigma$ -algebra  $B(E)$ . Suppose also that the equality

$$\mu(X) = \sup\{\mu(K) : K \text{ is compact \& } K \subseteq X\}$$

holds for every  $X \in B(E)$ . Show that  $\mu$  is a Radon measure on  $E$ .

**Exercise 4.15** Equip  $\omega_1 + 1$  with the usual order topology. Prove that  $\omega_1 + 1$  is a compact topological space but it is not a Radon space.

**Exercise 4.16** Let  $E_1$  and  $E_2$  be two topological spaces and let one of these spaces have a countable base. Show that the equality

$$B(E_1) \otimes B(E_2) = B(E_1 \times E_2)$$

holds. Show also that

$$B(\omega_1 + 1) \otimes B(\omega_1 + 1) \neq B((\omega_1 + 1) \times (\omega_1 + 1)),$$

where  $\omega_1 + 1$  is equipped with its order topology.

**Exercise 4.17** Let  $E$  be a topological space. By the symbol  $Ba(E)$  we denote the smallest  $\sigma$ -algebra of subsets of  $E$  with respect to which all continuous real functions defined on  $E$  are measurable. It is clear that

$$Ba(E) \subseteq B(E).$$

Prove that if  $E$  is a perfectly normal topological space (i.e.  $E$  is a perfect and normal topological space), then the equality

$$Ba(E) = B(E).$$

holds. In particular, this equality is true for any metric space  $E$ .

Let  $\omega_1$  and  $\omega_1 + 1$  be equipped with the order topologies. Let

$$g : \omega_1 \rightarrow \mathbb{R}$$

be an arbitrary continuous function. Prove that there exists an ordinal  $\xi < \omega_1$  such that

$$g \upharpoonright [\xi, \omega_1[ = \text{const.}$$

Deduce from this fact that

$$Ba(\omega_1) \neq B(\omega_1), \quad Ba(\omega_1 + 1) \neq B(\omega_1 + 1).$$

**Exercise 4.18** Let  $(E_i)_{i \in I}$  be an arbitrary family of compact topological spaces. Using Stone-Weierstrass theorem on approximation prove that

$$Ba\left(\prod_{i \in I} E_i\right) = \otimes_{i \in I} Ba(E_i).$$

**Exercise 4.19** Let  $E_1$  and  $E_2$  be any two  $\sigma$ -compact locally compact topological spaces. Show that the equality

$$Ba(E_1) \otimes Ba(E_2) = Ba(E_1 \times E_2)$$

holds.

**Exercise 4.20** Let  $E$  be a locally compact topological space and let  $\mu$  be a  $\sigma$ -finite measure defined on the  $\sigma$ -algebra  $Ba(E)$ . Prove that for any set  $X \in Ba(E)$  we have

$$\mu(X) = \sup\{\mu(K) : K \text{ is compact } G_\delta \text{-subset of } E \text{ \& } K \subseteq X\}.$$

Deduce from this fact that the measure  $\mu$  can be uniquely extended to a Radon measure on  $E$ .

**Exercise 4.21** Let  $E$  be a locally compact topological space. Let  $\mu$  and  $\nu$  be any two  $\sigma$ -finite Radon measures on  $E$ . Suppose that for every continuous function

$$f : E \rightarrow \mathbb{R}$$

with a compact support the equality

$$\int f d\mu = \int f d\nu$$

holds. Show that  $\mu = \nu$ .

**Exercise 4.22** Let  $\Gamma$  be an arbitrary compact topological group. Let  $\mu$  and  $\nu$  be any two probability Radon measures on  $\Gamma$  invariant under all left translations of  $\Gamma$ . Using the result of the previous exercise and applying the Fubini theorem prove that  $\mu = \nu$ . In fact, we have here the uniqueness theorem for the Haar measure on a compact group.

**Exercise 4.23** Let  $E$  be a separable Hilbert space. Let  $\mu$  and  $\nu$  be two probability Borel measures on  $E$  such that

$$\mu(V) = \nu(V)$$

for every open ball  $V$  in  $E$ . Show that  $\mu = \nu$ .

**Exercise 4.24** Let  $E$  be a finite-dimensional vector space over  $\mathbb{R}$  and let  $\Gamma$  be some group of linear transformations of  $E$ . Prove that the following two sentences are equivalent:

- a) the group  $\Gamma$  has the compact closure in the space  $GL(E)$  of all linear transformations of  $E$  equipped with the standard topology;
- b) there exists a scalar product  $\langle, \rangle$  on  $E$  for which every element  $g \in \Gamma$  is an orthogonal transformation of  $(E, \langle, \rangle)$ .

Find a necessary and sufficient conditions on  $\Gamma$  for the uniqueness of the mentioned scalar product (with exactness to a constant coefficient).

**Exercise 4.25** Let  $B(\mathbb{R})$  be the Borel  $\sigma$ -algebra of  $\mathbb{R}$ . Let  $K(\mathbb{R})$  be the  $\sigma$ -ideal of all first category subsets of  $\mathbb{R}$  and let  $L(\lambda)$  be the  $\sigma$ -ideal of all Lebesgue measure zero subsets of  $\mathbb{R}$ . Show that the quotient Boolean algebras  $B(\mathbb{R})/K(\mathbb{R})$  and  $B(\mathbb{R})/L(\lambda)$  are not isomorphic.

**Exercise 4.26** Let  $E$  be a basic set and let  $\Phi$  be a family of real functions defined on  $E$ . We say that the family  $\Phi$  separates the points of  $E$  if for any two distinct points  $x \in E$  and  $y \in E$  there exists a function  $\varphi \in \Phi$  such that

$$\varphi(x) \neq \varphi(y).$$

By the symbol  $\sigma(\Phi)$  we denote the smallest  $\sigma$ -algebra of subsets of  $E$  with respect to which all functions from  $\Phi$  are measurable.

Let  $E$  be a Polish topological space and let  $\Phi$  be a countable family of Borel real functions defined on  $E$ . Show that the following two sentences are equivalent:

- a) the family  $\Phi$  separates the points of  $E$ ;
- b)  $\sigma(\Phi) = B(E)$ .

In particular, we see that the equality

$$\sigma((X_i)_{i \in I}) = B(E)$$

holds for a countable family  $(X_i)_{i \in I} \subseteq B(E)$  if and only if the family of characteristic functions  $(1_{X_i})_{i \in I}$  separates the points of the space  $E$ .





## Chapter 5

# Choquet's Theorem and its Applications

In this Chapter we discuss some questions connected with a well known Choquet's theorem on capacities and give its applications to some problems concerning measurability of various sets and functions. First of all we shall formulate and prove this theorem. We shall do it in a general and an abstract form. Let  $E$  be a basic set and let  $\Phi$  be some class of subsets of  $E$ .

Suppose also that a function

$$\nu : P(E) \rightarrow \mathbb{R}$$

is given where  $P(E)$  denotes, as usual, the family of all subsets of the basic set  $E$ . We shall say that the function  $\nu$  is a **capacity for the class  $\Phi$**  (or *with respect to the class  $\Phi$* ) if the following conditions hold:

- 1) if  $X \subseteq Y \subseteq E$  then we have  $\nu(X) \leq \nu(Y)$ ;
- 2) if a sequence  $(X_n)_{n \in \mathbb{N}}$  of subsets of  $E$  is increasing by the inclusion then we have

$$\nu\left(\bigcup_n X_n\right) = \lim_n \nu(X_n);$$

- 3) if a sequence  $(X_n)_{n \in \mathbb{N}} \subseteq \Phi$  is decreasing by the inclusion then we have

$$\nu\left(\bigcap_n X_n\right) = \lim_n \nu(X_n).$$

The next example vividly shows that capacities can be frequently met in measure theory.

**Example 1.** Let  $(E, S, \mu)$  be a measure space with finite measure  $\mu$ . We take as  $\Phi$  any class of sets contained in the  $\sigma$ -algebra  $S$ . Let us define the function  $\nu$  by the formula

$$\nu(X) = \mu^*(X) \quad (X \subseteq E),$$

where  $\mu^*$  is the outer measure associated with the given measure  $\mu$ . It is easy to check that the function  $\nu$  is a capacity with respect to the class  $\Phi$ . As a rule, the class  $\Phi$  is taken in such a way that  $S = \sigma(\Phi)$ .

**Theorem 5.1 (Choquet)** . *Let  $E$  be a basic set and let  $\Phi$  be some class of subsets of  $E$  closed under finite unions and finite intersections. Moreover, let  $\nu$  be any*

capacity for the class  $\Phi$ . Then for every set  $X$  from the analytic class  $(A)(\Phi)$  the following relation holds :

$$\nu(X) = \sup\{\nu(Y) : Y \in \Phi_\delta \text{ \& } Y \subseteq X\},$$

where  $\Phi_\delta$  denotes the class of all countable intersections of sets from the original class  $\Phi$ .

**Proof.** Let  $X \in (A)(\Phi)$ . We can write

$$X = \bigcup_{t \in \mathbb{N}^\omega} \left( \bigcap_k F_{t_0 \dots t_k} \right),$$

where  $(F_t)_{t \in \mathbb{N}^{<\omega}}$  is some countable system of sets from the class  $\Phi$ . Since the original class  $\Phi$  is closed under finite intersections we can assume that the mentioned system of sets is regular. It is clear that

$$X = \bigcup_{n \in \mathbb{N}} \left( \bigcup_{\{t: t_0 \leq n\}} F_t \right),$$

where symbol  $F_t$  denotes the set

$$F_{t_0} \cap F_{t_0 t_1} \cap \dots \cap F_{t_0 \dots t_k} \cap \dots .$$

Let us remark that the last set may not belong to the class  $\Phi$ . Taking into account properties 1) and 2) of capacity  $\nu$  for any real number  $\epsilon > 0$  we can find a natural index  $n_0$  such that

$$\nu(X) - \epsilon < \nu(G_0),$$

where symbol  $G_0$  denotes the set

$$\bigcup_{\{t: t_0 \leq n_0\}} F_t.$$

Next we continue our construction by recursion. Suppose that a finite sequence

$$(n_0, n_1, \dots, n_k)$$

of natural numbers and a finite sequence

$$(G_0, G_1, \dots, G_k)$$

of sets are constructed in such a way that

$$G_r = \bigcup_{\{t: t_0 \leq n_0, \dots, t_r \leq n_r\}} F_t \quad (r = 0, \dots, k),$$

and the relations

$$\nu(X) - \epsilon < \nu(G_r) \quad (r = 0, \dots, k)$$

hold. Let us consider the set  $G_k$ . We can write

$$G_k = \bigcup_n \bigcup_{\{t: t_0 \leq n_0, \dots, t_k \leq n_k, t_{k+1} \leq n\}} F_t.$$

From the inequality

$$\nu(X) - \epsilon < \nu(G_k)$$

and properties 1) and 2) of the capacity  $\nu$  it follows that we can find a natural index  $n_{k+1}$  such that for the set

$$G_{k+1} = \bigcup_n \bigcup_{\{t: t_0 \leq n_0, \dots, t_k \leq n_k, t_{k+1} \leq n_{k+1}\}} F_t.$$

the inequality

$$\nu(X) - \epsilon < \nu(G_{k+1})$$

holds, too. In this way we shall construct two infinite sequences

$$(n_0, n_1, \dots, n_k, \dots),$$

$$(G_0, G_1, \dots, G_k, \dots).$$

Next, for every natural number  $k$  we put

$$H_k = \bigcup_{t_0 \leq n_0, \dots, t_k \leq n_k} F_{t_0 \dots t_k}.$$

Obviously, in the above formula we use only finite unions. Therefore,  $H_k \in \Phi$ . Moreover, it is easy to check that the sequence  $(H_k)_{k \in \mathbb{N}}$  is decreasing with respect to the inclusion and

$$G_k \subseteq H_k \quad (k \in \mathbb{N}).$$

So, condition 3) for the function  $\nu$  implies that

$$\nu(X) - \epsilon \leq \inf_k \nu(G_k) \leq \lim_k \nu(H_k) = \nu\left(\bigcap_k H_k\right).$$

Now, using the regularity of the system  $(F_t)_{t \in \mathbb{N}^{<\omega}}$  and König's theorem on  $\omega$ -trees we get the inclusion

$$\bigcap_k H_k \subseteq X.$$

So, we see that the set  $H = \bigcap_k H_k$  belongs to the class  $\Phi_\delta$  and

$$H \subseteq X, \quad \nu(X) - \epsilon < \nu(H).$$

Since the real number  $\epsilon > 0$  was taken at the beginning of the proof arbitrarily, we immediately obtain the thesis of Choquet's theorem.

Choquet's theorem is often formulated as follows : all analytic sets over the original class  $\Phi$  are *capacitable* with respect to any capacity for the class  $\Phi$ .

**Example 2.** Let  $(E, S, \mu)$  be a measure space with finite (or more generally,  $\sigma$ -finite) measure  $\mu$  and let  $\Phi$  be any class of subsets of  $E$  which generates the  $\sigma$ -algebra  $S$ . In Example 1 we have seen that the function  $\mu^*$  is a capacity for the class  $\Phi$ . Therefore, by Choquet's theorem, any analytic set over class  $\Phi$  is capacitable with respect to  $\mu^*$ . But, as it is easy to check, this means the following: any analytic set over class  $\Phi$  is measurable with respect to measure  $\bar{\mu}$ , which is the completion of the original measure  $\mu$ . It is also clear that if the original measure  $\mu$  is complete then we simply obtain the measurability with respect to  $\mu$  of any analytic set over the class  $\Phi$ . Now, let  $E$  be an arbitrary Polish topological space, let  $\mu$  be an arbitrary  $\sigma$ -finite Borel measure on  $E$  and let  $\Phi$  be the algebra of sets generated by the family of all closed subsets of  $E$ . Applying the above considerations to this particular situation we see that every analytic subset of the space  $E$  is  $\bar{\mu}$ -measurable. If  $E = \mathbb{R}$ , then we immediately obtain the old, classical result which says that all analytic subsets

of the real line  $\mathbb{R}$  are Lebesgue measurable. Hence, all complements of the analytic subsets of  $\mathbb{R}$  are also Lebesgue measurable. Thus, we see that analytic sets and their complements are good from the measure-theoretical point of view. Later we shall also check that the same sets are good from the topological point of view, because they have the Baire property and, moreover, they have the Baire property in the restricted sense.

Now we can give some other applications of Choquet's theorem.

**Theorem 5.2** *Let  $(E, S, \mu)$  be a complete probability space (or, more generally, a complete measure space with a  $\sigma$ -finite measure  $\mu$ ). Let  $K$  be a locally compact topological space with a countable base and let  $(K, B(K))$  be the measurable space canonically associated with  $K$ , i.e. the space  $K$  is equipped with its Borel  $\sigma$ -algebra. Finally, let*

$$pr_1 : E \times K \rightarrow E$$

*be the canonical projection. Then for any set  $Z \subseteq E \times K$  from the product  $\sigma$ -algebra  $S \otimes B(K)$  the set  $pr_1(Z)$  belongs to the  $\sigma$ -algebra  $S$ .*

**Proof.** Let us consider the family  $Comp(K)$  of all compact subsets of the topological space  $K$ . It is evident that the class  $Comp(K)$  is closed under finite unions and finite intersections. It is also clear that the Borel  $\sigma$ -algebra  $B(K)$  is generated by the class  $Comp(K)$ . Moreover, let us consider in the product  $E \times K$  the family of all sets of the form

$$\bigcup_{1 \leq i \leq m} (X_i \times Y_i),$$

where  $m$  is an arbitrary natural number,  $X_i$  are elements of the  $\sigma$ -algebra  $S$  and  $Y_i$  are members of the class  $Comp(K)$ . We denote the class of sets described in this way by the symbol  $\Phi$ . The class  $\Phi$  is closed under finite unions and finite intersections, too. It is also obvious that the  $\sigma$ -algebra generated by the class  $\Phi$  coincides with the product  $\sigma$ -algebra  $S \otimes B(K)$ . Let us define a real function  $\nu$  on the family of all subsets of  $E \times K$  by the formula

$$\nu(Z) = \mu^*(pr_1(Z)) \quad (Z \subseteq E \times K).$$

Let us check that the function  $\nu$  is a capacity on the  $E \times K$  with respect to the class  $\Phi$ . Indeed, it is clear that

$$\nu(Z_1) \leq \nu(Z_2)$$

if  $Z_1 \subseteq Z_2 \subseteq E \times K$ . Furthermore, if  $(Z_n)_{n \in \mathbb{N}}$  is an increasing sequence of subsets of  $E \times K$  and

$$Z = \bigcup_n Z_n,$$

then we have

$$pr_1(Z) = \bigcup_n pr_1(Z_n),$$

$$\nu(Z) = \mu^*(pr_1(Z)) = \lim_n \mu^*(pr_1(Z_n)) = \lim_n \nu(Z_n).$$

The last equalities immediately follow from the well known properties of the outer measure  $\mu^*$ . Finally, we must check that if  $(Z_n)_{n \in \mathbb{N}}$  is a decreasing sequence of sets from the class  $\Phi$  then

$$\nu\left(\bigcap_n Z_n\right) = \lim_n \nu(Z_n).$$

For this aim we shall first prove the equality

$$pr_1\left(\bigcap_n Z_n\right) = \bigcap_n pr_1(Z_n).$$

It is clear that it is enough to establish the inclusion

$$pr_1\left(\bigcap_n Z_n\right) \supseteq \bigcap_n pr_1(Z_n).$$

Let  $x \in \bigcap_n pr_1(Z_n)$ . This means that for any natural number  $n$  there exists a point  $y_n$  such that

$$(x, y_n) \in X_{n,i} \times Y_{n,i},$$

where  $X_{n,i} \times Y_{n,i}$  is a component of the representation of the set  $Z_n$  :

$$Z_n = \bigcup_{1 \leq i \leq m(n)} (X_{n,i} \times Y_{n,i}).$$

In particular,  $y_n \in Y_{n,i}$  and hence  $Y_{n,i} \neq \emptyset$ . Moreover, since the sequence of sets  $Z_n (n \in \mathbb{N})$  is decreasing by the inclusion then, without loss of generality, we may assume that the families of sets

$$(X_{n+1,i})_{1 \leq i \leq m(n+1)}, \quad (Y_{n+1,i})_{1 \leq i \leq m(n+1)}$$

are respectively inscribed into the families

$$(X_{n,i})_{1 \leq i \leq m(n)}, \quad (Y_{n,i})_{1 \leq i \leq m(n)}.$$

Now, applying König's theorem on  $\omega$ -trees, we can easily find a decreasing sequence of sets

$$(X_{n,i(n)} \times Y_{n,i(n)})_{n \in \mathbb{N}}$$

such that all the sets  $Y_{n,i(n)}$  are non-empty and the relation

$$x \in X_{n,i(n)}$$

holds for every  $n \in \mathbb{N}$ . Since all the sets  $Y_{n,i(n)}$  are compact, we have

$$\bigcap_n Y_{n,i(n)} \neq \emptyset.$$

Let  $y$  be any element of the last non-empty intersection. Then it is clear that

$$(x, y) \in \bigcap_n Z_n,$$

and so we have

$$x \in pr_1\left(\bigcap_n Z_n\right).$$

Hence, the required equality is proved. Now, taking into account the  $\mu$ -measurability of all sets from the family

$$(pr_1(Z_n))_{n \in \mathbb{N}},$$

we can write

$$\nu\left(\bigcap_n Z_n\right) = \mu^*\left(pr_1\left(\bigcap_n Z_n\right)\right) = \mu\left(\bigcap_n pr_1(Z_n)\right) =$$

$$= \lim_n \mu(pr_1(Z_n)) = \lim_n \nu(Z_n).$$

In this way we have checked that the function  $\nu$  is a capacity with respect to the class  $\Phi$ . Choquet's theorem implies now that any set from the class  $(A)(\Phi)$  is capacitable with respect to  $\nu$ . But, since  $K$  is a locally compact topological space with a countable base, it is easy to see that the following inclusion holds:

$$S \otimes B(K) \subseteq (A)(\Phi).$$

Therefore, for any set  $Z \in S \otimes B(K)$ , we have

$$\nu(Z) = \sup\{\nu(D) : D \in \Phi_\delta \text{ \& } D \subseteq Z\}.$$

In other words, we have

$$\mu^*(pr_1(Z)) = \sup\{\mu(pr_1(D)) : D \in \Phi_\delta \text{ \& } D \subseteq Z\},$$

which immediately gives us the measurability of the set  $pr_1(Z)$  with respect to the original measure  $\mu$ .

The theorem proved above is sometimes called the *theorem on measurable projection*.

Now, let us consider one application of Choquet's theorem to the question about the existence of some measurable selectors. First we introduce one definition and prove one auxiliary assertion.

Let  $(E, S, \mu)$  be a complete probability space (or, more generally, a complete measure space with a  $\sigma$ -finite measure  $\mu$ ) and let  $(K, B(K))$  be a measurable topological space where, as above,  $K$  is a locally compact space with a countable base and  $B(K)$  is the Borel  $\sigma$ -algebra of  $K$ . Now, let  $Z$  be a subset of the Cartesian product  $E \times K$ . We say that the set  $Z$  is a **measurable graph** if

- a)  $Z \in S \otimes B(K)$ ;
- b) for every  $e \in E$  the section  $Z(e)$  contains at most one point.

The next result gives us the characterization of measurable graphs in terms of measurable mappings from  $E$  into  $K$ .

**Theorem 5.3** *Let  $Z \subseteq E \times K$ . Then the following relations are equivalent:*

- 1) *the set  $Z$  is a measurable graph in  $E \times K$  ;*
- 2) *there exist a set  $X \in S$  and a measurable (with respect to the  $\sigma$ -algebras  $S$  and  $B(K)$ ) mapping*

$$g : X \rightarrow K,$$

*such that the equality*

$$Z = \{(x, y) \in X \times K : g(x) = y\}$$

*holds.*

**Proof.** First we prove implication 2)  $\rightarrow$  1). Let condition 2) hold and let us consider the mapping

$$\psi : X \times K \rightarrow K \times K$$

defined by the formula

$$\psi(x, y) = (g(x), y).$$

Let us observe that the mapping  $\psi$  is measurable with respect to the  $\sigma$ -algebras

$$(S | X) \otimes B(K), \quad B(K) \otimes B(K),$$

where  $S | X$  is the restriction of the  $\sigma$ -algebra  $S$  onto the set  $X$ , i.e.

$$S | X = \{X \cap Y : Y \in S\}.$$

Taking into account that the topological space  $K$  has a countable base and is Hausdorff, we see that the diagonal set  $diag(K \times K)$  is measurable with respect to the  $\sigma$ -algebra  $B(K) \otimes B(K)$ . Hence, the set

$$\psi^{-1}(diag(K \times K))$$

is measurable with respect to the  $\sigma$ -algebra  $(S|X) \otimes B(K)$ . But it is obvious that

$$\psi^{-1}(diag(K \times K)) = \{(x, y) \in X \times K : g(x) = y\} = Z.$$

This immediately shows us that  $Z$  is a measurable graph in the product  $E \times K$ .

Now, we shall prove implication 1)  $\rightarrow$  2). Suppose that condition 1) holds. Let us put

$$X = pr_1(Z).$$

The preceding theorem about measurable projection says that the set  $X$  is measurable with respect to the  $\sigma$ -algebra  $S$ . Let us define the mapping  $g$  from the set  $X$  into the topological space  $K$  putting for every  $x \in X$  the value  $g(x)$  equal to the unique point from the intersection

$$Z \cap (\{x\} \times K).$$

Let us check that this mapping is measurable with respect to the  $\sigma$ -algebras  $S | X$  and  $B(K)$ . Let  $D$  be any Borel subset of the topological space  $K$ . Then we can write

$$g^{-1}(D) = pr_1(Z \cap (E \times D)).$$

But it is clear that the set  $Z \cap (E \times D)$  is measurable with respect to the product  $\sigma$ -algebra  $S \otimes B(K)$ . Using the theorem about measurable projection once more we get that the set  $g^{-1}(D)$  is measurable with respect to the  $\sigma$ -algebra  $S$ , so the measurability of the mapping  $g$  is established.

For various applications of theorems proved above most important is the case when the topological space  $K$  coincides with the real line  $\mathbb{R}$  or with the positive half-line

$$\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}.$$

We notice here that the last case can be frequently met in the random processes theory. It is convenient to formulate the results given below for the topological space  $\mathbb{R}^+$ . But taking into account the existence of a Borel isomorphism between  $\mathbb{R}^+$  and any uncountable Polish topological space  $K$ , it can easily be seen that analogous results may be proved for  $K$ , too.

As usual, we assume that the half-line  $\mathbb{R}^+$  is equipped with its Borel  $\sigma$ -algebra  $B(\mathbb{R}^+)$ . Let  $(E, S, \mu)$  be any complete probability space (or, more generally, a complete measure space with a  $\sigma$ -finite measure  $\mu$ ). Let  $Z$  be a subset of the Cartesian product  $E \times \mathbb{R}^+$  such that

$$pr_1(Z) = E.$$



Let us call a **debut** of the set  $Z$  the real function

$$\phi_Z : E \rightarrow \mathbb{R}^+$$

defined by the formula

$$\phi_Z(e) = \inf\{t \in \mathbb{R}^+ : (e, t) \in Z\}.$$

The following easy proposition is true.

**Theorem 5.4** *Let  $(E, S, \mu)$  be a complete probability space (or, more generally, a complete measure space with a  $\sigma$ -finite measure  $\mu$ ) and let  $Z$  be a subset of the Cartesian product  $E \times \mathbb{R}^+$  which is measurable with respect to the product  $\sigma$ -algebra  $S \otimes B(\mathbb{R}^+)$ . Suppose also that*

$$pr_1(Z) = E.$$

*Then the debut  $\phi_Z$  of the set  $Z$  is a real function measurable with respect to the  $\sigma$ -algebras  $S$  and  $B(\mathbb{R}^+)$ .*

**Proof.** Indeed, for any real number  $t > 0$  the set

$$\phi_Z^{-1}([0, t]) = \{e \in E : \phi_Z(e) < t\}$$

is an image of the set

$$Z \cap (E \times [0, t])$$

by the projection  $pr_1$ . Since the last set is measurable with respect to the  $\sigma$ -algebra  $S \otimes B(\mathbb{R}^+)$  then, by the theorem about measurable projection, the set  $\phi_Z^{-1}([0, t])$  is measurable with respect to  $S$ . Hence, the function  $\phi_Z$  is measurable.

The next result is connected with a large group of the so called *uniformization theorems* or *theorems about measurable selectors*. We shall discuss some of these theorems later on in our book.

**Theorem 5.5** *Let  $(E, S, \mu)$  be any complete probability space (or, more generally, a complete measure space with a  $\sigma$ -finite measure  $\mu$ ) and let  $Z$  be a subset of the Cartesian product  $E \times \mathbb{R}^+$  which is measurable with respect to the product  $\sigma$ -algebra  $S \otimes B(\mathbb{R}^+)$ . Suppose also that*

$$pr_1(Z) = E.$$

*Then there exists a real function*

$$g : E \rightarrow \mathbb{R}^+$$

*measurable with respect to the  $\sigma$ -algebras  $S$  and  $B(\mathbb{R}^+)$  such that its graph is contained in the given set  $Z$ .*

**Proof.** It is sufficient to show that for every real number  $\epsilon > 0$  there exist a set  $X_\epsilon \in S$  and a measurable real function

$$g_\epsilon : X_\epsilon \rightarrow \mathbb{R}^+,$$

such that the following conditions hold :

- a)  $\mu(E \setminus X_\epsilon) < \epsilon$  ;
- b) the graph of the function  $g_\epsilon$  is contained in the given set  $Z$  .

It is clear that if we prove this fact then the rest of the proof will be evident: after finding a set  $X_\epsilon$  and a function  $g_\epsilon$  we move to the measure space

$$(E \setminus X_\epsilon, S|(E \setminus X_\epsilon), \mu|(S|(E \setminus X_\epsilon)))$$

and to the set

$$Z \cap ((E \setminus X_\epsilon) \times \mathbb{R}^+)$$

and continue analogous process for the real number  $\frac{\epsilon}{2}$ , the new measure space and the new set. After a countable number of steps we can fill the space  $E$  (except, may be, for to a  $\mu$ -measure zero set) with a countable family of constructed  $\mu$ -measurable sets and obtain the required function  $g$  from the functions of type  $g_\epsilon$  and a function defined on a  $\mu$ -measure zero set.

So let  $\epsilon$  be any strictly positive real number. Let us consider the class  $Comp(\mathbb{R}^+)$  of all compact subsets of the real half-line  $\mathbb{R}^+$ . Moreover, let us consider the class  $\Phi$  of all sets  $Z \subseteq E \times \mathbb{R}^+$  which can be represented in the form

$$Z = \bigcup_{1 \leq i \leq m} (X_i \times Y_i),$$

where  $m$  is an arbitrary natural number,  $X_i$  are members of  $S$  and  $Y_i$  are members of  $Comp(\mathbb{R}^+)$ . It is clear that the class  $\Phi$  generates the product  $\sigma$ -algebra  $S \otimes B(\mathbb{R}^+)$  and that the same class  $\Phi$  is closed under finite unions and finite intersections. Finally, it is easy to see that

$$S \otimes B(\mathbb{R}^+) \subseteq A(\Phi).$$

Using the same arguments as in the proof of the theorem about measurable projection, we can find for the given  $\epsilon$  and the set  $Z$  a set

$$Z^* \subseteq Z$$

from the class  $\Phi_\delta$  (i.e. from the class of all countable intersections of members of the class  $\Phi$ ) such that

$$\mu(pr_1(Z) \setminus pr_1(Z^*)) = \mu(E \setminus pr_1(Z^*)) < \epsilon.$$

Let  $\phi_{Z^*}$  be a debut of the set  $Z^*$  considered only on the set  $pr_1(Z^*)$  which is measurable with respect to the  $\sigma$ -algebra  $S$ . Notice now that for each element  $x \in pr_1(Z^*)$  the section

$$Z^*(x) = \{t \in \mathbb{R}^+ : (x, t) \in Z^*\}$$

is a non-empty compact subset of  $\mathbb{R}^+$  (because it is an intersection of some countable family of compact subsets of  $\mathbb{R}^+$ ). So we have

$$\inf(Z^*(x)) \in Z^*(x).$$

From this fact it follows that the graph of the function  $\phi_{Z^*}$  is contained in the set  $Z^*$ , so it is contained in the original set  $Z$ , too. It remains to remember that the function  $\phi_{Z^*}$  is measurable with respect to the  $\sigma$ -algebras  $S$  and  $B(\mathbb{R}^+)$ . So we can put

$$X_\epsilon = pr_1(Z^*), \quad g_\epsilon = \phi_{Z^*}.$$

Thus, theorem 5 is proved.

It is obvious that the notion of the debut of a given set lying in the Cartesian product  $E \times \mathbb{R}^+$  can be generalized to such subsets of  $E \times \mathbb{R}^+$ , the first projections

of which are not necessarily the whole space  $E$ . After this generalization it will be clear that if the given set  $Z \subseteq E \times \mathbb{R}^+$  is measurable with respect to the product  $\sigma$ -algebra  $S \otimes B(\mathbb{R}^+)$  then its debut is a measurable function from the set  $pr_1(Z)$  into  $\mathbb{R}^+$ . From this fact it is also clear how to generalize Theorems 4 and 5 to any set  $Z \subseteq E \times \mathbb{R}^+$  measurable with respect to the product  $\sigma$ -algebra  $S \otimes B(\mathbb{R}^+)$ .

Using the same arguments as in the proof of Theorem 5 we can obtain the following classical result.

**Theorem 5.6** *Let  $E$  be a Polish topological space and let  $Z$  be an analytic subset of the topological product  $E \times \mathbb{R}^+$ . Suppose that a Borel finite (or, more generally,  $\sigma$ -finite) measure  $\mu$  on  $E$  is given and let  $\bar{\mu}$  be the standard completion of the measure  $\mu$ . Then there exists a real function*

$$g : pr_1(Z) \rightarrow \mathbb{R}^+$$

such that

- 1)  $g$  is measurable with respect to the  $\sigma$ -algebras  $dom(\bar{\mu})$  and  $B(\mathbb{R}^+)$ ;
- 2) the graph of  $g$  is contained in the set  $Z$ .

Now, let  $P$  be an arbitrary uncountable Polish topological space. As we know there exists a Borel isomorphism

$$\varphi : (\mathbb{R}^+, B(\mathbb{R}^+)) \rightarrow (P, B(P)).$$

It is easy to see that for this isomorphism the equality

$$\varphi(A(\mathbb{R}^+)) = A(P)$$

holds. Taking into account this remark, we can immediately deduce the following result from Theorem 6.

**Theorem 5.7** *Let  $E_1$  and  $E_2$  be two Polish topological spaces and let  $Z$  be an analytic subset of the topological product  $E_1 \times E_2$ . Suppose also that a Borel finite (or  $\sigma$ -finite) measure  $\mu$  on  $E_1$  is given and let  $\bar{\mu}$  be the standard completion of the measure  $\mu$ . Then there exists a mapping*

$$g : pr_1(Z) \rightarrow E_2$$

such that

- 1)  $g$  is measurable with respect to the  $\sigma$ -algebras  $dom(\bar{\mu})$  and  $B(E_2)$ ;
- 2) the graph of  $g$  is contained in the set  $Z$ .

In many cases the theorem formulated above is sufficient for various applications. But sometimes we need more subtle formulations of theorems of this type. Let us remark that in Appendix C of this book we shall consider some much stronger results than Theorem 7.

We have shown in this Chapter that all analytic subsets of the real line  $\mathbb{R}$  are Lebesgue measurable. This classical fact was first proved by Luzin. Since there exists a deep analogy between the Lebesgue measurability and the Baire property on the real line, one can formulate the following natural question: do the analytic subsets of  $\mathbb{R}$  have the Baire property? It has turned out that the answer to this question is positive. Moreover, the stronger result is true: all analytic subsets of

$\mathbb{R}$  have the Baire property in the restricted sense. This classical result, in fact, is due to Luzin and Sierpiński. We decided to prove mentioned Luzin–Sierpiński’s theorem in this Chapter, too. Of course, we cannot base the proof of their result on Choquet’s theorem, which in some sense is of an approximation type (actually, the Baire property cannot be approximated downward by compact sets). In our proof we will use Sierpiński’s theorem about representation of analytic sets (see Theorem 5 of Chapter 3).

**Theorem 5.8** *Let  $E$  be any topological space which satisfies the Suslin condition. Then the family of all sets in  $E$  with the Baire property is invariant under the  $(A)$ -operation.*

**Proof.** Let

$$(X_s)_{s \in \mathbb{N}^{<\omega}}$$

be any family of sets in  $E$  with the Baire property. As we know

$$X = (A)((X_s)_{s \in \mathbb{N}^{<\omega}}) = \bigcap_{\xi < \omega_1} Y_\xi = \bigcup_{\xi < \omega_1} Z_\xi,$$

where

- 1)  $Y_\xi = \bigcup_n X_n^\xi,$
- 2)  $T_\xi = \bigcup_s (X_s^\xi \setminus X_s^{\xi+1}),$
- 3)  $Z_\xi = Y_\xi \setminus T_\xi.$

and the sets

$$X_s^\xi \quad (\xi < \omega_1, s \in \mathbb{N}^{<\omega})$$

are defined in Chapter 3 by the transfinite recursion on the index  $\xi$ . There we notice also that all sets

$$X_s^\xi, \quad Y_\xi, \quad T_\xi, \quad Z_\xi$$

belong to the  $\sigma$ -algebra of sets generated by the original family  $(X_s)_{s \in \mathbb{N}^{<\omega}}$ . Therefore, all the mentioned sets have the Baire property, too. Let us put

$$D_s^\xi = X_s^\xi \setminus X_s^{\xi+1} \quad (\xi < \omega_1, s \in \mathbb{N}^{<\omega}).$$

Since for every  $s \in \mathbb{N}^{<\omega}$  and  $\zeta \leq \xi < \omega_1$  we have

$$X_s^\xi \subseteq X_s^\zeta,$$

we see that

$$(D_s^\xi)_{\xi < \omega_1}$$

is a family of pairwise disjoint sets for every fixed  $s \in \mathbb{N}^{<\omega}$ . Moreover, all sets  $D_s^\xi$  have the Baire property. Since the space  $E$  satisfies the Suslin condition, there exists  $\xi(s) < \omega_1$  such that for any  $\xi > \xi(s)$  the set  $D_s^\xi$  is a first category set in  $E$ . But the family  $\mathbb{N}^{<\omega}$  is countable, so there exists some ordinal  $\alpha < \omega_1$  such that

$$(\forall s \in \mathbb{N}^{<\omega})(\xi(s) < \alpha).$$

Hence, all sets of the family

$$(D_s^\alpha)_{s \in \mathbb{N}^{<\omega}}$$

are of the first category. Notice now that

$$T_\alpha = \bigcup_s D_s^\alpha.$$

Therefore,  $T_\alpha$  is also a first category set in  $E$ . Moreover, it is easy to see that the following inclusions hold:

$$X \setminus Z_\alpha \subseteq Y_\alpha \setminus Z_\alpha \subseteq T_\alpha,$$

hence, the set  $X \setminus Z_\alpha$  is also a first category subset of  $E$ . But it is obvious that

$$X = (X \setminus Z_\alpha) \cup Z_\alpha,$$

and we obtain that the set  $X$  has the Baire property.

In the same way we can prove that in every topological space  $E$  which hereditarily satisfies the Suslin condition (i.e. any subspace of  $E$  satisfies the Suslin condition) the family of all sets with the Baire property in the restricted sense is invariant under  $(A)$ -operation. Indeed, it is sufficient to consider a system of sets

$$(E' \cap X_s)_{s \in \mathbb{N}^{<\omega}},$$

where  $E'$  is any subspace of  $E$  and all sets  $X_s$  ( $s \in \mathbb{N}^{<\omega}$ ) have the Baire property in the restricted sense. Applying the previous theorem to this system of sets we get a required result.

Notice now that any subspace of a Polish topological space evidently satisfies the Suslin condition (and, moreover, is separable). Hence, we obtain the following proposition:

**Theorem 5.9** *In every Polish topological space  $E$  all analytic subsets of  $E$  have the Baire property in the restricted sense.*

Let us remark at the end of this Chapter that the last results about the Baire property of analytic subsets of the topological spaces are true also for spaces which do not satisfy the Suslin condition. However, the proof of this fact requires another method which will be considered in Part 2 of this book.

## Exercises

**Exercise 5.1** *Let  $E$  be any basic set and let  $\Phi$  be some class of subsets of  $E$ . We say that  $\Phi$  is a **quasi-compact class** (in the sense of Marczewski) if for every countable family*

$$(Z_n)_{n \in \mathbb{N}} \subseteq \Phi$$

*the relation  $\bigcap_n Z_n = \emptyset$  implies that there exists a finite subfamily of  $(Z_n)_{n \in \mathbb{N}}$  with empty intersection, too. For instance, if  $E$  is an arbitrary Hausdorff topological space and  $\Phi$  is the class of all compact subsets of  $E$ , then it is easy to see that  $\Phi$  is a quasi-compact class of sets.*

*Now, let  $\Phi$  be any quasi-compact class of subsets of the basic set  $E$  and let  $\Phi^*$  be the class of all finite unions of elements from the class  $\Phi$ . Prove that the class  $\Phi^*$  is also a quasi-compact class.*

**Exercise 5.2** *Give generalizations of some results presented in this Chapter to the case when the locally compact topological space  $K$  with a countable base is replaced by a basic set  $F$  with a quasi-compact class  $\Phi \subseteq P(F)$ .*

**Exercise 5.3** *Show that any locally compact topological space with a countable base is homeomorphic to such a subset of a compact metric space the complement of which is a singleton. In particular, deduce from this fact that any locally compact topological space with a countable base is a Polish space.*

**Exercise 5.4** Let  $(E, S, \mu)$  be an arbitrary measure space with a  $\sigma$ -finite measure  $\mu$  and let  $X$  be a subset of  $E$ . We say that the set  $X$  has the **uniqueness property** with respect to the measure  $\mu$  if the following two conditions hold:

a) there exists a measure  $\nu$  extending the measure  $\mu$  such that

$$X \in \text{dom}(\nu);$$

b) for any two measures  $\nu_1$  and  $\nu_2$  which extend  $\mu$  and satisfy the relation

$$X \in \text{dom}(\nu_1) \cap \text{dom}(\nu_2)$$

we have the equality

$$\nu_1(X) = \nu_2(X).$$

Let us denote by the symbol  $Un(\mu)$  the class of all subsets of the basic set  $E$  which have the uniqueness property with respect to the original measure  $\mu$ . Let  $\bar{\mu}$  be, as usual, the completion of the measure  $\mu$ . Prove that the following equality holds:

$$Un(\mu) = \text{dom}(\bar{\mu}).$$

**Exercise 5.5** Let  $(E, S, \mu)$  be a measure space with a  $\sigma$ -finite measure  $\mu$  and let  $X_1$  be any subset of  $E$ . Prove that there exists a measure  $\mu_1$  defined on some  $\sigma$ -algebra of subsets of  $E$  such that

a)  $\mu_1$  is an extension of  $\mu$ ;

b)  $X_1 \in \text{dom}(\mu_1)$ .

Deduce from this fact that for an arbitrary finite family  $\{X_1, \dots, X_n\}$  of subsets of the basic set  $E$  there exists a measure  $\nu$  defined on some  $\sigma$ -algebra of subsets of  $E$  such that

c)  $\nu$  is an extension of  $\mu$ ;

d)  $\{X_1, \dots, X_n\} \subseteq \text{dom}(\nu)$ .

In the further considerations we will see that an analogous result is not longer true if we deal with a countable family  $\{X_1, \dots, X_n, \dots\}$  of subsets of  $E$ .

Now, let  $\mu$  be a  $\sigma$ -finite complete measure on  $E$ , let  $S(\mu) = \text{dom}(\mu)$  and let  $I(\mu)$  be the  $\sigma$ -ideal of all  $\mu$ -measure zero subsets of  $E$ . Thus, we have a measure Boolean algebra  $A(\mu) = S(\mu)/I(\mu)$  canonically associated with  $\mu$ . Let  $X$  be an arbitrary subset of  $E$  non-measurable with respect to  $\mu$ . Prove that there exists a measure  $\nu$  on  $E$  such that

a)  $\nu$  is an extension of  $\mu$ ;

b)  $X$  is measurable with respect to  $\nu$ ;

c) the canonical embedding of the Boolean algebra  $A(\mu)$  into the Boolean algebra  $A(\nu)$  is not a surjection, i.e.  $A(\mu)$  is a proper subalgebra of  $A(\nu)$ .

**Exercise 5.6** Starting with the fact that there are two disjoint sets in  $\mathbb{R}$  which are complements of analytic sets and are not separated by any two Borel subsets of  $\mathbb{R}$ , prove the following proposition: there exists a Borel set  $Z \subseteq \mathbb{R}^2$  such that

a)  $pr_1(Z) = \mathbb{R}$ ;

b) there is no Borel function  $f : \mathbb{R} \rightarrow \mathbb{R}$  the graph of which is contained in the set  $Z$ .

This classical result is due to Luzin and Novikov and shows us that in the class of all Borel subsets of the plane  $\mathbb{R}^2$  the uniformization problem, in general, has a negative solution. This fact also explains why in Theorem 5 of this Chapter we consider measurable, but not Borel, functions. The result of Luzin and Novikov will be proved in Part 2 of this book.

**Exercise 5.7** Let  $(E, S)$  be a measurable space and let  $X$  be a subset of  $E$ . We say that  $X$  is **absolutely measurable** with respect to the  $\sigma$ -algebra  $S$  if for each  $\sigma$ -finite measure  $\mu$  defined on  $S$  the following relation holds:

$$X \in \text{dom}(\bar{\mu}),$$

where  $\bar{\mu}$ , as usual, is the completion of the measure  $\mu$ . Deduce from Choquet's theorem that all members of the analytic class  $(A)(S)$  are absolutely measurable with respect to the  $\sigma$ -algebra  $S$ .

Moreover, let us consider the particular case when

$$(E, S) = (\mathbb{R}, B(\mathbb{R})).$$

In this case give an example of a Lebesgue measurable subset of  $\mathbb{R}$  which is not absolutely measurable with respect to the Borel  $\sigma$ -algebra  $B(\mathbb{R})$ .

**Exercise 5.8** Let  $E$  be an arbitrary Hausdorff topological space and let  $X$  be a subset of  $E$  which is a Radon space with respect to the induced topology. Show that the set  $X$  is absolutely measurable with respect to the Borel  $\sigma$ -algebra  $B(E)$ .

**Exercise 5.9** Let  $E$  be an arbitrary Radon topological space and let  $X$  be a subset of  $E$  which is absolutely measurable with respect to the Borel  $\sigma$ -algebra  $B(E)$ . Show that the set  $X$  equipped with the induced topology is a Radon topological space. In particular, every analytic subset of a Polish topological space is a Radon space. The same is true for the complement of any analytic subset of a Polish space.

**Exercise 5.10** Does there exist a subset of the real line  $\mathbb{R}$  which is absolutely measurable with respect to the Borel  $\sigma$ -algebra  $B(\mathbb{R})$  and which has not the Baire property in  $\mathbb{R}$ ?

**Exercise 5.11** Starting with Choquet's theorem prove directly Alexandrov-Hausdorff theorem which states that any uncountable Borel subset of a Polish topological space has the cardinality continuum.

**Exercise 5.12** Can one deduce from Choquet's theorem the classical Caratheodory theorem on extensions of measures from algebras to  $\sigma$ -algebras?

**Exercise 5.13** Let  $I$  be an arbitrary set of indices. For any index  $i \in I$  let us put

$$(E_i, S_i) = (\mathbb{R}, B(\mathbb{R})).$$

Furthermore, suppose that for every finite set  $\{i_1, \dots, i_n\}$  of pairwise distinct indices from  $I$  there is a corresponding probability Borel measure

$$\mu_{\{i_1, \dots, i_n\}}$$

on the  $n$ -dimensional Euclidean space  $E_{i_1} \times \dots \times E_{i_n}$ . Finally, let us suppose that for any two finite sets

$$\{i_1, \dots, i_n\} \subseteq I, \quad \{j_1, \dots, j_m\} \subseteq I$$

such that

$$\{i_1, \dots, i_n\} \subseteq \{j_1, \dots, j_m\}$$

the equality

$$\mu_{\{i_1, \dots, i_n\}}(X) = \mu_{\{j_1, \dots, j_m\}}(\{x \in E_{j_1} \times \dots \times E_{j_m} : (x_{i_1}, \dots, x_{i_n}) \in X\})$$

holds, whatever  $X \in B(E_{i_1} \times \dots \times E_{i_n})$  is taken. Prove that there exists a probability measure  $\mu$  defined on the  $\sigma$ -algebra

$$S = \otimes_{i \in I} S_i$$

and such that for every finite set  $\{x_{i_1}, \dots, x_{i_n}\} \subseteq I$  we have

$$\mu(\{x \in \prod_{i \in I} E_i : (x_{i_1}, \dots, x_{i_n}) \in X\}) = \mu_{\{i_1, \dots, i_n\}}(X),$$

whatever  $X \in B(E_{i_1} \times \dots \times E_{i_n})$  is taken.

This classical result is due to Kolmogorov.

**Exercise 5.14** Generalize the result of the previous exercise to the case when for any index  $i \in I$  we have

$$(E_i, S_i) = (K_i, B(K_i)),$$

where  $K_i$  is a Radon topological space.

**Exercise 5.15** Let  $E$  be a basic set and let  $\Phi$  be a class of subsets of  $E$ . Suppose that  $E \in \Phi$  and  $\Phi$  is closed under countable unions and under countable intersections. Suppose also that a real function

$$\nu : \Phi \rightarrow \mathbb{R}$$

is given and for any increasing or decreasing (with respect to the inclusion) sequence of sets  $(X_n)_{n \in \mathbb{N}} \subseteq \Phi$  we have

$$\nu(\lim_n X_n) = \lim_n \nu(X_n).$$

A function  $f : E \rightarrow \mathbb{R}$  is called  $\nu$ -measurable if  $f^{-1}(A) \in \Phi$  for every Borel subset  $A$  of  $\mathbb{R}$ . Let  $(g_n)_{n \in \mathbb{N}}$  be an arbitrary sequence of  $\nu$ -measurable real functions convergent at any point  $x \in E$ . Let us put

$$g(x) = \lim_n g_n(x) \quad (x \in E).$$

Show that the function  $g$  is also  $\nu$ -measurable. Prove that for each real number  $\epsilon > 0$  there exists a set  $X \in \Phi$  such that

a)  $\nu(X) > \nu(E) - \epsilon;$

b) the sequence  $(g_n)_{n \in \mathbb{N}}$  converges uniformly to  $g$  on  $X$ .

Deduce from this fact the classical Egorov theorem from measure theory. Then as a corollary prove the classical Lebesgue theorem stating that for any finite measure  $\mu$  the convergence almost everywhere of a sequence of  $\mu$ -measurable real functions implies the convergence in measure  $\mu$  of this sequence.



**Exercise 5.16** Let us consider a measure space  $(\mathbb{R}, \text{dom}(\lambda), \lambda)$  where  $\lambda$  is the classical Lebesgue measure on  $\mathbb{R}$ . Let  $P$  be an arbitrary projective subset of  $\mathbb{R}$  and let  $B(P)$  be the Borel  $\sigma$ -algebra of the topological space  $P$ . Let us put

$$S = \text{dom}(\lambda) \otimes B(P)$$

and consider the canonical projection

$$\text{pr}_1 : \mathbb{R} \times P \rightarrow \mathbb{R}.$$

Assuming that all projective subsets of  $\mathbb{R}$  are Lebesgue measurable, prove that

$$(\forall X \in S)(\text{pr}_1(X) \in \text{dom}(\lambda)).$$

Similarly, let us put

$$T = \bar{B}(\mathbb{R}) \otimes B(P),$$

where  $\bar{B}(\mathbb{R})$  is the  $\sigma$ -algebra of all subsets of  $\mathbb{R}$  which have the Baire property. Assuming that all projective subsets of  $\mathbb{R}$  have the Baire property prove that

$$(\forall X \in T)(\text{pr}_1(X) \in \bar{B}(\mathbb{R})).$$

Recall that the Lebesgue measurability and the Baire property of all projective subsets of  $\mathbb{R}$  follow from the Axiom of Projective Determinacy (see Part 2 of this book).

**Exercise 5.17** Let  $E$  be any topological space. Prove that the Baire property (respectively, the Baire property in the restricted sense) is preserved in  $E$  by the  $A$ -operation.

**Exercise 5.18** Use the method of the proof of Theorem 8 from this Chapter and show that in any topological space  $E$  all analytic sets are absolutely measurable with respect to the Borel  $\sigma$ -algebra  $B(E)$ .

Notice also that this fact is an immediate consequence of the result of the previous exercise. Namely, if we consider the von Neumann topology (see Appendix B) associated with the completion  $\bar{\mu}$  of a  $\sigma$ -finite Borel measure  $\mu$  defined on our topological space  $E$ , then we easily obtain the required fact.

**Exercise 5.19** Let  $X$  be an analytic subset of a Polish topological space, let  $E$  be a metric space equipped with a  $\sigma$ -finite Borel measure  $\mu$  and let  $f : X \rightarrow E$  be a Borel mapping. Show that, for every real number  $\epsilon > 0$ , there exists a subset  $Y$  of  $X$  such that

- a)  $Y$  is a compact space;
- b) the restriction of  $f$  to  $Y$  is a homeomorphism between the spaces  $Y$  and  $f(Y)$ ;
- c)  $\mu(f(X) \setminus f(Y)) < \epsilon$ .

## Chapter 6

# The Structure of the Real Line

The real line  $\mathbb{R}$  is undoubtedly a very important mathematical object. A precise construction of real numbers from rational numbers was independently done in the second part of the nineteenth century by Cantor, Meray, Dedekind and Weierstrass. We assume that the reader knows the standard construction of real numbers which starts with the set of all natural numbers, then goes on to integers, next to rational numbers and the last step is achieved by some method of completion of the set of all rationals. Of course, here we are not going to discuss in detail all these constructions. The aim of this Chapter is to present some characteristic properties of the reals.

First, let us recall several notions and definitions from the theory of partially ordered sets. Let  $(E, \leq)$  be any partially ordered set. We say that a set  $X \subseteq E$  is **upward bounded** if

$$(\exists e \in E)(\forall x \in X)(x \leq e).$$

Similarly, we say that a set  $X \subseteq E$  is **downward bounded** if

$$(\exists e \in E)(\forall x \in X)(e \leq x).$$

Furthermore, we say that a set  $X \subseteq E$  has a **supremum** if there exists an element  $e \in E$  such that

$$\begin{aligned} &(\forall x \in X)(x \leq e), \\ &(\forall y \in E)((\forall x \in X)(x \leq y) \rightarrow e \leq y). \end{aligned}$$

It is easy to see that such an element  $e$  is unique and we write  $e = \sup(X)$ . In an analogous way we define an **infimum** of a set  $X \subseteq E$ .

Now let  $(E, \leq)$  be an arbitrary linearly ordered set. An ordered pair  $(A, B)$  of subsets of the basic set  $E$  is called a **Dedekind cut** of  $(E, \leq)$  if

$$A \cup B = E,$$

$$(\forall a \in A)(\forall b \in B)(a < b).$$

In particular, we see that sets  $A$  and  $B$  are disjoint. If they are also non-empty then the pair  $(A, B)$  is called a **bounded Dedekind cut** of  $(E, \leq)$ .

We say that a linearly ordered set  $(E, \leq)$  is **Dedekind complete** if for every bounded Dedekind cut  $(A, B)$  of  $E$  there exists an element  $e \in E$  such that

$$(\forall a \in A)(a \leq e) \ \& \ (\forall b \in B)(e \leq b).$$

Of course, the last notion is fundamental in the theory of linearly ordered sets. This notion gives us a possibility to prove an important theorem (due to Dedekind) on embedding a linearly ordered set  $(E, \leq)$  into a Dedekind complete linearly ordered set. The most interesting case is obtained when the initial linearly ordered set  $(E, \leq)$  is a **dense linearly ordered** set. Since we will need this notion in the sequel, we recall it now.

We say that a subset  $X$  of  $E$  is dense in  $(E, \leq)$  if for any non-empty open interval  $]a, b[ \subseteq E$  we have

$$]a, b[ \cap X \neq \emptyset.$$

In other words,  $X$  is dense in  $(E, \leq)$  if and only if  $X$  is a dense subset of the basic set  $E$  equipped with its order topology generated by all open intervals in  $E$ . Finally, we say that  $(E, \leq)$  is a dense linearly ordered set if for any two elements  $a \in E$  and  $b \in E$  such that  $a < b$  we have

$$]a, b[ \neq \emptyset.$$

Now, let us denote by  $D(E)$  the set of all Dedekind cuts of a dense linearly ordered set  $(E, \leq)$ . The set  $D(E)$  is also linearly ordered by the following relation:

$$(A, B) \leq_* (A', B') \iff A \subseteq A'.$$

It is easy to see that the mapping

$$\Phi : (E, \leq) \rightarrow (D(E), \leq_*)$$

defined by the formula

$$\Phi(e) = (\{x \in E : x \leq e\}, \{x \in E : x > e\})$$

is a monomorphism from the linearly ordered set  $(E, \leq)$  into the linearly ordered set  $(D(E), \leq_*)$ . Moreover, it is not difficult to check that the set  $\text{ran}(\Phi)$  is a dense subset of  $(D(E), \leq_*)$  and that  $(D(E), \leq_*)$  is a complete linearly ordered set (in particular, Dedekind complete linearly ordered set). Taking into account these facts we formulate the following definition. If the initial linearly ordered set  $(E, \leq)$  has neither the least element, nor the largest element (shortly, has not end-points), then the linearly ordered set

$$(D(E) \setminus \{(\emptyset, E), (E, \emptyset)\}, \leq_*)$$

is called a **Dedekind completion** of the initial linearly ordered set  $(E, \leq)$ . For example, the real line  $\mathbb{R}$  equipped with its standard ordering is the Dedekind completion of the set of all rational numbers  $\mathbb{Q}$  also equipped with the standard ordering.

Of course, the standard linear ordering is one of the fundamental structures on the real line  $\mathbb{R}$ . There are also two fundamental binary algebraic operations defined on  $\mathbb{R}$ : addition and multiplication. As we know, the real line is an algebraic (commutative) field with respect to these operations. The connection between the ordering structure and the algebraic operations converts  $\mathbb{R}$  into an ordered field. Let us recall that the structure

$$\mathcal{K} = (K, +, \cdot, 0, 1, \leq)$$

is a *linearly ordered field* (or simply **ordered field**) if  $(K, +, \cdot, 0, 1)$  is a field and  $\leq$  is a linear ordering of the basic set  $K$  such that the following two properties hold:

$$(\forall x, y, z \in K)(x \leq y \rightarrow x + z \leq y + z);$$

$$(\forall x, y \in K)((0 \leq x \ \& \ 0 \leq y) \rightarrow 0 \leq x \cdot y).$$

We assume that the reader knows some simple consequences of these properties. For example, it is easy to check that

$$0 \leq x^2$$

for every element  $x \in K$ .

As stated above, the field of all real numbers is the canonical and very important example of an ordered field. Obviously, any subfield of  $\mathbb{R}$  (in particular, the field  $\mathbb{Q}$ ) is an ordered field. There are also some natural examples of ordered fields which extend the field of all real numbers.

**Example 1.** Let  $\mathbb{R}[x]$  denote the field of all rational functions of a real variable  $x$  with real coefficients. In other words, any element of  $\mathbb{R}[x]$  can be represented in the form

$$x \mapsto \frac{p_1(x)}{p_2(x)} \quad (x \in \mathbb{R}),$$

where  $p_1$  and  $p_2$  are some polynomials with the real coefficients. Of course, we have a canonical injection

$$\phi : \mathbb{R} \rightarrow \mathbb{R}[x]$$

defined by the formula

$$\phi(a)(x) = a \quad (x \in \mathbb{R})$$

for every  $a \in \mathbb{R}$ . This injection is a monomorphism from the field  $\mathbb{R}$  into the field  $\mathbb{R}[x]$ . Using this monomorphism we may identify  $\mathbb{R}$  with  $\phi(\mathbb{R})$ . Now let us define a binary relation  $\leq_*$  on the field  $\mathbb{R}[x]$  by the following formula:

$$f \leq_* g \iff (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x \leq y \rightarrow f(y) \leq g(y)).$$

It is easy to check that  $\leq_*$  is a linear ordering of  $\mathbb{R}[x]$  and the tuple

$$(\mathbb{R}[x], +, \cdot, 0, 1, \leq_*)$$

is a linearly ordered field. In this ordered field we have the inequalities

$$0 \leq_* \dots \leq_* \frac{1}{x^n} \leq_* \dots \leq_* \frac{1}{x^2} \leq_* \frac{1}{x} \leq_* 1,$$

$$1 \leq_* 2 \leq_* \dots \leq_* x \leq_* x^2 \leq_* \dots \leq_* x^n \leq_* \dots$$

Now we want to formulate and prove the first characterization of the real line in terms of ordered fields.

**Theorem 6.1** *Suppose that*

$$\mathcal{K} = (K, +_K, \cdot_K, 0_K, 1_K, \leq_K)$$

*is an ordered field such that one of the following two conditions holds :*

- 1) *any non-empty upward bounded subset of  $K$  has a supremum in  $K$  (the Weierstrass Axiom);*
- 2)  *$(K, \leq_K)$  is Dedekind complete (the Dedekind Axiom).*

*Then  $\mathcal{K}$  is isomorphic with the structure  $(\mathbb{R}, +, \cdot, 0, 1, \leq)$ .*

**Proof.** First let us observe that  $0_K <_K 1_K$ . Indeed, if we assume otherwise then from the inequality  $1_K <_K 0_K$  we would get

$$0_K = 1_K + (-1_K) <_K 0_K + (-1_K) = -1_K,$$

$$0_K = 0_K \cdot 0_K <_K (-1_K) \cdot (-1_K) = 1_K.$$

So we obtain a contradiction. Therefore, the inequality  $0_K <_K 1_K$  holds. Hence, for any strictly positive natural number  $n \in \mathbb{N}$  we have

$$0_K \leq_K n \cdot 1_K.$$

In particular, our field  $K$  has the characteristic zero. Let us put

$$F = \left\{ \frac{n \cdot 1_K}{m \cdot 1_K} : n \in \mathbb{Z} \text{ \& } m \in \mathbb{N} \setminus \{0\} \right\},$$

where  $\mathbb{Z}$  denotes the set of all integers. It is not difficult to check that  $F$  is some subfield of the field  $K$ . Moreover, the function

$$\Phi : \mathbb{Q} \rightarrow F$$

defined by the formula

$$\Phi\left(\frac{n}{m}\right) = \frac{n \cdot 1_K}{m \cdot 1_K} \quad \left(\frac{n}{m} \in \mathbb{Q}\right)$$

is the canonical isomorphism between the structure

$$(\mathbb{Q}, +, \cdot, 0, 1, \leq)$$

and the structure

$$(F, +_F, \cdot_F, 0_F, 1_F, \leq_F).$$

Now, suppose that for our initial structure  $\mathcal{K}$  the Weierstrass Axiom is valid. Let us try to extend the function  $\Phi$  to such a bijection

$$\bar{\Phi} : \mathbb{R} \rightarrow K$$

which preserves the algebraic operations and ordering. For this purpose let us put

$$\bar{\Phi}(x) = \sup\{\Phi(q) : q \in \mathbb{Q} \text{ \& } q < x\} \quad (x \in \mathbb{R}).$$

It is clear that  $\bar{\Phi}$  is an injection. We want to establish that the function  $\bar{\Phi}$  is also a surjection. Take any  $z \in K$ . We assert that

$$z = \sup\{q \cdot 1_K : q \in \mathbb{Q} \text{ \& } q \cdot 1_K <_K z\}.$$

Indeed, suppose that this equality is not true and let

$$z' = \sup\{q \cdot 1_K : q \in \mathbb{Q} \text{ \& } q \cdot 1_K <_K z\}.$$

Then we have  $z - z' > 0$ . Furthermore, it is not difficult to check that

$$]0, z - z'[ \cap \{q \cdot 1_K : q \in \mathbb{Q}\} = \emptyset.$$

Let us put

$$a = \frac{1_K}{z - z'}.$$

Obviously, the relation

$$\{x \in K : a <_K x\} \cap \{q \cdot 1_K : q \in \mathbb{Q}\} = \emptyset$$

holds. In particular, this relation implies that the set

$$\{n \cdot 1_K : n \in \mathbb{N}\}$$

is upward bounded. Let

$$t = \sup\{n \cdot 1_K : n \in \mathbb{N}\}.$$

Let  $m$  be such a natural number that

$$t - \frac{1_K}{2 \cdot 1_K} <_K m \cdot 1_K \leq_K t.$$

Then we have the relation

$$t <_K (m+1) \cdot 1_K,$$

which gives us a contradiction. Hence, any element  $z \in K$  is equal to

$$\sup\{q \cdot 1_K : q \in \mathbb{Q} \text{ \& } q \cdot 1_K <_K z\}.$$

From this fact it immediately follows that the mapping  $\bar{\Phi}$  is a surjection and thus it is a bijection. Since the mapping  $\bar{\Phi}$  preserves algebraic operations and ordering it is easy to see that the mapping  $\bar{\Phi}$  also preserves algebraic operations and ordering. Therefore,  $\bar{\Phi}$  is an isomorphism between these ordered fields.

Finally, it remains to prove that the Dedekind Axiom implies the thesis of the theorem, too. For this aim it is sufficient to establish the implication

the Dedekind axiom  $\rightarrow$  the Weierstrass axiom.

Assume that the Dedekind axiom holds for the ordered field  $\mathcal{K}$ . Let  $X \subseteq K$  be a non-empty upward bounded set. Let us put

$$A = \{a \in K : (\exists x \in X)(a \leq_K x)\}.$$

$$B = K \setminus A.$$

Then both sets  $A$  and  $B$  are non-empty and

$$(\forall a \in A)(\forall b \in B)(a <_K b)$$

Hence,  $(A, B)$  is a bounded Dedekind cut in  $(K, \leq_K)$ . Let  $z \in K$  be such that

$$(\forall a \in A)(a \leq z) \text{ \& } (\forall b \in B)(z \leq b).$$

Then it is easy to see that  $z$  is the supremum of the set  $X$ . So the Weierstrass axiom holds for the ordered field  $\mathcal{K}$ , and Theorem 1 is proved.

It is worth remarking here that the theorem proved above is closely connected with the next important result about countable linearly ordered sets.

**Theorem 6.2 (Cantor)** *Any countable linearly ordered set  $(E, \leq_E)$  can be monomorphically imbedded into the linearly ordered set  $(\mathbb{Q}, \leq)$  of all rational numbers. Moreover, if  $(E, \leq_E)$  is non-empty dense and has neither the least nor the largest elements then  $(E, \leq_E)$  is isomorphic with  $(\mathbb{Q}, \leq)$ .*

**Proof.** Suppose that  $(E, \leq_E)$  is a countable linearly ordered set. Hence we can write the basic set  $E$  in the form

$$E = \{e_1, e_2, \dots, e_n, \dots\}.$$

We shall define the required monomorphism

$$\Phi : (E, \leq_E) \rightarrow (\mathbb{Q}, \leq)$$

by mathematical recursion. Let us suppose that  $k \in \mathbb{N}$  and that we have already defined an injective partial mapping

$$\Phi_k : \{e_1, \dots, e_k\} \rightarrow \mathbb{Q}$$

which preserves ordering. Let us consider the element  $e_{k+1}$ . Of course, we can find a permutation  $\phi$  of the set  $\{1, \dots, k\}$  such that

$$e_{\phi(1)} <_E e_{\phi(2)} <_E \dots <_E e_{\phi(k)}.$$

There are only three possible cases:

- 1)  $e_{k+1} < e_{\phi(1)}$ ,
- 2)  $e_{\phi(k)} < e_{k+1}$ ,
- 3)  $e_{\phi(i)} < e_{k+1} < e_{\phi(i+1)}$  for some index  $i$  from the set  $\{1, \dots, k-1\}$ .

It is clear that using the density of  $(\mathbb{Q}, \leq)$  and the fact that there are neither the least nor the largest elements in  $(\mathbb{Q}, \leq)$  in any of these three cases we can find an extension

$$\Phi_{k+1} : \{e_1, \dots, e_{k+1}\} \rightarrow \mathbb{Q}$$

of the mapping  $\Phi_k$  which preserves ordering. In this way we construct an increasing by the inclusion sequence

$$(\Phi_k)_{k \geq 1}$$

of partial monomorphisms. At the end we put

$$\Phi = \bigcup_k \Phi_k.$$

It is not difficult to check that  $\Phi$  is the required monomorphism from  $(E, \leq_E)$  into  $(\mathbb{Q}, \leq)$ . If our basic linearly ordered set  $(E, \leq_E)$  is non-empty and does not have the least and the largest elements, then a simple modification of the construction of the monomorphism  $\Phi$  will give us a surjection from  $E$  onto  $\mathbb{Q}$  and hence, the resulting mapping  $\Phi$  will be an isomorphism between  $(E, \leq_E)$  and  $(\mathbb{Q}, \leq)$ . Thus, the Cantor theorem is proved.

Despite its simplicity, the theorem proved above is a very important result which has a lot of generalizations (see exercises to this Chapter) and has a lot of applications. The next result, also due to Cantor, is an immediate application of this theorem.

**Theorem 6.3 (Cantor)** *Suppose that  $(E, \leq_E)$  is a non-empty dense Dedekind complete linearly ordered set without end-points containing some countable dense subset. Then  $(E, \leq_E)$  is order isomorphic with  $(\mathbb{R}, \leq)$ .*

**Proof.** Let  $D$  be a countable dense subset of  $E$ . According to the previous theorem, there exists an isomorphism

$$\phi : (D, \leq_E) \rightarrow (\mathbb{Q}, \leq).$$

For every element  $e \in E$  let us put

$$\Phi(e) = \sup\{\phi(x) : x <_E e\}.$$

Now it is easy to check that  $\Phi$  is a required isomorphism between  $(E, \leq_E)$  and  $(\mathbb{R}, \leq)$ .

So we see that Theorem 3 gives us another characterization of the real line  $\mathbb{R}$ . This characterization is formulated only in terms of the theory of linearly ordered sets and does not touch the algebraic structure on  $\mathbb{R}$ . Namely, let us write once more the properties of a partially ordered set  $(P, \leq_P)$  which ensure the existence of an isomorphism between  $(P, \leq_P)$  and  $(\mathbb{R}, \leq)$ . These properties are as follows:

- 1)  $P$  is non-empty;
- 2)  $(P, \leq_P)$  is a Dedekind complete dense linear ordering;
- 3)  $P$  has no end-points;
- 4)  $P$  has a countable dense subset.

As we know the last property means that the order topology on  $P$  is a topology of a separable space. We also know that every separable topological space satisfies the Suslin condition (i.e. countable chain condition). Let us replace property 4) by the following, weaker property:

- 4')  $P$  satisfies the Suslin condition.

In 1920 in the first volume of the journal *Fundamenta Mathematicae*, the young Russian mathematician (the disciple of Luzin), M. Suslin posed the problem whether properties 1), 2), 3) and 4') also give an isomorphism between structures  $(P, \leq_P)$  and  $(\mathbb{R}, \leq)$ . This problem played a prominent role in the further development of set theory and was an object of intensive investigations.

The following definition is important for our further considerations. We say that  $(P, \leq_P)$  is a **Suslin line** if it satisfies properties 1), 2), 3) and 4') but does not contain in itself a countable dense subset. Thus, we see that Suslin's question can be reformulated as follows : does a Suslin line exist?

The **Suslin Hypothesis** is the following assertion: there is no Suslin line. After many years of developments it turned out that the Suslin Hypothesis, similarly to the Continuum Hypothesis, is undecidable on the basis of the usual axioms of set theory. Now we discuss this problem more thoroughly.

Suppose that  $(S, \preceq_*)$  is a Suslin line. We may obviously assume that  $S \cap \mathbb{R} = \emptyset$ . Let us define the ordering  $\preceq_+$  on the set  $S \cup \mathbb{R}$  as follows:

$$x \preceq_+ y \iff (x \in S \ \& \ y \in \mathbb{R}) \vee \\ (x \in S \ \& \ y \in S \ \& \ x \preceq_* y) \vee (x \in \mathbb{R} \ \& \ y \in \mathbb{R} \ \& \ x \leq y).$$

It is easy to see that the linearly ordered set  $(S \cup \mathbb{R}, \preceq_+)$  is a Suslin line, too. Hence, we conclude that a Suslin line may contain subintervals which are similar to the real line  $\mathbb{R}$ . However, from such a Suslin line we can construct another one which does not contain subintervals of the type  $\mathbb{R}$  (i.e. subintervals which contain countable dense subsets). Indeed, let  $(S, \preceq)$  be an arbitrary Suslin line. Let us consider the maximal (with respect to the inclusion) family  $\mathcal{D}$  of pairwise disjoint non-empty



open intervals in  $S$  such that any interval from this family contains a countable dense subset. Since  $(S, \preceq)$  satisfies the countable chain condition, we have

$$\text{card}(\mathcal{D}) \leq \omega.$$

Let us put

$$D = \bigcup \mathcal{D}.$$

Then  $D$  contains a countable dense subset, too. Denote this subset by the symbol  $D_0$ . We claim that there are elements  $a, b \in S$  such that

$$a \prec b, \quad ]a, b[ \cap D_0 = \emptyset.$$

This fact holds because otherwise the set  $D_0$  would be a dense subset of  $(S, \preceq)$ . Let  $K = ]a, b[$ . No non-empty open subinterval of  $K$  has a countable dense subset since otherwise we would obtain a contradiction with maximality of the family  $\mathcal{D}$ . Hence,

$$(K, \preceq|_{K \times K})$$

is a **regular Suslin line**, i.e. such a Suslin line that no non-empty open subinterval of it contains a dense countable subset.

A first proposition presented here and connected with a question on the existence of a Suslin line gives a positive result concerning the Suslin Hypothesis. Namely, it turns out that set theory with the Suslin Hypothesis is consistent.

**Theorem 6.4** *If Martin's Axiom and the negation of the Continuum Hypothesis hold, then there is no Suslin line.*

**Proof.** Suppose that Martin's Axiom and the negation of the Continuum Hypothesis hold. Let  $(S, \preceq)$  be a Suslin line. Without loss of generality we may assume that  $(S, \preceq)$  is a regular Suslin line.

We shall define an  $\omega_1$ -sequence

$$(\ ]a_\alpha, b_\alpha[ )_{\alpha < \omega_1}$$

of non-empty open subintervals of  $S$  such that

- 1)  $(\forall \alpha, \beta < \omega_1)(\alpha < \beta \rightarrow (I_\alpha \cap I_\beta = \emptyset \vee \text{cl}(I_\beta) \subseteq I_\alpha))$ ;
- 2)  $(\forall \alpha < \omega_1)(\exists \beta < \omega_1)(\alpha < \beta \ \& \ \text{cl}(I_\beta) \subseteq I_\alpha)$ ,

where  $I_\alpha$  denotes the open interval  $]a_\alpha, b_\alpha[$  and  $\text{cl}(I_\alpha)$  is the corresponding segment  $[a_\alpha, b_\alpha]$ .

In order to construct the required  $\omega_1$ -sequence let us fix an arbitrary function  $f : \omega_1 \rightarrow \omega_1$  such that

$$(\forall \alpha < \omega_1)(\forall \beta < \omega_1)(\exists \zeta > \beta)(f(\zeta) = \alpha).$$

Such a function can be constructed easily. Namely, let us take any partition  $(X_\alpha)_{\alpha < \omega_1}$  of  $\omega_1$  into sets of cardinality  $\omega_1$  and put

$$f(\zeta) = \alpha \iff \zeta \in X_\alpha \quad (\alpha, \zeta \in \omega_1).$$

Suppose now that  $\alpha < \omega_1$  and that the partial sequence  $(\ ]a_\beta, b_\beta[ )_{\beta < \alpha}$  of intervals has already been constructed. Let  $f(\alpha) = \xi$ . If  $\xi < \alpha$ , then we put  $J = I_\xi$  and if  $\xi \geq \alpha$ , then we put  $J = S$ . Since the set  $J$  does not contain a countable dense subset, we can find two elements  $a$  and  $b$  in  $J$  such that

$$a \prec b, \quad ]a, b[ \cap \{a_\beta, b_\beta : \beta < \alpha\} = \emptyset.$$

We put  $a_\alpha = a$  and  $b_\alpha = b$ . Obviously, the  $\omega_1$ -sequence constructed in this way satisfies relations 1) and 2).

At the next step of the proof we need some partial ordering for which we could use Martin's Axiom. Let us define a partial ordering  $\ll$  on  $\omega_1$  by the following formula:

$$\beta \ll \alpha \longleftrightarrow (\alpha < \beta) \ \& \ (cl(I_\beta) \subseteq I_\alpha).$$

From relation 1) and from the definition of a Suslin line it follows that  $(\omega_1, \ll)$  satisfies the countable chain condition. For any ordinal  $\alpha < \omega_1$  let us consider a subset

$$D_\alpha = \{\beta < \omega_1 : \alpha < \beta\}.$$

of  $\omega_1$ . Relation 2) implies that  $(D_\alpha)_{\alpha < \omega_1}$  is a family of coinital subsets of the partially ordered set  $(\omega_1, \ll)$ . Notice also that we have assumed the negation of the Continuum Hypothesis, i.e.  $\omega_1 < \mathfrak{c}$ , where  $\mathfrak{c}$ , as usual, denotes the cardinality continuum. Hence, we may apply Martin's Axiom to this partial order and to the family  $(D_\alpha)_{\alpha < \omega_1}$  and find a filter  $F \subseteq \omega_1$  in  $(\omega_1, \ll)$  such that

$$(\forall \alpha < \omega_1)(F \cap D_\alpha \neq \emptyset).$$

Let

$$\phi : \omega_1 \rightarrow \omega_1$$

be any increasing function such that  $ran(\phi) = F$ . Then we see that

$$(\forall \alpha, \beta)(\alpha < \beta < \omega_1 \rightarrow cl(I_{\phi(\beta)}) \subseteq I_{\phi(\alpha)}).$$

But then we conclude that

$$(\bigcap a_{\phi(\alpha)}, a_{\phi(\alpha+1)})_{\alpha < \omega_1}$$

is a family of pairwise disjoint non-empty open intervals in  $S$ . But since  $S$  satisfies the countable chain condition, it is impossible. Hence, Theorem 4 is proved.

Theorem 4 shows us that some additional axioms of set theory decide negatively the question on the existence of a Suslin line (so, these axioms decide positively the Suslin Hypothesis). However, there are other additional axioms of set theory which imply the existence of a Suslin line. One of such axioms is the so called **diamond principle** (denoted usually by  $\diamond$ ). This principle says that there exists an  $\omega_1$ -sequence of sets  $(X_\alpha)_{\alpha < \omega_1}$  such that

- 1)  $(\forall \alpha < \omega_1)(X_\alpha \subseteq \alpha)$ ;
- 2) for each set  $X \subseteq \omega_1$  the set

$$\{\alpha < \omega_1 : X \cap \alpha = X_\alpha\}$$

is a stationary subset of  $\omega_1$ .

The family of sets  $(X_\alpha)_{\alpha < \omega_1}$  mentioned above sometimes is called a diamond  $\omega_1$ -sequence.

The diamond principle is true in the famous Gödel Constructible Universe and hence, is consistent with usual axioms of set theory. This principle was discovered by Jensen as a very powerful combinatorial assertion which follows from the Axiom of Constructibility (this axiom identifies the von Neumann Universe with the Gödel Universe). Since the Axiom of Constructibility is meaningless for most mathematicians because of its logical contents, some combinatorial statements, like  $\diamond$ , are quite popular to make some sophisticated constructions. Let us mention also

that there are quite simple models of set theory **ZFC** where the diamond principle holds. Hence, the consistency of the diamond principle can be proved without the Axiom of Constructibility. We shall discuss this problem more deeply in Part 2 of this book.

We can think about the diamond principle as of a strong enumeration principle of subsets of  $\omega_1$ . Namely, during recursive constructions of length  $\omega_1$  it allows us take into account stationary many times any subset of  $\omega_1$ . For example, remark that the diamond principle implies the Continuum Hypothesis (see exercises after this Chapter).

We will use in the construction of a Suslin line some seemingly stronger form of the diamond principle. Namely, we will need a family of sets  $(T_\alpha)_{\alpha < \omega_1}$  such that

- 3)  $(\forall \alpha < \omega_1)(T_\alpha \subseteq \alpha \times \alpha)$ ;
- 4) for any subset  $X$  of  $\omega_1 \times \omega_1$  the set

$$\{\alpha < \omega_1 : T_\alpha = X \cap (\alpha \times \alpha)\}$$

is a stationary subset of  $\omega_1$ .

The existence of such a family easily follows from the diamond principle. Indeed, let

$$f : \omega_1 \times \omega_1 \rightarrow \omega_1$$

be an arbitrary bijection. Then the set

$$C = \{\alpha < \omega_1 : f^{-1}(\alpha) = \alpha \times \alpha\}$$

is a closed and unbounded subset of  $\omega_1$ . Let us take a family  $(X_\alpha)_{\alpha < \omega_1}$  the existence of which is postulated by the diamond principle. For each  $\alpha < \omega_1$  we put

$$T_\alpha = f^{-1}(X_\alpha) \cap (\alpha \times \alpha).$$

Suppose now that  $X \subseteq \omega_1 \times \omega_1$ . Then  $f(X)$  is a subset of  $\omega_1$  and the set

$$S = \{\alpha < \omega_1 : f(X) \cap \alpha = X_\alpha\}$$

is a stationary subset of  $\omega_1$ . But the set  $C \cap S$  is also a stationary subset of  $\omega_1$ . Hence, for any  $\alpha \in C \cap S$  we have

$$f^{-1}(f(X) \cap \alpha) = f^{-1}(f(X)) \cap f^{-1}(\alpha) = X \cap (\alpha \times \alpha)$$

and

$$f^{-1}(X_\alpha) = f^{-1}(X_\alpha \cap \alpha) = f^{-1}(X_\alpha) \cap (\alpha \times \alpha) = T_\alpha.$$

Therefore, the family  $(T_\alpha)_{\alpha < \omega_1}$  satisfies relations 3) and 4).

**Theorem 6.5** *The diamond principle implies that there exists a Suslin line.*

**Proof.** We shall construct a linear ordering  $\preceq$  on  $\omega_1$  such that the pair  $(\omega_1, \preceq)$  is a Suslin line. The construction of this ordering will be done by the transfinite recursion of length  $\omega_1$  which starts from  $\alpha = 1$ . We shall define an  $\omega_1$ -sequence  $(\preceq_\alpha)_{1 \leq \alpha < \omega_1}$  which satisfies, among others, the following properties:

- 1) for each non-zero  $\alpha < \omega_1$  the relation  $\preceq_\alpha$  is a dense linear ordering without end-points on the ordinal product  $\omega \cdot \alpha$ ;
- 2)  $(\forall \alpha, \beta < \omega_1)(0 < \alpha < \beta \rightarrow \preceq_\alpha = \preceq_\beta \cap (\omega \cdot \alpha))$ .

First of all let us fix some sequence  $(T_\alpha)_{\alpha < \omega_1}$  which satisfies conditions 3) and 4).

We begin our construction with taking as  $\preceq_1$  any dense linear ordering without end-points on  $\omega$ . Suppose now that  $1 < \alpha < \omega_1$  and that the sequence  $(\preceq_\beta)_{1 \leq \beta < \alpha}$  is constructed. If  $\alpha$  is a limit ordinal number, then we put

$$\preceq_\alpha = \bigcup_{1 \leq \beta < \alpha} \preceq_\beta.$$

If  $\alpha = \beta + 1$ , then the linear ordering  $\preceq_\beta$  on  $\omega \cdot \beta$  has already been constructed. We call a subset  $X$  of  $\omega \cdot \beta \times \omega \cdot \beta$  a  $\beta$ -foe if it satisfies the following two conditions:

$$\begin{aligned} & (\forall (x, y), (u, v) \in X)((x, y) \neq (u, v) \rightarrow ]x, y[ \cap ]u, v[ = \emptyset); \\ & (\forall a, b \in \omega \cdot \beta)(a \prec_\beta b \rightarrow (\exists (x, y) \in X)(]a, b[ \cap ]x, y[ \neq \emptyset)). \end{aligned}$$

Notice that  $\preceq_\beta$  is a dense linear ordering without end-points on the countable set  $\omega \cdot \beta$ . Therefore, the linear ordering  $(\omega \cdot \beta, \preceq_\beta)$  can be identified with the set  $(\mathbb{Q}, \leq)$  of all rational numbers and its Dedekind completion with the real line  $\mathbb{R}$ . Let us consider the family

$$T = \{T_\zeta : \zeta \leq \beta \text{ \& } T_\zeta \text{ is a } \beta\text{-foe}\}.$$

Since  $\beta$  is countable the family  $T$  is countable, too. Notice that each  $\beta$ -foe  $X$  defines an open dense subset

$$\tilde{X} = \bigcup \{]x, y[ : (x, y) \in X\}$$

of the Dedekind completion of the linear ordering  $(\omega \cdot \beta, \preceq_\beta)$ . So, we may apply the Baire theorem to the family  $\{\tilde{X} : X \in T\}$  and find a bounded Dedekind cut  $(A, B)$  in  $(\omega \cdot \beta, \preceq_\beta)$  such that  $A$  has no maximal element,  $B$  has no minimal element and

$$(A, B) \in \bigcap \{\tilde{X} : X \in T\}.$$

What we want to do now is to put a copy of  $\mathbb{Q}$  into the gap between  $A$  and  $B$ . More precisely, we define an ordering  $\preceq_\alpha$  on

$$\omega \cdot \alpha = \omega \cdot \beta + \omega \cdot \{\beta\}$$

in such a way that

a) the ordering

$$(\omega \cdot \{\beta\}, \preceq_\alpha \cap (\omega \cdot \{\beta\} \times \omega \cdot \{\beta\}))$$

is isomorphic with  $(\mathbb{Q}, \leq)$ ;

b)  $(\forall x \in A)(\forall y \in \omega \cdot \{\beta\})(x \preceq_\alpha y)$ ;

c)  $(\forall x \in \omega \cdot \{\beta\})(\forall y \in B)(x \preceq_\alpha y)$ .

The relation  $\preceq_\alpha$  is a linear dense ordering without end-points on  $\omega \cdot \alpha$ . Moreover, every  $\beta$ -foe from the family  $T$  remains an  $\alpha$ -foe, too. This fact will play a crucial role in our further considerations.

We put at the end

$$\preceq = \bigcup_{\alpha < \omega_1} \preceq_\alpha$$

and assert that the Dedekind completion of  $(\omega_1, \preceq)$  is a Suslin line.

It follows from our construction that  $\preceq$  is a dense linear ordering on  $\omega_1$  without end-points. So, the same holds for the Dedekind completion of  $(\omega_1, \preceq)$ .

It is also easy to observe that there is no countable dense subset of  $(\omega_1, \preceq)$ . Indeed, suppose that  $D$  is any countable subset of  $\omega_1$ . The regularity of  $\omega_1$  implies that there is an ordinal  $\beta < \omega_1$  such that  $D \subseteq \omega \cdot \beta$ . But then the whole interval  $\omega \cdot \{\beta\}$  is disjoint with the set  $D$ . So,  $D$  is not dense in  $(\omega_1, \preceq)$ . But from this fact we easily deduce that there is no countable dense subset in the Dedekind completion of  $(\omega_1, \preceq)$ , too.

It remains to show that the Dedekind completion of  $(\omega_1, \preceq)$  satisfies the countable chain condition. We shall prove that  $(\omega_1, \preceq)$  satisfies the countable chain condition. This fact, clearly implies the countable chain condition for the Dedekind completion of  $(\omega_1, \preceq)$ .

Hence, suppose that  $(]a_i, b_i])_{i \in I}$  is a maximal (with respect to inclusion) family of pairwise disjoint non-empty open subintervals of  $(\omega_1, \preceq)$ . Let us put

$$F = \{(a_i, b_i) : i \in I\}$$

and let us consider the set

$$U = \{\alpha < \omega_1 : \omega \cdot \alpha = \alpha \text{ \& } F \cap (\alpha \times \alpha) \text{ is an } \alpha\text{-foe}\}.$$

It is not difficult to check that the set  $U$  is a closed and unbounded subset of  $\omega_1$  (this is the standard part of almost any proof which uses the diamond principle). But the set

$$V = \{\alpha < \omega_1 : T_\alpha = F \cap (\alpha \times \alpha)\}$$

is a stationary subset of  $\omega_1$ , and hence

$$U \cap V \neq \emptyset.$$

Let us take any  $\xi \in U \cap V$ . Then we have

$$T_\xi = F \cap (\xi \times \xi).$$

Thus,  $T_\xi$  is a  $\xi$ -foe and we know that  $T_\xi$  was taken into account during the construction of  $\preceq_{\xi+1}$ . We know also that  $T_\xi$  remains a  $(\xi+1)$ -foe. Moreover, we claim that for every ordinal  $\alpha$  such that  $\xi \leq \alpha < \omega_1$  the set  $T_\xi$  is an  $\alpha$ -foe. Indeed, if  $T_\xi$  is a  $\zeta$ -foe and  $\xi < \zeta < \omega_1$ , then  $T_\xi$  is a  $(\zeta+1)$ -foe, too. The limit step is also clear. Namely, if  $\xi < \zeta < \omega_1$  and  $\zeta$  is a limit ordinal and for all  $\eta$  satisfying the inequalities  $\xi < \eta < \zeta$  the set  $T_\xi$  is an  $\eta$ -foe, then  $T_\xi$  is a  $\zeta$ -foe, too. Hence,

$$\{]u, v[ : (u, v) \in T_\xi\}$$

is a maximal family of pairwise disjoint open intervals, so  $F = T_\xi$  and therefore,

$$\text{card}(I) = \text{card}(T_\xi) \leq \omega.$$

Thus, Theorem 5 is proved.

As we know any non-empty Dedekind complete dense linear order without endpoints and with a countable dense subset is isomorphic with the real line  $\mathbb{R}$ . Hence, the real line  $\mathbb{R}$  is unique in a large class of Dedekind complete dense linear orders. We have another situation with Suslin lines. Namely, it can be proved that the square of any Suslin line is not a c.c.c. topological space. But it is possible to construct (using the diamond principle) two Suslin lines  $P$  and  $S$  such that their topological product satisfies c.c.c. This fact shows us that Suslin lines may strongly differ from each other. This is probably one of the reasons why set theoretical and topological properties of Suslin line are not studied as intensively as of the real line. No we are going to prove some basic facts about Suslin lines.

First, let us make one important observation concerning a Suslin line (which was, in fact, used in the above discussion). Suppose that  $(S, \preceq_S)$  is a Suslin line. Note that if  $\nu$  is an ordinal and  $(x_\alpha)_{\alpha < \nu}$  is a strictly increasing (or strictly decreasing) sequence in  $S$ , then  $\nu < \omega_1$ . Indeed, otherwise

$$(\ ]x_\alpha, x_{\alpha+1}[ )_{\alpha < \nu}$$

would be an uncountable family of pairwise disjoint non-empty open intervals in  $S$ , which is impossible.

Let us recall that  $B(S)$  denotes the class of all Borel subsets of the topological space  $S$ .

**Theorem 6.6** *Let  $(S, \preceq_S)$  be an arbitrary Suslin line. Then*

- 1)  $S$  has a dense subset of cardinality  $\omega_1$ ;
- 2)  $\text{card}(S) = 2^\omega$ ;
- 3)  $\text{card}(B(S)) = 2^\omega$ ;
- 4) the intersection of any countable family of dense open subsets of  $S$  has a non-empty interior.

**Proof.** Suppose that  $(S, \preceq)$  is an arbitrary Suslin line. We define by transfinite recursion a sequence  $(D_\alpha)_{\alpha < \omega_1}$  of countable subsets of  $S$  such that the union

$$D = \bigcup_{\alpha < \omega_1} D_\alpha$$

is a dense subset of  $S$ .

Let  $D_0$  consist of one point from  $S$ . Suppose that  $\alpha < \omega_1$  and that a partial sequence  $(D_\beta)_{\beta < \alpha}$  has already been defined. Let us consider the set

$$A = \bigcup_{\beta < \alpha} D_\beta.$$

Since  $A$  is a countable set, it is not dense in  $S$ . Hence, the set  $S \setminus \text{cl}(A)$  is the union of a non-empty family  $\mathcal{W}_\alpha$  of pairwise disjoint non-empty open intervals. Of course, the family  $\mathcal{W}_\alpha$  is countable. Let  $D_\alpha$  be a subset of  $S$  which intersects each interval from  $\mathcal{W}_\alpha$  exactly at one point. In this way we define the whole sequence  $(D_\alpha)_{\alpha < \omega_1}$  and, finally, we put

$$D = \bigcup_{\alpha < \omega_1} D_\alpha.$$

Obviously,  $\text{card}(D_\alpha) = \omega_1$ . We will show that  $D$  is a dense subset of  $(S, \preceq)$ . Suppose otherwise. Then there exists a point

$$p \in S \setminus \text{cl}(D).$$

For each ordinal  $\alpha < \omega_1$  we can find an interval  $]a_\alpha, b_\alpha[ \in \mathcal{W}_\alpha$  such that

$$p \in ]a_\alpha, b_\alpha[.$$

Of course, for any  $\alpha < \beta < \omega_1$  we have

$$]a_\beta, b_\beta[ \subset ]a_\alpha, b_\alpha[.$$

Hence, the sequence  $(a_\alpha)_{\alpha < \omega_1}$  is strictly increasing. But this is impossible by the remark preceding the theorem. Thus,  $D$  is a dense subset of  $(S, \preceq)$ .

Now let us fix an arbitrary dense subset  $D$  of  $S$  of cardinality  $\omega_1$ . Notice that for any point  $p \in S$  we can find  $\nu < \omega_1$  and a strictly increasing sequence  $(x_\alpha)_{\alpha < \nu}$  of points from  $D$  such that

$$x = \sup\{x_\alpha : \alpha < \nu\}.$$

Hence, we have

$$\text{card}(S) \leq (\text{card}(D))^\omega = 2^\omega.$$

The converse inequality  $\text{card}(S) \geq 2^\omega$  is clear since  $S$  is complete.

It is easy to construct an infinite countable family of pairwise disjoint non-empty open intervals in  $S$ . Hence, there are at least  $2^\omega$  different open subsets of  $S$ . Furthermore, since every open set in  $S$  is a disjoint union of a family of open intervals and any such family is countable we see that there are precisely  $2^\omega$  open subsets of  $S$ . From this fact we immediately conclude that the equality

$$\text{card}(B(S)) = 2^\omega$$

holds, too.

Suppose now that  $(D_n)_{n \in \mathbb{N}}$  is a countable family of open dense subsets of  $S$ . Then every set  $D_n$  is the union of a countable family  $\mathcal{O}_n$  of open intervals. Let  $E_n$  be the set consisting of all end-points of intervals from  $\mathcal{O}_n$  and let

$$E = \bigcup_{n \in \mathbb{N}} E_n.$$

Since  $E$  is countable, we can find a non-empty open interval  $V$  in  $S$  such that  $V \cap E = \emptyset$ . From the density of every set  $D_n$  we immediately deduce that  $V \subseteq D_n$  for each  $n \in \mathbb{N}$ , so

$$V \subseteq \bigcap_{n \in \mathbb{N}} D_n.$$

Hence, Theorem 6 is proved.

At this place we finish the discussion of Suslin lines and the Suslin Hypothesis. We will use Suslin lines in several examples in Part 2 of this book.

We started this Chapter with some characterizations of the real line  $\mathbb{R}$ . There are many other characterizations of this important mathematical object. For example, according to the famous Pontrjagin theorem, any locally compact connected topological field is isomorphic with the field of all real numbers  $\mathbb{R}$ , or with the field of all complex numbers  $\mathbb{C}$  or with the non-commutative field of all quaternions. From this theorem we immediately get the next characterization of the real line. Namely,  $\mathbb{R}$  can be characterized as a one-dimensional connected locally compact topological field. We shall not prove these results because they are outside the scope of our book.

Now we shall discuss a more important question for us. This question is connected with generalized limits on the real line  $\mathbb{R}$ . We shall denote by the symbol  $E$  the vector space of all convergent sequences of real numbers. Obviously, the space  $E$  is partially ordered by the relation

$$(x_n)_{n \in \mathbb{N}} \preceq (y_n)_{n \in \mathbb{N}} \iff (\forall n \in \mathbb{N})(x_n \leq y_n).$$

For any element

$$x = (x_n)_{n \in \mathbb{N}} \in E$$

we put

$$\|x\| = \sup_n |x_n|.$$

It is clear that the functional

$$\|\cdot\| : E \rightarrow \mathbb{R}$$

is a norm on the vector space  $E$ . Moreover, the pair  $(E, \|\cdot\|)$  is a separable Banach space. We shall consider a natural continuous positive linear functional

$$\lim : E \rightarrow \mathbb{R},$$

defined by the equality

$$\lim(x) = \lim_n x_n$$

for each  $x = (x_n)_{n \in \mathbb{N}}$  from the space  $E$ . This functional does not depend on finitely many values  $x_n$ , i.e. if  $x \in E$ ,  $y \in E$  and

$$\text{card}(\{n \in \mathbb{N} : x_n \neq y_n\}) < \omega,$$

then we have

$$\lim_n x_n = \lim_n y_n.$$

Let us denote by the symbol  $l_\infty$  the vector space of all bounded sequences of real numbers ordered by the same formula as the space  $E$ . It is clear that the inclusion

$$E \subseteq l_\infty$$

holds. The space  $l_\infty$  is a non-separable Banach space with the norm

$$\|x\| = \sup_n |x_n| \quad (x = (x_n)_{n \in \mathbb{N}} \in l_\infty).$$

We say that a mapping

$$Lim : l_\infty \rightarrow \mathbb{R}$$

is a **Banach limit** (or a **generalized limit**) if

- 1)  $Lim$  is a positive linear functional on  $l_\infty$  (hence, it is continuous);
- 2)  $Lim$  is an extension of the linear functional  $\lim$ .

We finish this Chapter with the following result which establishes, among others, the existence of Banach limits.

**Theorem 6.7** *Let  $\mu$  be any non-negative finitely additive set function defined on the family  $P(\mathbb{N})$  of all subsets of  $\mathbb{N}$  such that*

$$\mu(\{n\}) = 0 \quad (n \in \mathbb{N}),$$

$$\mu(\mathbb{N}) = 1.$$

*Then the mapping*

$$x \mapsto \int x d\mu \quad (x \in l_\infty)$$

*is a Banach limit.*

Let us remark, in order to avoid misunderstandings, that by the definition

$$\int x d\mu = \int x^+ d\mu - \int x^- d\mu,$$

where, for example, the symbol  $\int x^+ d\mu$  denotes the upper bound of all sums of the form

$$\sum_{j \in J} t_j \mu(P_j),$$



where  $(P_j)_{j \in J}$  is any finite partition of the set  $\mathbb{N}$  and for every  $j \in J$  the real number  $t_j$  is defined as

$$t_j = \inf(x^+|_{P_j}),$$

(the definition of  $\int x^- d\mu$  is analogous).

We leave the simple proof of this theorem to the reader. We shall make only one additional remark. Let  $\Phi$  be any non-principal ultrafilter on the set  $\mathbb{N}$ . Then there exists a set function  $\mu$  canonically associated with the ultrafilter  $\Phi$  and defined on  $P(\mathbb{N})$  by the following formula:

$$\mu(A) = \begin{cases} 0 & \text{if } A \notin \Phi, \\ 1 & \text{if } A \in \Phi. \end{cases}$$

It is clear that this function is non-negative and finitely additive. Hence, by the theorem formulated above this set function gives us a Banach limit. It is worth remarking that in this case the Banach limit is characterized by the following formula:

$$\text{Lim}(x) = a \iff (\forall \epsilon > 0)(\{n : |x_n - a| < \epsilon\} \in \Phi).$$

Therefore, we see that in theory **(ZF)** & **(DC)** from the existence of a non-principal ultrafilter on  $\mathbb{N}$  we can deduce the existence of a Banach limit.

## Exercises

**Exercise 6.1** Show that there exist two irrational real numbers  $r$  and  $t$  such that the number  $r^t$  is rational.

**Exercise 6.2** Find two non-isomorphic Dedekind complete dense linearly ordered sets  $(E_1, \preceq_1)$  and  $(E_2, \preceq_2)$  having no end-points and satisfying the following relation:

$$\text{card}(E_1) = \text{card}(E_2) = \mathfrak{c}.$$

**Exercise 6.3** Let  $\mathbb{C}$  denote the field of all complex numbers. Show that there is no linear ordering  $\leq$  on  $\mathbb{C}$  such that the structure  $(\mathbb{C}, +, \cdot, 0, 1, \leq)$  is a linearly ordered field.

**Exercise 6.4** Let  $\alpha$  be a fixed ordinal number. A linearly ordered set  $(E, \preceq)$  is called an  $\eta_\alpha$ -set if for any two sets  $X \subseteq E$  and  $Y \subseteq E$  such that

$$\begin{aligned} \text{card}(X) < \omega_\alpha, \quad \text{card}(Y) < \omega_\alpha, \\ (\forall x \in X)(\forall y \in Y)(x < y) \end{aligned}$$

there exists an element  $e \in E$  satisfying the following relations:

$$(\forall x \in X)(x < e), \quad (\forall y \in Y)(e < y).$$

The notion of  $\eta_\alpha$ -set is due to Hausdorff. In particular, the linearly ordered sets

$$(\mathbb{Q}, \leq), \quad (\mathbb{R}, \leq)$$

are  $\eta_0$ -sets. Show that

- if  $(E, \leq)$  is an  $\eta_\alpha$ -set, then  $E$  has not end-points and  $\text{card}(E) \geq \omega_\alpha$ ;
- if  $(E, \leq)$  is an  $\eta_\alpha$ -set, then every linearly ordered set  $(P, \leq_P)$  such that  $\text{card}(P) \leq \omega_\alpha$  can be monomorphically embedded into  $(E, \leq)$ ;
- any two linearly ordered  $\eta_\alpha$ -sets of cardinality  $\omega_\alpha$  are isomorphic.

Moreover, prove that for every ordinal  $\alpha$  there exists an  $\eta_{\alpha+1}$ -set of cardinality  $2^{\omega_\alpha}$ .

These results, in particular, show us that if the Continuum Hypothesis holds, then there exists a linearly ordered set of the cardinality continuum which is universal in the sense that any other linearly ordered set of the cardinality continuum can be monomorphically embedded into it.

Let us remark also that  $\eta_\alpha$ -sets are a very particular case of the so called **saturated models** which are considered in general model theory.

**Exercise 6.5** Show that there exists a linearly ordered set  $(E, \leq)$  having (as a topological space in its order topology) the following properties:

- a)  $E$  is locally isomorphic with  $\mathbb{R}$  (hence,  $E$  is also locally homeomorphic with  $\mathbb{R}$ );
- b)  $E$  is connected;
- c)  $E$  is not separable.

Deduce from these facts that the space  $E \times E$  is connected non-separable and locally homeomorphic with the Euclidean plane  $\mathbb{R}^2$ . This space gives us an example of a two-dimensional topological, even differentiable, manifold without a countable base. In particular, the manifold  $E \times E$  does not admit a triangulation (in the usual sense of the classical combinatorial topology).

**Exercise 6.6** Let  $E$  be a connected metric space locally separable at each of its points (i.e. for every  $x \in E$  there exists a neighbourhood  $V(x)$  of  $x$  which is separable). Show that  $E$  has a countable base, i.e.  $E$  is separable.

**Exercise 6.7** Give a description of all closed subgroups of the additive group of the real line  $\mathbb{R}$ .

**Exercise 6.8** Give a description of all closed subgroups of the additive group of the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  ( $n \in \mathbb{N}$ ).

**Exercise 6.9** Suppose that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has two periods  $q_1$  and  $q_2$  such that the real number  $q_1/q_2$  is irrational. Show that if  $f$  is Lebesgue measurable, then  $f$  is equivalent to a constant function. Formulate and prove an analogous result for the Baire property.

**Exercise 6.10** Show that the diamond principle implies the Continuum Hypothesis.

**Exercise 6.11** Suppose that the diamond principle holds. Show that there exists a family  $(X_\alpha)_{\alpha < 2^{\omega_1}}$  of stationary subsets of  $\omega_1$  such that for any two distinct ordinals  $\alpha, \beta < 2^{\omega_1}$  the intersection  $X_\alpha \cap X_\beta$  is a non-stationary subset of  $\omega_1$ .

**Exercise 6.12** Let  $(S, \preceq)$  be an arbitrary Suslin line. Show that the topological square of  $S$  does not satisfy the countable chain condition.

This classical result is due to Kurepa. Deduce from this result that

$$\text{card}(B(S)) = 2^{\omega_1}.$$

**Exercise 6.13** Let  $\mu$  be any  $\sigma$ -finite Borel measure defined on a Suslin line  $(S, \preceq)$ . Show that the measure  $\mu$  has a separable support, i.e. that there exists a separable closed set  $F \subseteq S$  such that  $\mu(S \setminus F) = 0$ .

**Exercise 6.14** Let  $E$  be a topological space. We say that  $E$  is a Lindelof space if every open covering of  $E$  contains a countable subcovering of  $E$ . We say that  $E$  is a hereditarily Lindelof space if each subspace of  $E$  is a Lindelof space.

Prove that any Suslin line  $(S, \preceq)$  is a non-separable hereditarily Lindelof space. Prove also that a Suslin line  $S$  does not admit a group structure agreed with the order topology of  $S$ .

**Exercise 6.15** Assume the diamond principle. Construct two Suslin lines  $T$  and  $S$  such that the topological product  $T \times S$  satisfies the countable chain condition.

**Exercise 6.16** Show that if  $(S, \preceq)$  is a regular Suslin line, then the ideals of meager sets and of nowhere dense sets coincide. Moreover, show that

$$\text{non}(K(S)) = \omega_1.$$

**Exercise 6.17** Using the Erdős-Rado combinatorial theorem (see Appendix A) prove that the cardinality of any non-separable linearly ordered set satisfying c.c.c. does not exceed the cardinality continuum.

Deduce the same result from the Arhangel'ski theorem stating that the cardinality of any compact topological space satisfying the first countability axiom does not exceed the cardinality continuum.

**Exercise 6.18** Let  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  be any two sequences of real numbers. Show that the following inequality holds:

$$\limsup_n (x_n + y_n) \leq \limsup_n x_n + \limsup_n y_n.$$

**Exercise 6.19** Applying the previous exercise and the Hahn-Banach theorem about extensions of a linear functional show that there exists a mapping

$$\text{Lim} : l_\infty \rightarrow \mathbb{R}$$

with the following properties:

- $\text{Lim}$  is a linear positive functional (hence, it is continuous);
- $\text{Lim}$  extends the mapping  $\lim$ ;
- the mapping  $\text{Lim}$  is invariant under the shift, i.e. the equality

$$\text{Lim}((x_n)_{n \in \mathbb{N}}) = \text{Lim}((x_{n+1})_{n \in \mathbb{N}}) \quad ((x_n)_{n \in \mathbb{N}} \in l_\infty)$$

holds.

The mapping  $\text{Lim}$  is called an **invariant Banach limit** on the space  $l_\infty$ .

**Exercise 6.20** Show that the cardinality of the family of all Banach limits is equal to  $2^c$ .

**Exercise 6.21** Prove that any one-dimensional connected differentiable manifold with a countable base is diffeomorphic either with the one-dimensional torus, or with the real line  $\mathbb{R}$ . Hence, we obtain that any non-compact one-dimensional connected differentiable manifold with a countable base is diffeomorphic with the real line.

**Exercise 6.22** Let  $X$  be a subset of the real line  $\mathbb{R}$ . Denote by  $Is(X)$  the family of all those subsets of  $\mathbb{R}$  which are order isomorphic with  $X$ . Show that for non-empty set  $X$  we have

$$\text{card}(Is(X)) = c.$$

**Exercise 6.23** *Prove that every infinite countable linearly ordered set  $(E, \preceq)$  is isomorphic to a certain proper subset of  $E$ . Give an example of a subset  $X$  of the real line such that  $\text{card}(X) = \mathfrak{c}$  and  $X$  is not isomorphic to any proper subset of  $X$  (of course, we mean here that  $X$  is equipped with the order induced by the standard order of  $\mathbb{R}$ ). These results are due to Dushnik and Muller.*



## Chapter 7

# Measure and Category on the Real Line

In this Chapter we investigate the  $\sigma$ -ideal  $\mathbb{L}$  of all Lebesgue measure zero subsets of the real line  $\mathbb{R}$  and the  $\sigma$ -ideal  $\mathbb{K}$  of all first category subsets of  $\mathbb{R}$ . Hence, the basic set of our considerations is  $\mathbb{R}$ . However, after having acquainted himself with our discussions in Chapters 4 and 5, the reader will find that all further considerations have a much wider range of applications.

Let us introduce and recall some notations. By  $\mathbb{B}$  we denote here the  $\sigma$ -algebra  $B(\mathbb{R})$  of all Borel subsets of  $\mathbb{R}$ . The standard one-dimensional Lebesgue measure on  $\mathbb{R}$  we shall denote here by  $\lambda$ . By  $\mathbb{Q}$  we denote the set of all rational numbers.

We say that  $I$  is an ideal of subsets of a basic set  $E$  if  $I$  is an ideal and  $\bigcup I = E$ , i.e. if  $[E]^{<\omega} \subseteq I$ . Let  $J$  be an ideal of subsets of  $\mathbb{R}$ . We say that  $J$  has a **Borel base** if

$$(\forall X \in J)(\exists Y \in J)(X \subseteq Y \text{ \& } Y \in \mathbb{B}).$$

Note that for every ideal  $J$  with a Borel base the inequality

$$\text{cof}(J) \leq \mathfrak{c}$$

holds, where  $\mathfrak{c}$  denotes, as usual, the cardinality continuum. Note also that both ideals  $\mathbb{L}$  and  $\mathbb{K}$  have Borel bases.

Let us recall that any subset of  $\mathbb{R}$  with the Baire property can be represented in the form  $B \Delta K$ , where  $B$  is a Borel set in  $\mathbb{R}$  and  $K$  is a first category subset of  $\mathbb{R}$ . An analogous fact is true for Lebesgue measurable sets in  $\mathbb{R}$ . Namely, any Lebesgue measurable subset of  $\mathbb{R}$  can be represented in the form  $B \Delta L$ , where  $B$  is a Borel set in  $\mathbb{R}$  and  $L$  is a Lebesgue measure zero subset of  $\mathbb{R}$ . Hence, the family of all subsets of  $\mathbb{R}$  with the Baire property coincides with the  $\sigma$ -algebra generated by  $\mathbb{B} \cup \mathbb{K}$  and the family of all Lebesgue measurable subsets of  $\mathbb{R}$  coincides with the  $\sigma$ -algebra generated by  $\mathbb{B} \cup \mathbb{L}$ . These simple observations establish the basic similarity between the ideals  $\mathbb{K}$  and  $\mathbb{L}$  and between the  $\sigma$ -algebras mentioned above.

Let  $(G, \cdot)$  be an arbitrary group. For any sets  $X, Y \subseteq G$  and any element  $g \in G$  we define the following sets:

- 1)  $X \cdot g = \{x \cdot g : x \in X\}$ ,
- 2)  $X \cdot Y = \{x \cdot y : x \in X \text{ \& } y \in Y\}$ ,
- 3)  $X^{-1} = \{x^{-1} : x \in X\}$ .

Suppose that  $I$  is an ideal of subsets of the group  $(G, \cdot)$ . We say that this ideal is **right invariant under translations** if

$$(\forall X \in I)(\forall g \in G)(X \cdot g \in I).$$

We say that this ideal is **symmetric** if

$$(\forall X \in I)(X^{-1} \in I).$$

If the group  $G$  is commutative we shall use additive notation, e.g. we shall write  $X + g$ ,  $X + Y$ ,  $-X$  instead of  $X \cdot g$ ,  $X \cdot Y$ ,  $X^{-1}$ .

Let us notice that both ideals  $\mathbb{K}$  and  $\mathbb{L}$  are symmetric and invariant under all translations from the group  $(\mathbb{R}, +)$ .

We start with some results about ideals on groups which are similar to ideals  $\mathbb{K}$  and  $\mathbb{L}$  from the algebraic point of view.

First we shall show that no ideal on a group symmetric and invariant under translations of this group can be determined by one set.

**Theorem 7.1** *Suppose that  $I$  is a proper ideal of subsets of a commutative group  $(G, +)$  and suppose that  $I$  is symmetric and invariant under all translations from this group. Then there is no set  $A \in I$  such that*

$$(\forall B \in I)(\exists g \in G)(B \subseteq A + g).$$

**Proof.** Suppose that such a set  $A$  exists. Without loss of generality we may assume that  $A$  is a symmetric set, i.e. that  $A = -A$ . Since the ideal  $I$  is proper there exists a set  $B \in I$  satisfying the relations  $A \subseteq B$  and  $A \neq B$ . Obviously, we may also assume that  $B = -B$ . Let  $g$  be such an element of  $G$  that  $B \subseteq A + g$ . Then we have

$$-B \subseteq -A + g,$$

and therefore,

$$B \subseteq A - g,$$

Thus, we obtain

$$A \subset B \subseteq A + g \subseteq B + g \subseteq (A - g) + g = A.$$

So, we get a contradiction which proves our theorem.

The assumption that the ideal  $I$  is symmetric is essential in this theorem. Namely, let us consider the ideal  $J$  of subsets of  $\mathbb{R}$  defined by the equality

$$J = \{B \subseteq \mathbb{R} : (\exists x \in \mathbb{R})(\forall y \in B)(y \leq x)\}.$$

The ideal  $J$  is invariant under all translations of  $\mathbb{R}$ . Let us put  $A = ]-\infty, 0]$ . Then it is easy to see that

$$(\forall B \in J)(\exists x \in \mathbb{R})(B \subseteq A + x).$$

So, we conclude that the ideal  $J$  is determined by the set  $A$ .

**Theorem 7.2** *Suppose that  $I$  is a proper ideal on a commutative group  $(G, +)$  invariant under all translations from this group. Let  $H \subseteq I$  and let  $\text{card}(H) < \text{cov}(I)$ . Then we have*

$$\text{card}(G \setminus \bigcup H) = \text{card}(G).$$

**Proof.** Suppose otherwise that there exists a set  $H \subseteq I$  such that

$$\text{card}(H) < \text{cov}(I),$$

$$\text{card}(G \setminus \bigcup H) < \text{card}(G).$$

Let us put

$$X = G \setminus \bigcup H.$$

We fix an element  $g \in G \setminus (X - X)$ . It is easy to check that

$$g + X \subseteq \bigcup H.$$

Let us put

$$S = H \cup \{A - g : A \in H\}.$$

Since the ideal  $I$  is invariant, we see that  $S \subseteq I$ . It is also clear that  $\text{card}(S) < \text{cov}(I)$ . But we have

$$\bigcup S = G,$$

so we obtain a contradiction.

Recall that two ideals  $I$  and  $J$  of subsets of a basic set  $E$  are orthogonal if there exists a partition  $\{A, B\}$  of the set  $E$  such that  $A \in I$  and  $B \in J$ . As we know the ideals  $\mathbb{K}$  and  $\mathbb{L}$  on the real line are orthogonal.

**Theorem 7.3 (Rothberger)** *Suppose that  $I$  and  $J$  are two orthogonal ideals on a commutative group  $(G, +)$  invariant under all translations from this group. Then the following equality holds:*

$$\text{cov}(I) \leq \text{non}(J).$$

**Proof.** Let  $\{A, B\}$  be a partition of the group  $G$  such that  $A \in I$  and  $B \in J$ . Let  $X$  be a subset of  $G$  such that  $X \notin J$  and  $\text{card}(X) = \text{non}(J)$ . Then

$$(\forall g \in G)((g + X) \cap A \neq \emptyset).$$

Hence, we have

$$(\forall g \in G)(\exists x \in X)(\exists a \in A)(g + x = a).$$

But the last relation means that

$$G = \bigcup \{A - x : x \in X\}.$$

Thus, we obtain

$$\text{cov}(I) \leq \text{card}(X) = \text{non}(J)$$

and the theorem is proved.

In particular, we see that

$$\text{cov}(\mathbb{K}) \leq \text{non}(\mathbb{L}), \quad \text{cov}(\mathbb{L}) \leq \text{non}(\mathbb{K}).$$

These two inequalities give us a non-trivial correlation between the main cardinal functions describing the properties of ideals  $\mathbb{K}$  and  $\mathbb{L}$ . The second part of this book will be devoted for more detailed discussion of properties of the mentioned ideals. Now, let us recall that if  $J = \mathbb{K}$  or  $J = \mathbb{L}$ , then

$$\omega_1 \leq \text{add}(J) \leq \text{non}(J), \text{cov}(J) \leq \text{cof}(J) \leq \mathfrak{c}.$$

Hence, if the Continuum Hypothesis holds, then all of these cardinal numbers are equal to the cardinality continuum. The next result shows us that Martin's Axiom also implies that these cardinal numbers for  $\mathbb{K}$  and  $\mathbb{L}$  are equal to  $\mathfrak{c}$ .

**Theorem 7.4** *If Martin's Axiom holds, then*

$$\text{add}(\mathbb{K}) = \text{add}(\mathbb{L}) = \mathfrak{c}.$$



**Proof.** Assume that Martin's Axiom holds. First we shall prove that the equality  $add(\mathbb{R}) = \mathfrak{c}$  holds.

Suppose that  $\kappa < \mathfrak{c}$  and that  $(N_\alpha)_{\alpha < \kappa}$  is a family of nowhere dense subsets of  $\mathbb{R}$ . We must show that  $\bigcup_{\alpha < \kappa} N_\alpha \in \mathbb{K}$ . Let  $I(\mathbb{R})$  denote the family of all finite sequences of non-empty open intervals with rational end-points. Let

$$P = \{(f, U) : f \in I(\mathbb{R}) \text{ \& } U \text{ is a dense open subset of } \mathbb{R}\}.$$

We define a partial ordering  $\leq$  on the set  $P$ . Namely, we put

$$(f, U) \leq (g, V)$$

if the relation

$$(U \subseteq V) \text{ \& } (f \supseteq g) \text{ \& } (\forall i \in \text{dom}(f) \setminus \text{dom}(g))(f(i) \subseteq V)$$

holds.

Now we shall check that the partially ordered set  $(P, \leq)$  satisfies the countable chain condition. Suppose that  $((f_\alpha, V_\alpha))_{\alpha < \omega_1}$  is an arbitrary family of elements from  $P$ . Then there are ordinals  $\alpha < \beta < \omega_1$  such that  $f_\alpha = f_\beta$ . Let  $V = V_\alpha \cap V_\beta$ . Then  $V$  is a dense open subset of  $\mathbb{R}$  and

$$(f_\alpha, V) \leq (f_\alpha, V_\alpha), \quad (f_\alpha, V) \leq (f_\beta, V_\beta).$$

Hence, we see that  $(P, \leq)$  satisfies c.c.c., so we may apply Martin's Axiom to the partially ordered set  $(P, \leq)$ .

For every  $\alpha < \kappa$ , every  $n \in \mathbb{N}$  and any two numbers  $p, q \in \mathbb{Q}$  such that  $p < q$  we put

$$D_\alpha = \{(f, U) \in P : N_\alpha \cap U = \emptyset\}$$

and

$$E_{p,q}^n = \{(f, U) \in P : (\exists m > n)(m \in \text{dom}(f) \text{ \& } f(m) \cap ]p, q[ \neq \emptyset)\}.$$

It is easy to check that for each  $\alpha < \kappa$  the set  $D_\alpha$  is a coinital subset of  $(P, \leq)$  and that for each  $n \in \mathbb{N}$  and for all  $p, q$  from  $\mathbb{Q}$  such that  $p < q$  the set  $E_{p,q}^n$  is coinital, too. Let us put

$$S = \{D_\alpha : \alpha < \kappa\} \cup \{E_{p,q}^n : n \in \mathbb{N} \text{ \& } p, q \in \mathbb{Q} \text{ \& } p < q\}.$$

We see that  $\text{card}(S) < \kappa \cdot \omega < \mathfrak{c}$ . So there exists a filter  $F$  in  $(P, \leq)$  which intersects all sets from the family  $S$ . We define

$$I = \bigcup \{f : (\exists U)((f, U) \in F)\}.$$

Since  $F \cap E_{p,q}^n \neq \emptyset$  for every  $n \in \mathbb{N}$  and any rational numbers  $p, q$  such that  $p < q$ , we see that  $\text{dom}(I) = \mathbb{N}$ . For each natural number  $n$  we put

$$U_n = \bigcup_{n < m} I(m)$$

and

$$H = \bigcap_n U_n.$$

It is clear that for each  $n \in \mathbb{N}$  the set  $U_n$  is open in  $\mathbb{R}$ . Moreover, suppose that  $p, q$  are rational numbers and  $p < q$ . Then

$$F \cap E_{p,q}^n \neq \emptyset.$$

Hence, there exists  $m > n$  such that

$$I(m) \cap ]p, q[ \neq \emptyset.$$

Thus, for every  $n \in \mathbb{N}$  the set  $U_n$  is dense and open in  $\mathbb{R}$ . Therefore, the set  $H$  is a dense  $G_\delta$ -subset of  $\mathbb{R}$ .

Notice, at the end, that if  $\alpha < \kappa$ , then  $F \cap D_\alpha \neq \emptyset$ , so there exists an element  $(f, U)$  of  $F$  such that  $N_\alpha \cap U = \emptyset$ . But, since  $F$  is a filter, it is not difficult to check that

$$H \subseteq U,$$

so  $N_\alpha \cap H = \emptyset$ , too. Therefore, we have

$$\bigcup_{\alpha < \kappa} N_\alpha \subseteq \mathbb{R} \setminus H.$$

In particular, we see that  $\bigcup_{\alpha < \kappa} N_\alpha \in \mathbb{K}$ .

Now we shall prove the second part of the theorem, i.e. that Martin's Axiom implies the equality  $add(\mathbb{L}) = \mathbf{c}$ . Suppose that  $\kappa < \mathbf{c}$  and that  $(L_\alpha)_{\alpha < \kappa} \subseteq \mathbb{L}$ . Let us fix a real number  $\epsilon > 0$ . We shall show that there exists an open set  $U \subseteq \mathbb{R}$  such that  $\lambda(U) \leq \epsilon$  and

$$\bigcup_{\alpha < \kappa} L_\alpha \subseteq U.$$

Let

$$P = \{V \subseteq \mathbb{R} : V \text{ is open set } \& \lambda(V) < \epsilon\}.$$

We consider the partial ordering of  $P$  defined by the formula

$$U \leq V \iff U \supseteq V.$$

First we shall establish that  $(P, \leq)$  satisfies the countable chain condition. In order to show this suppose that  $\{V_\alpha\}_{\alpha < \omega_1}$  is an uncountable subfamily of  $P$ . Then there exist a strictly positive number  $\epsilon_1 < \epsilon$  and an uncountable subset  $T$  of  $\omega_1$  such that for each  $\alpha \in T$  we have  $\lambda(V_\alpha) < \epsilon_1$ . For every  $\alpha \in T$  let  $I_\alpha$  be a finite union of intervals with rational end-points such that

$$\lambda(V_\alpha \Delta I_\alpha) < (\epsilon - \epsilon_1)/2.$$

Then there are different  $\alpha, \beta \in T$  such that  $I_\alpha = I_\beta$ . Obviously, we have

$$V_\alpha \cup V_\beta \subseteq V_\alpha \cup (I_\alpha \setminus V_\alpha) \cup (V_\beta \setminus I_\beta).$$

Therefore,

$$\lambda(V_\alpha \cup V_\beta) < \lambda(V_\alpha) + \lambda(I_\alpha \setminus V_\alpha) + \lambda(V_\beta \setminus I_\beta) < \epsilon.$$

Thus,  $P$  satisfies c.c.c. and we may apply Martin's Axiom to the partial ordering  $P$ .

Note that for every  $\alpha < \kappa$  the set

$$G_\alpha = \{V \in P : L_\alpha \subseteq V\},$$

is a coinital subset of  $P$ . Hence, there exists a filter  $G$  in  $P$  such that

$$(\forall \alpha < \kappa)(G_\alpha \cap G) \neq \emptyset.$$

Let us consider the set

$$U = \bigcup G.$$

It is easy to see that

$$(\forall \alpha < \kappa)(L_\alpha \subseteq U).$$

Note that every element of  $G$  is an open set and we can find a countable family  $(U_n)_{n \in \mathbb{N}}$  of elements from  $G$  such that  $U = \bigcup_n U_n$ . Moreover,  $G$  is a directed family of sets. Hence, if  $U_1, \dots, U_n$  are elements of  $G$ , then  $U_1 \cup \dots \cup U_n$  is also an element of  $G$ , so

$$\lambda(U_1 \cup \dots \cup U_n) < \epsilon.$$

This fact immediately implies that

$$\lambda(U) = \lambda\left(\bigcup_n U_n\right) \leq \epsilon.$$

Since  $\epsilon$  is an arbitrary strictly positive number, the theorem is proved.

It is well known that theory **ZFC** & (MA) &  $(2^\omega > \omega_1)$  is consistent. Hence, it is consistent that  $\text{add}(\mathbb{K}) > \omega_1$  and that  $\text{add}(\mathbb{L}) > \omega_1$ . In the second part of this book we shall see that theory

$$\mathbf{ZFC} \ \& \ (2^\omega = \omega_2) \ \& \ (\text{add}(\mathbb{K}) = \text{add}(\mathbb{L}) = \omega_1)$$

is consistent, too. So, the values of these cardinal functions are not uniquely determined by theory **ZFC**. However, there are natural examples of topological and measure spaces for which these functions are precisely determined.

**Example 1.** Let us consider  $\omega_1$  as a topological space with the discrete topology and let  $X$  denote the product topological space  $\omega_1^\omega$ . For every ordinal  $\alpha < \omega_1$  the set

$$A_\alpha = \{f \in X : (\forall n \in \mathbb{N})(f(n) \leq \alpha)\}$$

is closed and nowhere dense in  $X$ . Moreover, we have

$$\bigcup_{\alpha < \omega_1} A_\alpha = X.$$

Hence, in this situation  $\text{cov}(K(X)) = \omega_1$ . Remark also that  $X$  is a topological space metrizable by a complete metric.

**Example 2.** For an arbitrary infinite cardinal  $\kappa$  let us consider the measure product space and topological product space  $\{0, 1\}^\kappa$ . From the Kuratowski-Ulam theorem and from the Fubini theorem we can easily deduce that

$$\text{add}(K(\{0, 1\}^\kappa)) \leq \text{add}(K(\{0, 1\}^\omega)),$$

$$\text{add}(L(\{0, 1\}^\kappa)) \leq \text{add}(L(\{0, 1\}^\omega)).$$

In particular, let us put  $\kappa = \omega_1$ . For each ordinal  $\alpha < \omega_1$  let

$$A_\alpha = \{f \in \{0, 1\}^{\omega_1} : (\forall \beta > \alpha)(f(\beta) = 0)\}.$$

It is easy to check that for  $\alpha < \omega_1$  we have

$$A_\alpha \in K(\{0, 1\}^{\omega_1}) \cap L(\{0, 1\}^{\omega_1})$$

but

$$\bigcup_{\alpha < \omega_1} A_\alpha \notin K(\{0, 1\}^{\omega_1}) \cup L(\{0, 1\}^{\omega_1}).$$

Thus, for  $\kappa = \omega_1$  we have

$$\text{add}(K(\{0, 1\}^\kappa)) = \text{add}(L(\{0, 1\}^\kappa)) = \omega_1.$$

The above examples show us that the flexibility of the cardinal functions connected with ideals (as we have in the case of  $\mathbb{K}$  and  $\mathbb{L}$ ) is an exception rather than a rule.

The following two results give us another similarity between measure and category.

**Theorem 7.5** *Let  $(G, \cdot)$  be a locally compact topological group and let  $\mu$  be a left invariant Haar measure on  $G$ . If  $A$  is a  $\mu$ -measurable subset of  $G$  and  $\mu(A) > 0$ , then there exists a neighbourhood  $U$  of the neutral element of  $G$  such that*

$$(\forall h \in U)(\mu(h \cdot A \cap A) > 0).$$

**Proof.** Without loss of generality we may assume that  $\mu(A) < \infty$ . Let  $0 < \varepsilon < \frac{\mu(A)}{3}$ . Since  $\mu$  is a Radon measure we can find an open set  $V$  and a compact set  $K$  such that

- 1)  $K \subseteq A \subseteq V$ ;
- 2)  $\mu(V \setminus A) < \varepsilon$ ;
- 2)  $\mu(A \setminus K) < \varepsilon$ .

From the compactness of the set  $K$  we can easily deduce that there exists a neighbourhood  $U$  of the neutral element of  $G$  such that

$$U \cdot K \subseteq V.$$

Notice that

$$(\forall h \in G)(h \cdot K \cap K \subseteq h \cdot A \cap A).$$

Therefore, it is sufficient to show that

$$(\forall h \in U)(\mu(h \cdot K \cap K) > 0).$$

Suppose that this relation is not true. Then there exists  $h \in U$  such that

$$\mu(h \cdot K \cap K) = 0.$$

But  $K \subseteq V$  and  $h \cdot K \subseteq V$ . Thus, we have

$$\mu(V) \geq \mu(h \cdot K \cup K) = \mu(h \cdot K) + \mu(K) = 2\mu(K).$$

From this relation we conclude that

$$2(\mu(A) - \varepsilon) \leq \mu(A) + \varepsilon,$$

and, finally, we obtain

$$\mu(A) \leq 3\varepsilon < \mu(A).$$

This contradiction proves the theorem.

A careful analysis of the proof of the last theorem shows us that it can be done in theory **(ZF)** & **(DC)**. Moreover, a little modification of the proof presented above gives us that for every  $\mu$ -measurable subset  $A$  of the group  $G$  we have

$$\lim_h \mu(h \cdot A \cap A) = \mu(A),$$

where  $h \in G$  converges to the neutral element of  $G$ . This property of the Haar measure  $\mu$  is called the **Steinhaus property** of  $\mu$ . In particular, we see that the classical Lebesgue measure on  $\mathbb{R}$  has the Steinhaus property. Moreover, from the Steinhaus property of  $\mu$  and from the uniqueness theorem for a Haar measure we can easily deduce another important property of  $\mu$ . We mean here the so called **metric transitivity** of  $\mu$ . Namely, suppose that  $(G, \cdot)$  is a  $\sigma$ -compact locally compact topological group,  $H$  is a countable dense subset of  $G$  and  $A$  is a  $\mu$ -measurable subset of  $G$  such that  $\mu(A) > 0$ . Then we have

$$\mu(G \setminus \cup\{h \cdot A : h \in H\}) = 0.$$

Of course, we can apply this result to the Lebesgue measure on  $\mathbb{R}$ .

Now we shall consider a weak analogue of the Steinhaus property for category.

**Theorem 7.6** *Let  $(G, \cdot)$  be a topological group and let  $A$  be a non-meager subset of  $G$  having the Baire property. Then there exists a neighbourhood  $U$  of the neutral element of  $G$  such that*

$$(\forall h \in U)(h \cdot A \cap A \text{ is non-meager}).$$

**Proof.** From the Banach theorem about open first category sets (see Chapter 2) we can immediately deduce that any non-empty open subset of our group  $G$  is non-meager. Let us find an open subset  $V$  of  $G$  such that  $A \Delta V$  is meager and let us fix an element  $g \in V$ . From the continuity of basic operations on the group  $(G, \cdot)$  we get a neighbourhood  $U$  of the neutral element of  $G$  such that

$$(U^{-1}) \cdot U \cdot g \subseteq V.$$

We claim that

$$(\forall h \in U)(U \cdot g \subseteq (h \cdot V \cap V)).$$

Indeed, let  $h \in U$  and  $t \in U$ . Then we have  $t \cdot g \in V$ . But we also have  $h^{-1} \cdot t \cdot g \in V$ , so  $t \cdot g \in h \cdot V$ . Therefore,  $t \cdot g \in (h \cdot V \cap V)$ . Thus, for each element  $h \in U$  the set  $h \cdot V \cap V$  is non-meager. Now it is obvious that the same holds for the set  $A$ , i.e.  $h \cdot A \cap A$  is non-meager.

In the proof of the last theorem we used the classical Banach theorem about open first category sets, which is based on the Axiom of Choice. Hence, we see that the proof of the last theorem is done in theory **ZFC**. We want to remark that for some groups  $(G, \cdot)$  the Axiom of Choice can be replaced by a much weaker axiom **DC** (see exercises after this Chapter).

Let  $I$  be an ideal on a topological group  $(G, \cdot)$  and let  $I$  possess a Borel base. We say that  $I$  has the Steinhaus property if for any Borel set  $A$  which does not belong to  $I$  the set

$$A \cdot A^{-1} = \{ab^{-1} : a, b \in A\}$$

has a non-empty interior.

Now suppose that  $A$  and  $U$  are subsets of a group  $(G, \cdot)$  such that

$$(\forall h \in U)(h \cdot A \cap A \neq \emptyset).$$

Then it is clear that

$$(\forall h \in U)(\exists a, b \in A)(h \cdot a = b).$$

Therefore, we have

$$U \subseteq A \cdot A^{-1}.$$

Taking into account this simple remark and the results of two previous theorems we conclude that both ideals  $\mathbb{K}$  and  $\mathbb{L}$  have the Steinhaus property.

Theorems 5 and 6 proved above have various applications. In particular, the following example gives us a simple application of these theorems.

**Example 3.** Suppose that  $G$  is a Lebesgue measurable subgroup of the group  $(\mathbb{R}, +)$ . Then  $G = \mathbb{R}$  or  $G \in \mathbb{L}$ .

Indeed, suppose that  $\lambda(G) > 0$ . Then by Theorem 5 there exists an  $\varepsilon > 0$  such that

$$] - \varepsilon, \varepsilon[ \subseteq G - G = G.$$

But then we have

$$\bigcup_{n \in \mathbb{N}} n ] - \varepsilon, \varepsilon[ \subseteq G,$$

hence  $G = \mathbb{R}$ . An analogous fact is obviously true for the Baire property.

All results presented above show a big similarity between ideals  $\mathbb{K}$  and  $\mathbb{L}$ . Both are symmetric, invariant under translations, both possess Borel bases and have the Steinhaus property, both are  $\sigma$ -additive. Moreover, Martin's Axiom implies that these ideals are  $\mathbf{c}$ -additive.

We shall give another result which also shows a similarity between ideals  $\mathbb{K}$  and  $\mathbb{L}$ . In order to formulate and prove this result we need one auxiliary assertion concerning  $\kappa$ -additive ideals on an infinite cardinal  $\kappa$ .

Suppose that  $I_1$  is an ideal of subsets of a basic set  $E_1$  and that  $I_2$  is an ideal of subsets of a basic set  $E_2$ . We say that a bijection

$$f : E_1 \rightarrow E_2$$

is an *isomorphism* between  $I_1$  and  $I_2$  if

$$(\forall X \subseteq E_1)(X \in I_1 \iff f(X) \in I_2).$$

We say that ideals  $I_1$  and  $I_2$  are *isomorphic* if there exists an isomorphism between  $I_1$  and  $I_2$ .

Let  $\kappa$  be an arbitrary infinite cardinal. We define three simple ideals by the following relations:

- 1)  $I_0(\kappa) = [\kappa]^{<\kappa}$ ;
- 2)  $I_1(\kappa) = P(\kappa \times \{0\}) \cup [\kappa \times \{1\}]^{<\kappa}$ ;
- 3)  $I_2(\kappa) = \{A \subseteq \kappa \times \kappa : \text{card}(pr_1(A)) < \kappa\}$ .

Note that the ideal  $I_0$  is an ideal of subsets of  $\kappa$ , the ideal  $I_1$  is an ideal of subsets of  $(\kappa \times \{0\}) \cup (\kappa \times \{1\})$  and the ideal  $I_3$  is an ideal of subsets of  $\kappa \times \kappa$ . Since we have

$$\kappa = \text{card}((\kappa \times \{0\}) \cup (\kappa \times \{1\})) = \text{card}(\kappa \times \kappa),$$

all these ideals are defined on sets of cardinality  $\kappa$ .

**Theorem 7.7** *Let  $\kappa$  be an infinite cardinal and let  $J$  be an ideal of subsets of a basic set of cardinality  $\kappa$ . Suppose also that  $\text{add}(J) = \kappa$  and  $\text{cof}(J) = \kappa$ . Then  $J$  is isomorphic with one of the ideals  $I_0(\kappa), I_1(\kappa), I_2(\kappa)$ .*

**Proof.** We may assume that  $J$  is an ideal on the basic set  $X = \bigcup J$ . Taking into account the equality  $\text{add}(J) = \kappa$  we see that  $\kappa$  is a regular cardinal and  $[X]^{<\kappa} \subseteq J$ .

If the equality  $[X]^{<\kappa} = J$  holds, then  $J$  is isomorphic with the ideal  $I_0(\kappa)$ . Suppose now that  $J \setminus [X]^{<\kappa} \neq \emptyset$ . If there exists a set  $A \in J$  such that for any another set  $B \in J$  we have  $\text{card}(B \setminus A) < \kappa$ , then

$$J = P(A) \cup [X \setminus A]^{<\kappa},$$

hence,  $J$  is isomorphic with  $I_1(\kappa)$ .

Thus, we may assume that

$$(\forall A \in J)(\exists B \in J)(\text{card}(B \setminus A) = \kappa).$$

Let  $(B_\alpha)_{\alpha < \kappa} \subseteq J$  be a base of the ideal  $J$ . For each ordinal  $\alpha < \kappa$  we put

$$C_\alpha = \bigcup_{\zeta < \alpha} B_\zeta.$$

Then  $\{C_\alpha\}_{\alpha < \kappa}$  is also a base of  $J$ . Moreover, if  $\alpha < \beta$ , then  $C_\alpha \subseteq C_\beta$ . Our assumption implies that for every  $\alpha < \kappa$  there exists  $\beta < \kappa$  such that  $\text{card}(C_\beta \setminus C_\alpha) = \kappa$ . Without loss of generality we can suppose that  $\text{card}(C_{\alpha+1} \setminus C_\alpha) = \kappa$  for each  $\alpha < \kappa$ . Now let us put

$$C_{\alpha+1} \setminus C_\alpha = \{c_\beta^\alpha : \beta < \kappa\}.$$

Then the mapping

$$f : \kappa \times \kappa \rightarrow X$$

defined by the formula

$$f((\alpha, \beta)) = c_\beta^\alpha$$

gives us an isomorphism between the ideal  $J$  and the ideal  $I_2(\kappa)$ .

The following result is known as the Erdős-Sierpiński **Duality Principle**.

**Theorem 7.8 (Erdős-Sierpiński)** *Suppose that  $\text{add}(\mathbb{K}) = \text{add}(\mathbb{L}) = \mathfrak{c}$ . Then the ideals  $\mathbb{K}$  and  $\mathbb{L}$  are isomorphic. Moreover, there exists an isomorphism  $f : \mathbb{R} \rightarrow \mathbb{R}$  between  $\mathbb{K}$  and  $\mathbb{L}$  such that  $(f \circ f)(x) = x$  for every  $x \in \mathbb{R}$ .*

**Proof.** Let us observe that both ideals  $\mathbb{K}$  and  $\mathbb{L}$  satisfy assumptions of the previous theorem. Therefore, each of these ideals is isomorphic with one of the ideals  $I_0(\mathfrak{c}), I_1(\mathfrak{c}), I_2(\mathfrak{c})$ . Note that ideals  $\mathbb{K}$  and  $\mathbb{L}$  contain sets of the cardinality continuum. Hence, we can eliminate the ideal  $I_0(\mathfrak{c})$  as a possible isomorphic image of  $\mathbb{K}$  or  $\mathbb{L}$ . Notice now that for every  $A \in \mathbb{K}$  there exists a non-empty perfect set  $P \in \mathbb{K}$  such that  $A \cap P = \emptyset$ . The same fact is true for the ideal  $\mathbb{L}$ . But the set  $\mathfrak{c} \times \{0\}$  is in the ideal  $I_1(\mathfrak{c})$  and there is no  $P \in I_1(\mathfrak{c})$  such that  $\text{card}(P) = \mathfrak{c}$  and  $(\mathfrak{c} \times \{0\}) \cap P = \emptyset$ . Hence, we can also eliminate the ideal  $I_1(\mathfrak{c})$  as a possible isomorphic image of  $\mathbb{K}$  or  $\mathbb{L}$ . Therefore, we see that both ideals  $\mathbb{K}$  and  $\mathbb{L}$  are isomorphic to the ideal  $I_2(\mathfrak{c})$ . In particular, we obtain that ideals  $\mathbb{K}$  and  $\mathbb{L}$  are isomorphic.

Now let  $\{K, L\}$  be a partition of the real line  $\mathbb{R}$  such that  $K \in \mathbb{K}$  and  $K$  is an  $F_\sigma$ -set and  $L \in \mathbb{L}$  and  $L$  is a  $G_\delta$ -set. The results of Chapters 3 and 4 imply that there exists a Borel isomorphism between the ideals  $\mathbb{K}$  and  $\mathbb{K} \cap P(L)$  and there exists a Borel isomorphism between the ideals  $\mathbb{L}$  and  $\mathbb{L} \cap P(K)$ . Hence, the ideals  $\mathbb{K} \cap P(L)$  and  $\mathbb{L} \cap P(K)$  are isomorphic, too. Let  $\pi : L \rightarrow K$  be such an isomorphism. Then

it is not difficult to check that the function  $f = \pi \cup \pi^{-1}$  is a required isomorphism between  $\mathbb{K}$  and  $\mathbb{L}$ .

Now we shall consider some properties of two classical Boolean algebras closely connected with the ideals  $\mathbb{K}$  and  $\mathbb{L}$ . Let  $\mathcal{C}$  denote the quotient Boolean algebra  $\mathbb{B}/(\mathbb{B} \cap \mathbb{K})$  and let  $\mathcal{R}$  denote the quotient Boolean algebra  $\mathbb{B}/(\mathbb{B} \cap \mathbb{L})$ . The Boolean algebra  $\mathcal{C}$  is called the *Cohen algebra* and the Boolean algebra  $\mathcal{R}$  is called the *Solovay algebra*.

Let  $B$  be a Boolean algebra and let  $X \subseteq B$ . A supremum of  $X$  in  $B$ , if it exists, is denoted by the symbol  $\sum X$ . Similarly, an infimum of  $X$  in  $B$ , if it exists, is denoted by the symbol  $\prod X$ .

Let  $\kappa$  be an uncountable cardinal number. We say that  $B$  is a  $\kappa$ -**complete** Boolean algebra if every subset of  $B$  of cardinality less than  $\kappa$  has a supremum in  $B$ . Notice that  $\omega_1$ -complete Boolean algebras are also called  $\sigma$ -complete algebras. It is easy to check that if a Boolean algebra  $B$  is  $\kappa$ -complete and  $J$  is a  $\kappa$ -complete ideal in  $B$ , then the quotient algebra  $B/J$  is  $\kappa$ -complete, too. In particular, Boolean algebras  $\mathcal{C}$  and  $\mathcal{R}$  are  $\sigma$ -complete.

Recall that a Boolean algebra  $B$  is complete if every subset  $X$  of  $B$  has a supremum in  $B$ , i.e. if  $B$  is  $\kappa$ -complete for any uncountable cardinal  $\kappa$ . We say that a Boolean algebra  $B$  satisfies the countable chain condition (c.c.c) if every family  $X \subseteq B$  of pairwise disjoint elements (i.e.  $(\forall x, y \in X)(x \neq y \rightarrow x \wedge y = 0)$ ) is at most countable.

It is easy to check that both algebras  $\mathcal{C}$  and  $\mathcal{R}$  satisfy c.c.c.

**Theorem 7.9** *Suppose that  $B$  is a  $\sigma$ -complete Boolean algebra satisfying the countable chain condition. Then  $B$  is a complete algebra.*

**Proof.** Let  $X$  be an arbitrary family of elements of  $B$ . Let  $Y$  be a maximal (with respect to inclusion) subset of  $B \setminus \{0\}$  consisting of pairwise disjoint elements such that

$$(\forall y \in Y)(\exists x \in X)(y \leq x).$$

Then  $\text{card}(Y) \leq \omega$  and it is not difficult to show that

$$\sum Y = \sum X.$$

From this theorem we immediately conclude that both Boolean algebras  $\mathcal{C}$  and  $\mathcal{R}$  are complete nonatomic algebras satisfying c.c.c.

A Boolean algebra  $B$  is called a **measure Boolean algebra** if there exists a function

$$\mu : B \rightarrow [0, 1]$$

such that

- 1)  $\mu(1) = 1$ ;
- 2)  $(\forall b \in B)(\mu(b) = 0 \iff b = 0)$ ;
- 3) for every sequence  $(b_n)_{n \in \mathbb{N}}$  of pairwise disjoint elements from  $B$  we have

$$\mu\left(\sum_{n \in \mathbb{N}} b_n\right) = \sum_{n \in \mathbb{N}} \mu(b_n).$$



The function  $\mu$  is called a measure on the algebra  $B$ .

Obviously, there exists a probability diffused Borel measure  $\nu$  on the real line  $\mathbb{R}$  equivalent to the Lebesgue measure  $\lambda$ . It is clear that  $\nu$  canonically defines the corresponding measure  $\mu$  on the algebra  $\mathcal{R}$ . In particular, we obtain that  $\mathcal{R}$  is a measure Boolean algebra. Remark also that the measure  $\nu$  is Borel isomorphic with the restriction of the Lebesgue measure  $\lambda$  to the Borel  $\sigma$ -algebra of the segment  $[0, 1]$ . Indeed, as we know any two probability diffused Borel measures defined on Polish topological spaces are Borel isomorphic.

Let  $B$  be a Boolean algebra. We say that a set  $X \subseteq B \setminus \{0\}$  is *cofinal* in  $B$  if  $X$  is a cofinal subset of  $B \setminus \{0\}$  with respect to the ordering inverse to the ordering of  $B$ , i.e.

$$(\forall b \in B \setminus \{0\})(\exists x \in X)(x \leq b).$$

The *cofinality* of  $B$ , denoted by  $\text{cof}(B)$ , is defined by the equality

$$\text{cof}(B) = \min\{\text{card}(X) : X \subseteq B \setminus \{0\} \text{ \& } X \text{ is cofinal in } B\}.$$

Now we can formulate the following

**Theorem 7.10** *The algebras  $\mathcal{C}$  and  $\mathcal{R}$  are not isomorphic. In fact, we have  $\text{cof}(\mathcal{C}) = \omega$  and  $\text{cof}(\mathcal{R}) > \omega$ . Moreover, there is no Borel isomorphism between the ideals  $\mathbb{K}$  and  $\mathbb{L}$ .*

**Proof.** Let  $T$  be an arbitrary countable base of the standard topology on  $\mathbb{R}$ . Then it is easy to see that  $\{[U]_{\mathbb{K}} : U \in T\}$  is a cofinal set in  $\mathcal{C}$ . Hence,  $\text{cof}(\mathcal{C}) = \omega$ . Let us show that  $\text{cof}(\mathcal{R}) > \omega$ . Let  $\lambda$  be a probability measure on  $\mathcal{R}$  canonically associated with the Lebesgue measure and suppose that  $(a_n)_{n \in \mathbb{N}} \subseteq \mathcal{R} \setminus \{0\}$  is a countable cofinal family in  $\mathcal{R}$ . For each  $n \in \mathbb{N}$  let us choose an element  $b_n \in \mathcal{R} \setminus \{0\}$  such that  $b_n < a_n$  and  $0 < \lambda(b_n) < 2^{-n-2}$ . Then  $(b_n)_{n \in \mathbb{N}}$  is a cofinal subfamily of  $\mathcal{R} \setminus \{0\}$ , too. Let us put

$$c = \sum_{n \in \mathbb{N}} b_n.$$

Then we have  $\lambda(c) < \frac{1}{2}$ , so  $c' > 0$ . But there is no  $n \in \mathbb{N}$  such that  $b_n \leq c'$ . Thus, we obtain a contradiction. Hence,  $\text{cof}(\mathcal{R}) > \omega$ .

The inequality  $\text{cof}(\mathcal{C}) < \text{cof}(\mathcal{R})$  immediately implies that the algebras  $\mathcal{C}$  and  $\mathcal{R}$  are not isomorphic.

Suppose now that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a Borel bijection between ideals  $\mathbb{K}$  and  $\mathbb{L}$ . Then the mapping  $\varphi : \mathcal{R} \rightarrow \mathcal{C}$  defined by the formula

$$\varphi([A]_{\mathbb{L}}) = [f^{-1}(A)]_{\mathbb{K}} \quad (A \in \mathbb{B})$$

is an isomorphism between algebras  $\mathcal{R}$  and  $\mathcal{C}$ , and we get a contradiction.

Theorem 10 gives us the first important difference between measure and category on the real line. Moreover, this theorem shows that the isomorphism between  $\mathbb{K}$  and  $\mathbb{L}$  postulated by the Duality Principle is a very strange mapping which is far from being effective.

Let us discuss the second important difference between ideals  $\mathbb{K}$  and  $\mathbb{L}$ .

Let  $B$  be a  $\mathbf{c}^+$ -complete Boolean algebra. We say that  $B$  is **weakly**  $(\omega, \omega)$ -**distributive** if for every double sequence  $\{b_{nk}\}_{n,k \in \mathbb{N}}$  of elements of  $B$  the following equality holds:

$$\prod_n \sum_k b_{nk} = \sum_{f \in \omega^\omega} \prod_n \sum_{k < f(n)} b_{nk}.$$

The reader can treat this property as rather complicated. However, its meaning is completely clear, for example, in the language of forcing extensions of models of set theory. Note that the inequality

$$\prod_n \sum_k b_{nk} \geq \sum_{f \in \omega^\omega} \prod_n \sum_{k < f(n)} b_{nk}$$

holds in every  $\mathbf{c}^+$ -complete Boolean algebra.

**Theorem 7.11** *The algebra  $\mathcal{C}$  is not weakly  $(\omega, \omega)$ -distributive. The algebra  $\mathcal{R}$  is weakly  $(\omega, \omega)$ -distributive.*

**Proof.** Let us recall that for any two uncountable Polish spaces there exists a Borel isomorphism between them which preserves the first category subsets. Hence, the algebra  $\mathcal{C}$  is isomorphic to the algebra  $B(\mathbb{N}^\omega)/K(\mathbb{N}^\omega)$ , where  $\mathbb{N}^\omega$  denotes the standard Baire space.

For any natural numbers  $n$  and  $k$  let us put

$$B_{nk} = \{f \in \mathbb{N}^\omega : f(n) = k\}.$$

Then for each  $n \in \mathbb{N}$  we have

$$\sum_k [B_{nk}] = [\bigcup_k B_{nk}] = [\mathbb{N}^\omega] = 1.$$

Hence,

$$\prod_n \sum_k [B_{nk}] = 1.$$

Let  $f \in \omega^\omega$  be an arbitrary function. Then

$$\bigcap_n \bigcup_{k < f(n)} B_{nk}$$

is a closed nowhere dense subset of  $\mathbb{N}^\omega$ . Thus,

$$\prod_n \sum_{k < f(n)} [B_{nk}] = 0.$$

Therefore, we have

$$\sum_{f \in \omega^\omega} \prod_n \sum_{k < f(n)} [B_{nk}] = 0.$$

In this way we proved that the algebra  $\mathcal{C}$  is not weakly  $(\omega, \omega)$ -distributive.

Suppose now that  $\{b_{nk}\}_{n,k \in \mathbb{N}}$  is a double sequence of elements of  $\mathcal{R}$ . Let  $\lambda$  be the canonical probability measure on  $\mathcal{R}$  and let

$$b = \prod_n \sum_k b_{nk}.$$

Without loss of generality we may assume that  $\lambda(b) > 0$ . Fix a real number  $\epsilon > 0$ . Note that for every  $n \in \mathbb{N}$  we have

$$b \leq \sum_k b_{nk}.$$

Hence,

$$b = \sum_k b \cdot b_{nk}.$$

From this equality we get

$$\lambda(b) \leq \lim_k \lambda\left(\sum_{l < k} b \cdot b_{nl}\right).$$

Let  $f : \omega \rightarrow \omega$  be a function such that for each  $n \in \mathbb{N}$  we have

$$\lambda\left(\sum_{l < f(n)} b \cdot b_{nl}\right) \geq \lambda(b) \cdot \left(1 - \frac{\epsilon}{2^{n+1}}\right).$$

Then it is not difficult to check that

$$\lambda\left(\prod_n \sum_{l < f(n)} b \cdot b_{nl}\right) \geq \lambda(b) \cdot (1 - \epsilon).$$

But  $\epsilon > 0$  is arbitrary, so we see that

$$\prod_n \sum_k b_{nk} \leq \sum_{f \in \omega^\omega} \prod_n \sum_{k < f(n)} b_{nk}.$$

Thus, the theorem is proved.

Now we shall discuss some properties of the standard Baire space  $\mathbb{N}^\omega$ . Let us recall the order relation  $\preceq$  on  $\omega^\omega$  defined in Chapter 1 of the book:

$$f \preceq g \iff (\exists n \in \omega)(\forall m \geq n)(f(m) \leq g(m)).$$

In the same Chapter we defined also two cardinal numbers  $\mathbf{b}$  and  $\mathbf{d}$ . We shall show that these cardinal numbers are closely connected with cardinal functions describing the ideal  $\mathbb{K}$ .

**Theorem 7.12** *The following relations hold:*

- 1)  $\mathbf{b} \leq \text{non}(\mathbb{K})$ ;
- 2)  $\text{cov}(\mathbb{K}) \leq \mathbf{d}$ ;
- 3)  $\text{add}(\mathbb{K}) = \min\{\mathbf{b}, \text{cov}(\mathbb{K})\}$ ;
- 4)  $\text{cof}(\mathbb{K}) = \max\{\mathbf{d}, \text{cov}(\mathbb{K})\}$ .

**Proof.** For every function  $f \in \omega^\omega$  let us consider the set  $K_f$  defined by the equality

$$K_f = \{g \in \omega^\omega : g \preceq f\}.$$

Note that

$$K_f = \bigcup_n \bigcap_{k > n} \{g \in \omega^\omega : g(k) \leq f(k)\}.$$

From this formula we easily deduce that for every  $f \in \omega^\omega$  the set  $K_f$  is a first category subset of the space  $\mathbb{N}^\omega$ .

Suppose now that  $\kappa < \mathbf{b}$  and that  $\{f_\alpha : \alpha < \kappa\}$  is a subset of  $\omega^\omega$ . Then the set  $\{f_\alpha : \alpha < \kappa\}$  is not  $\preceq$ -unbounded, i.e. there exists a function  $f \in \omega^\omega$  such that

$$(\forall \alpha < \kappa)(f_\alpha \preceq f).$$

But then we have

$$\{f_\alpha : \alpha < \kappa\} \subseteq K_f,$$

hence,  $\text{non}(\mathbb{K}) \geq \mathbf{b}$ .

Suppose now that a set  $X \subseteq \omega^\omega$  is  $\preceq$ -cofinal, i.e. that

$$(\forall f \in \omega^\omega)(\exists g \in X)(f \preceq g).$$

Then we have

$$\omega^\omega = \bigcup_{g \in X} K_g,$$

and this relation implies the inequality  $\text{cov}(\mathbb{K}) \leq \mathbf{d}$ .

Now we want to show that

$$\min\{\mathbf{b}, \text{cov}(\mathbb{K})\} \leq \text{add}(\mathbb{K}).$$

Suppose that  $\kappa < \min\{\mathbf{b}, \text{cov}(\mathbb{K})\}$  and that  $\{K_\alpha : \alpha < \kappa\}$  is an arbitrary family of sets from the  $\sigma$ -ideal  $\mathbb{K}$ . For each  $\alpha < \kappa$  let  $(N_n^\alpha)_{n \in \mathbb{N}}$  be a sequence of nowhere dense subsets of  $\mathbb{R}$  such that

$$K_\alpha = \bigcup_n N_n^\alpha.$$

Let us put

$$\mathcal{S} = \{N_n^\alpha : \alpha < \kappa \text{ \& } n \in \mathbb{N}\}.$$

It is sufficient to establish that

$$\bigcup \mathcal{S} \in \mathbb{K}.$$

We shall do it in two steps.

First we observe that there exists a real number  $t_0$  such that for every  $N \in \mathcal{S}$  we have  $(t_0 + \mathbb{Q}) \cap N = \emptyset$ . Indeed, it is easy to check that any real number  $t_0$  from the set

$$\mathbb{R} \setminus \bigcup \{N - q : N \in \mathcal{S} \text{ \& } q \in \mathbb{Q}\}$$

is a required one. This set is non-empty, since

$$\text{card}(\{N - q : N \in \mathcal{S} \text{ \& } q \in \mathbb{Q}\}) \leq \kappa \cdot \omega < \text{cov}(\mathbb{K}).$$

Let us fix some enumeration  $(q_n)_{n \in \mathbb{N}}$  of the set  $\mathbb{Q} + t_0$ . For any set  $N \in \mathcal{S}$  let  $f_N$  be a function from  $\omega^\omega$  such that

$$N \cap \bigcup_n ]q_n - \frac{1}{f(n)}, q_n + \frac{1}{f(n)}[ = \emptyset.$$

Notice that

$$\text{card}(\{f_N : N \in \mathcal{S}\}) \leq \kappa \cdot \omega < \mathbf{b}.$$

Hence, there exists a function  $g \in \omega^\omega$  such that  $f_N \preceq g$  for every  $N \in \mathcal{S}$ . Evidently, we may assume that 0 does not belong to  $\text{ran}(g)$ . Let us put

$$H = \bigcap_n \bigcup_{k > n} ]q_k - \frac{1}{g(k)}, q_k + \frac{1}{g(k)}[.$$

Then  $H$  is a dense  $G_\delta$ -subset of  $\mathbb{R}$  and for every  $N \in \mathcal{S}$  we have  $N \cap H = \emptyset$ . So the union  $\bigcup \mathcal{S}$  is a first category subset of  $\mathbb{R}$ .

Now we can easily show that

$$\text{cof}(\mathbb{K}) \leq \max\{\mathbf{d}, \text{cov}(\mathbb{K})\}.$$

Indeed, let  $X$  be a subset of  $\mathbb{R}$  such that  $\text{card}(X) = \text{non}(\mathbb{K})$  and  $X \notin \mathbb{K}$ . Let  $F$  be a  $\preceq$ -cofinal subset of  $\omega^\omega$  such that  $\text{card}(F) = \mathbf{d}$ . We may assume that for each function  $f \in F$  the range of  $f$  does not contain 0. Finally, let  $(q_n)_{n \in \mathbb{N}}$  be a fixed enumeration of  $\mathbb{Q}$ . We put

$$\mathcal{T} = \{\mathbb{R} \setminus \bigcap_n \bigcup_{k > n} ]q_k + x - \frac{1}{f(k)}, q_k + x + \frac{1}{f(k)}[ : x \in X \ \& \ f \in F\}.$$

Using the previous arguments we deduce that  $\mathcal{T}$  is a cofinal subset of  $\mathbb{K}$ . Hence, we have the inequality  $\text{cof}(\mathbb{K}) \leq \max\{\mathbf{d}, \text{cov}(\mathbb{K})\}$ .

It remains to prove that

$$\text{add}(\mathbb{K}) \leq \mathbf{b}, \quad \mathbf{d} \leq \text{cof}(\mathbb{K}).$$

In order to prove these inequalities we need two notations. For every function  $f \in \omega^\omega$  let us put

$$T_f = \{x \in \{0, 1\}^\omega : (\forall n)(x(f(n)) = 0)\}.$$

Notice that if  $f \in \omega^\omega$  is a strictly increasing function, then  $T_f$  is a nowhere dense closed subset of the Cantor space  $\{0, 1\}^\omega$ . For any decreasing (with respect to inclusion) sequence  $G = (G_n)_{n \in \mathbb{N}}$  of dense open subsets of  $\{0, 1\}^\omega$  we define by recursion the following sequence  $(m_k)_{k \in \mathbb{N}}$  of natural numbers:

1.  $m_0 = 0$ ;
2.  $m_{k+1} = \min\{n > m_k : (\forall s \in \{0, 1\}^{m_k})(\exists t \in \{0, 1\}^n)(s \subseteq t \ \& \ [t] \subseteq G_k)\}$ ,

where  $[t]$  denotes the basic open subset of the space  $\{0, 1\}^\omega$  determined by the finite sequence  $t$ . Let us put

$$f_n(k) = m_{n+k}$$

for all  $n, k \in \mathbb{N}$ . Let  $h(G) \in \omega^\omega$  be a function such that

$$(\forall n \in \mathbb{N})(f_n \preceq h(G)).$$

We assert that for every decreasing sequence  $G = (G_n)_{n \in \mathbb{N}}$  of dense open subsets of  $\{0, 1\}^\omega$  and for every strictly increasing function  $g \in \omega^\omega$  the following implication holds:

$$T_g \cap \bigcap_n G_n = \emptyset \quad \rightarrow \quad g \preceq h(G).$$

Let us prove this implication. Suppose that the relation  $g \preceq h(G)$  does not hold. Then it is not difficult to check that the set

$$\{n \in \mathbb{N} : (\exists k \in \mathbb{N})(g(n) \leq m_k \ \& \ m_{k+1} < g(n+1))\}$$

is infinite. Indeed, suppose that there exists  $p \in \mathbb{N}$  such that for all  $n > p$  the interval  $[g(n), g(n+1)[$  contains at most one element from the set

$$\{m_0, m_1, \dots, m_k, \dots\}.$$

Let

$$k_0 = \min\{k : m_k > g(p)\}.$$

Then for each natural index  $i$  we have the inequality

$$g(p+i) \leq m_{k_0+i}.$$

This inequality gives us

$$g(p+i) \leq f_{k_0}(i) \leq f_{k_0}(p+i)$$

for all natural numbers  $i$ . Hence, we obtain  $g \preceq f_{k_0}$ , which is impossible. Thus, the mentioned set is infinite. Now, taking into account the definition of natural numbers  $m_k$  ( $k \in \mathbb{N}$ ), we can recursively construct an element  $x$  from the intersection  $T_g \cap \bigcap_n G_n$ . So, we proved our auxiliary assertion.

Suppose now that  $F \subseteq \omega^\omega$  is an  $\preceq$ -unbounded family of functions. We may assume that each function from  $F$  is strictly increasing. Suppose that the set

$$T = \bigcup \{T_f : f \in F\}$$

is a first category set in the space  $\{0, 1\}^\omega$  and let us take a decreasing sequence  $G = (G_n)_{n \in \mathbb{N}}$  of dense open subsets of  $\{0, 1\}^\omega$  such that

$$T \cap \bigcap_n G_n = \emptyset.$$

Then we have  $f \preceq h(G)$  for every  $f \in F$ . Hence, we conclude that  $F$  is bounded. This contradiction shows us that  $T$  is not a first category subset of  $\{0, 1\}^\omega$ . Thus,  $\text{add}(\mathbb{K}) \leq \mathbf{b}$ .

Now let us take any cofinal family  $(K_\alpha)_{\alpha < \text{cof}(\mathbb{K})}$  of first category sets in the space  $\{0, 1\}^\omega$ . For each  $\alpha < \text{cof}(\mathbb{K})$  let us fix a decreasing sequence

$$G_\alpha = (G_n^\alpha)_{n \in \mathbb{N}}$$

of dense open subsets of  $\{0, 1\}^\omega$  such that

$$K_\alpha \cap \bigcap_n G_n^\alpha = \emptyset.$$

It is easy to see that the family

$$\{h(G_\alpha) : \alpha < \text{cof}(\mathbb{K})\}$$

is a  $\preceq$ -cofinal family. Hence, we have  $\mathbf{d} \leq \text{cof}(\mathbb{K})$ , and the theorem is proved.

Let  $\mathbb{W}$  denote the ideal of nowhere dense subsets of  $\mathbb{R}$ . This ideal is an important mathematical object, because the  $\sigma$ -ideal  $\mathbb{K}$  is  $\sigma$ -generated by the ideal  $\mathbb{W}$ . It is obvious that

$$\text{add}(\mathbb{W}) = \text{non}(\mathbb{W}) = \omega$$

and that

$$\text{cov}(\mathbb{W}) = \text{cov}(\mathbb{K}).$$

We shall show that, in a certain sense, this ideal is as complicated as the ideal  $\mathbb{K}$ .

**Theorem 7.13**  $\text{cof}(\mathbb{W}) = \text{cof}(\mathbb{K})$ .

**Proof.** Let  $F$  be a cofinal subfamily of the ideal  $\mathbb{W}$ . We shall prove that the family

$$\{[0, 1] \cap \bigcup_{n \in \mathbb{N}} (Y - n) : Y \in F\}$$

is cofinal in the ideal  $\mathcal{K}([0, 1])$ . Suppose that  $A \subseteq [0, 1]$  is a first category set. Let  $(A_n)_{n \in \omega}$  be a countable family of nowhere dense subsets of  $[0, 1]$  such that  $A = \bigcup_n A_n$ . Let us put

$$B = \bigcup_n (A_n + n).$$

Then  $B$  is a nowhere dense subset of  $\mathbb{R}$ , so there exists a set  $Y \in F$  such that  $B \subseteq Y$ . But then we have

$$A \subseteq [0, 1] \cap \bigcup_n (Y - n).$$

This inclusion shows us that  $\text{cof}(\mathbb{K}) \leq \text{cof}(\mathbb{W})$ .

Suppose now that  $X \subseteq \mathbb{R}$  is a set of cardinality  $\text{non}(\mathbb{K})$  not belonging to the ideal  $\mathbb{K}$ . Let us take a set  $F \subseteq \omega^\omega$  of cardinality  $\mathfrak{d}$  such that for every function  $g \in \omega^\omega$  there exists a function  $f \in F$  satisfying the relation

$$(\forall n)(g(n) \leq f(n)).$$

Let  $(q_n)_{n \in \mathbb{N}}$  be a fixed enumeration of the set  $\mathbb{Q}$ . For each  $x \in \mathbb{R}$  and for each  $f \in \omega^\omega$  let us define

$$C_x^f = x + \bigcup_n ]q_n - \frac{1}{f(n) + 1}, q_n + \frac{1}{f(n) + 1}[.$$

We assert that the family

$$\{\mathbb{R} \setminus C_x^f : x \in X \text{ \& } f \in F\}$$

is a base of the ideal  $\mathbb{W}$ . Indeed, suppose that  $A \in \mathbb{W}$ . Let

$$x \in X \cap (\mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} (A - q)).$$

Then it is easy to check that  $(x + \mathbb{Q}) \cap A = \emptyset$ . So, we can take a function  $g \in \omega^\omega$  such that

$$A \cap \bigcup_n ]x + q_n - \frac{1}{g(n) + 1}, x + q_n + \frac{1}{g(n) + 1}[ = \emptyset.$$

Now let  $f$  be a function from  $F$  such that  $(\forall n)(g(n) \leq f(n))$ . Then  $A \cap C_x^f = \emptyset$ , so we see that

$$\text{cof}(\mathbb{W}) \leq \text{non}(\mathbb{K}) \cdot \mathfrak{d}.$$

Hence, by Theorem 12, we get the inequality  $\text{cof}(\mathbb{W}) \leq \text{cof}(\mathbb{K})$ . Finally, we have the required equality  $\text{cof}(\mathbb{W}) = \text{cof}(\mathbb{K})$ .

## Exercises

**Exercise 7.1** Find a proper  $\sigma$ -ideal of subsets of the real line  $\mathbb{R}$  which is invariant under all translations of  $\mathbb{R}$  and for which there exists a set  $A \in I$  satisfying the relation

$$(\forall B \in I)(\exists x \in \mathbb{R})(B \subseteq A + x).$$

Show that in Theorem 1 of this Chapter we can omit the assumption "I is symmetric" if our group  $(G, \cdot)$  satisfies the following relation:

$$(\forall g \in G)(\exists n \in \mathbb{N})(n \cdot g = 0),$$

where 0 denotes the neutral element of  $G$ .

**Exercise 7.2** Prove that there exist two first category subsets  $X$  and  $Y$  of the real line  $(\mathbb{R}, +)$  such that the set  $X + Y$  does not have the Baire property in  $\mathbb{R}$ . Conclude from this fact that the ideal  $\mathbb{K}$  is not closed with respect to the operation  $+$ .

Similarly, prove that there exist two Lebesgue measure zero subsets  $X$  and  $Y$  of  $\mathbb{R}$  such that the set  $X + Y$  is not measurable in the Lebesgue sense. Conclude from this fact that the ideal  $\mathbb{L}$  also is not closed with respect to the operation  $+$ .

**Exercise 7.3** a) Let  $(G, \cdot)$  be an arbitrary topological group and let  $A$  and  $B$  be any two non-meager subsets of  $G$  with the Baire property. Show that the set  $A \cdot B^{-1}$  has a non-empty interior in  $G$ .

b) Let  $(G, \cdot)$  be a locally compact topological group equipped with the Haar measure  $\mu$  and let  $A$  and  $B$  be  $\mu$ -measurable subsets of  $G$  such that  $\mu(A) > 0$ ,  $\mu(B) > 0$ . Show, in theory **ZF** & **DC**, that the set  $A \cdot B^{-1}$  has a non-empty interior in  $G$ .

**Exercise 7.4** Let  $(G, \cdot)$  be a topological group satisfying the first countability axiom, i.e. the neutral element of  $G$  has a countable fundamental system of neighbourhoods. Prove, in theory **ZF** & **DC**, an analogue of the Steinhaus property for  $(G, \cdot)$ .

**Exercise 7.5** Let  $(G, +)$  be a topological vector space (over the field of rational numbers). Prove, in theory **ZF** & **DC**, an analogue of the Steinhaus property for  $(G, +)$ .

**Exercise 7.6** Prove the metric transitivity of the Haar measure.

**Exercise 7.7** Formulate and prove an analogue of the metric transitivity for the Baire property in topological groups.

**Exercise 7.8** Let  $\kappa$  be an arbitrary uncountable cardinal and let  $\{0, 1\}^\kappa$  be the generalized Cantor discontinuum of the weight  $\kappa$ . Find the values of the following cardinal functions:

$$\text{add}(L(\{0, 1\}^\kappa)), \quad \text{add}(K(\{0, 1\}^\kappa)).$$

**Exercise 7.9** Recall that a Boolean algebra  $B$  is atomic if for every  $b \in B \setminus \{0\}$  there exists an atom  $a \in B$  such that  $a \leq b$ . Show that if  $B$  is a complete atomic Boolean algebra and

$$\text{At}(B) = \{a \in B : a \text{ is an atom}\},$$

then  $B$  is isomorphic with the power set algebra  $P(\text{At}(B))$ .

**Exercise 7.10** Let  $\kappa$  be an uncountable cardinal and let  $B$  be a  $\kappa$ -complete Boolean algebra. Suppose also that  $J \subseteq B$  is a  $\kappa$ -complete ideal in  $B$ . Show that the quotient Boolean algebra  $B/J$  is  $\kappa$ -complete, too.

**Exercise 7.11** Let  $\kappa$  be an infinite cardinal number. We say that a Boolean algebra  $B$  satisfies the  $\kappa$ -chain condition if for every family  $X \subseteq B$  of pairwise disjoint elements we have  $\text{card}(X) < \kappa$ .

Show that if an algebra  $B$  is  $\kappa$ -complete and satisfies the  $\kappa$ -chain condition, then  $B$  is a complete Boolean algebra.

In particular, let  $\kappa = \omega$ . Show that any Boolean algebra  $B$  satisfying the  $\omega$ -chain condition is finite.

**Exercise 7.12** Prove that if  $B$  is a complete non-atomic Boolean algebra and  $\text{cof}(B) = \omega$ , then  $B$  is isomorphic to the Cohen algebra  $\mathcal{C}$ .

**Exercise 7.13** We say that a Boolean algebra  $B$  has an  $(\omega_1, \omega)$ -caliber if for every  $X \in [B]^{\omega_1}$  there exists  $Y \in [X]^\omega$  such that  $\prod Y > 0$ . Show that the Solovay algebra  $\mathcal{R}$  has  $(\omega_1, \omega)$ -caliber.



**Exercise 7.14** Prove the Erdős-Sierpiński Duality Principle under the assumption that

$$\text{add}(\mathbb{K}) = \text{add}(\mathbb{L}) = \text{cof}(\mathbb{K}) = \text{cof}(\mathbb{L}).$$

**Exercise 7.15** Assume that Martin's Axiom holds. Let  $E$  be a topological space with a countable base and let  $\{X_i : i \in I\}$  be a family of first category subsets of  $E$  such that

$$\text{card}(I) < \mathfrak{c}.$$

Prove that the set  $\cup\{X_i : i \in I\}$  is also a first category subset of  $E$ .

**Exercise 7.16** Assume that Martin's Axiom holds. Let  $E$  be a topological space with a countable base such that every closed subset of  $E$  is  $G_\delta$ -set in  $E$ . Let  $\mu$  be an arbitrary  $\sigma$ -finite Borel measure on  $E$ . Prove that the completion of  $\mu$  is a  $\mathfrak{c}$ -additive measure.

## Chapter 8

# Some Classical Subsets of the Real Line

There are two main goals of this Chapter: to justify the title of the book and to acquaint the reader with some classical subsets of the real line  $\mathbb{R}$  which play an important role not only in the mathematical analysis but also in other fields of mathematics. We have already met one of such sets: this is the classical Cantor discontinuum  $C$  on the real line  $\mathbb{R}$  which, as a topological space, is homeomorphic with the product space  $\{0, 1\}^\omega$ , where the set  $\{0, 1\}$  is equipped with the discrete topology.

One of the most important properties of the Cantor discontinuum is the following: in the class of all non-empty compact metric spaces this space is in some sense an initial one. More precisely, any non-empty compact metric space is a continuous image of the Cantor discontinuum. As an illustration of the power of this fact let us remark that it implies the existence of various Peano-type mappings (continuous surjections from the segment  $[0, 1]$  onto the square  $[0, 1]^2$ ), the existence of a family of the cardinality continuum of non-empty pairwise disjoint perfect subsets of the segment  $[0, 1]$  and so on.

Let us remark also that for the **generalized Cantor discontinuum**, i.e. topological space of the form  $\{0, 1\}^\kappa$ , where  $\kappa$  is an arbitrary uncountable cardinal, we do not have any similar result. In other words, not every compact topological space is a continuous image of some generalized Cantor discontinuum. This fact (observed first by E. Marczewski) is implied by the Suslin condition (i.e. the countable chain condition) for the generalized Cantor discontinuum, which was discussed in Chapter 2. This property (i.e. c.c.c) evidently is preserved by continuous surjective mappings. Hence, if a topological compact space does not satisfy the Suslin condition (for example, the space  $\omega_1 + 1$  with the usual order topology), then it cannot be a continuous image of the generalized Cantor discontinuum. The fact mentioned above was an impulse for creating a whole domain in general topology, namely, the so called theory of dyadic compact spaces (i.e. such compact spaces which are continuous images of the generalized Cantor discontinuums).

We shall meet the Cantor discontinuum in future considerations many times. We shall see that there are a lot of non-trivial assertions and constructions which are closely related to this object.

The Cantor discontinuum  $C$  is one of the first uncountable mathematical objects which can be called a small set. We know that this set is small from the point of view of category, namely, this set is a nowhere dense subset of the real line  $\mathbb{R}$ . The same can be said from the point of view of the classical Lebesgue measure. Namely, the Lebesgue measure of  $C$  is equal to zero. However, the Cantor discontinuum is

not as small as one can think at the first observation. For example, it is easy to check that the set

$$C + C = \{x + y : x \in C \text{ \& } y \in C\}$$

contains a non-empty open subinterval of the real line  $\mathbb{R}$ . Actually, we have

$$C + C = [0, 2].$$

In particular, this fact shows us that neither ideals  $\mathbb{K}$  and  $\mathbb{L}$  on the real line  $\mathbb{R}$  are closed under vector sums of their elements. It is worth remarking here that on the real line there exist non-trivial invariant  $\sigma$ -ideals which are closed under vector sums of their elements (see exercises to this Chapter).

Now we shall construct a Lebesgue measurable subset of the real line  $\mathbb{R}$  which, together with its complement, is big from the Lebesgue measure point of view. Namely, we shall construct a Lebesgue measurable set  $X \subseteq \mathbb{R}$  such that

$$\lambda(X \cap V) > 0, \quad \lambda((\mathbb{R} \setminus X) \cap V) > 0$$

for every non-empty open set  $V \subseteq \mathbb{R}$ . Let  $(V_n)_{n \in \mathbb{N}}$  be any countable base of open subsets of  $\mathbb{R}$ . By the method of mathematical recursion we define two sequences

$$(X_n)_{n \in \mathbb{N}}, \quad (Y_n)_{n \in \mathbb{N}}$$

of subsets of  $\mathbb{R}$  such that for each index  $n \in \mathbb{N}$  the following relations hold:

- 1)  $X_n, Y_n$  are closed and nowhere dense subsets of  $\mathbb{R}$ ;
- 2)  $\lambda(X_n) > 0$  and  $\lambda(Y_n) > 0$ ;
- 3)  $X_n \cup Y_n \subseteq V_n$ ,

and, moreover,

- 4)  $X_n \cap Y_m = \emptyset$  for all  $n, m \in \mathbb{N}$ .

Suppose that  $n \in \mathbb{N}$  and that the partial sequences

$$(X_i)_{i < n}, \quad (Y_i)_{i < n}$$

are constructed and satisfy the above conditions. Let us consider the set

$$Z = \bigcup_{i < n} (X_i \cup Y_i).$$

This set is closed and nowhere dense in  $\mathbb{R}$ . Hence, there exists a non-empty open interval

$$U_n \subseteq V_n \setminus Z_n.$$

Now we can easily construct two closed nowhere dense sets

$$X_n \subseteq U_n, \quad Y_n \subseteq U_n$$

such that

$$X_n \cap Y_n = \emptyset, \quad \lambda(X_n) > 0, \quad \lambda(Y_n) > 0.$$

Thus, the sequences  $(X_n)_{n \in \mathbb{N}}$  and  $(Y_n)_{n \in \mathbb{N}}$  are constructed. Now it is sufficient to put

$$X = \bigcup_n X_n.$$

It is clear that for every non-degenerated segment  $[a, b] \subseteq \mathbb{R}$  we have

$$\lambda([a, b] \cap X) > 0, \quad \lambda([a, b] \cap (\mathbb{R} \setminus X)) > 0.$$

A non-essential modification of the above construction gives us a proof of the following, more general theorem.

**Theorem 8.1** *There exists a countable partition  $(T_n)_{n \in \mathbb{N}}$  of the real line  $\mathbb{R}$  with the following properties:*

- 1) *for each  $n \in \mathbb{N}$  the set  $T_n$  is Lebesgue measurable;*
- 2) *for any  $n \in \mathbb{N}$  and for any non-empty open set  $V \subseteq \mathbb{R}$  we have*

$$\lambda(V \cap T_n) > 0.$$

*In particular, we also have*

$$\lambda(V \cap (\mathbb{R} \setminus T_n)) > 0.$$

There are some generalizations of the above theorem in terms of the Baire category (see exercises after this Chapter). But there is no analog of this theorem for subsets of  $\mathbb{R}$  with the Baire property. In other words, there is no subset of the real line  $\mathbb{R}$  which has the Baire property such that for every non-empty open set  $V \subseteq \mathbb{R}$  both sets

$$V \cap X, \quad V \cap (\mathbb{R} \setminus X)$$

are not first category subsets of  $\mathbb{R}$ . We leave an easy proof of this fact to the reader.

**Example 1.** Let  $X$  be an  $F_\sigma$ -subset of the real line  $\mathbb{R}$  such that for every non-empty open set  $V \subseteq \mathbb{R}$  we have

$$\lambda(X \cap V) > 0, \quad \lambda((\mathbb{R} \setminus X) \cap V) > 0.$$

The construction of such a set was shown above. We shall consider the characteristic function  $\phi_X$  of this set. This function has a lot of interesting properties. Before we formulate one of such properties we recall some basic facts about the Baire classification of the real functions defined on the real line. We say that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is of the zero Baire class  $B_0(\mathbb{R}, \mathbb{R})$  if  $f$  is a continuous function, i.e.

$$B_0(\mathbb{R}, \mathbb{R}) = C(\mathbb{R}, \mathbb{R}).$$

Suppose that for a given ordinal number  $\alpha < \omega_1$  the Baire classes

$$B_\xi(\mathbb{R}, \mathbb{R})$$

are defined for all  $\xi < \alpha$ . Then the Baire class of the order  $\alpha$  is defined as a class of all functions  $f$  from  $\mathbb{R}$  into  $\mathbb{R}$  such that

$$f(x) = \lim_n f_n(x) \quad (x \in \mathbb{R}),$$

where  $(f_n)_{n \in \mathbb{N}}$  is some sequence of functions satisfying the inclusion

$$\{f_0, f_1, \dots\} \subseteq \bigcup_{\xi < \alpha} B_\xi(\mathbb{R}, \mathbb{R}).$$

In this way we can define the class

$$B_\alpha(\mathbb{R}, \mathbb{R})$$

of functions for every ordinal  $\alpha < \omega_1$ . Next we put

$$B(\mathbb{R}, \mathbb{R}) = \bigcup_{\alpha < \omega_1} B_\alpha(\mathbb{R}, \mathbb{R}).$$

Using the method of transfinite recursion it is not difficult to show that the class  $B(\mathbb{R}, \mathbb{R})$  coincides with the class of all Borel functions from  $\mathbb{R}$  into  $\mathbb{R}$ . Hence, the class of all Borel mappings from  $\mathbb{R}$  into  $\mathbb{R}$  can be characterized as the least class of functions from  $\mathbb{R}$  into  $\mathbb{R}$  which contains all continuous functions from  $\mathbb{R}$  into  $\mathbb{R}$  and is closed under limits of sequences of functions (obviously, we consider the pointwise convergence here). Let  $\alpha$  be any ordinal less than  $\omega_1$ . A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function **strictly of the class  $\alpha$**  if

$$f \in B_\alpha(\mathbb{R}, \mathbb{R}) \setminus \bigcup_{\xi < \alpha} B_\xi(\mathbb{R}, \mathbb{R}).$$

A famous result, due to Lebesgue, states that for every  $\alpha < \omega_1$  there exists a function strictly of the class  $\alpha$ . Our function  $\phi_X$  is, which is easy to prove, from the second Baire class. Moreover, since for every non-degenerated segment  $[a, b]$  we have

$$\lambda([a, b] \cap X) > 0, \quad \lambda([a, b] \cap (\mathbb{R} \setminus X)) > 0,$$

it is not difficult to check that  $\phi_X$  is not equivalent (with respect to the  $\sigma$ -ideal of  $\lambda$ -measure zero sets) to any function from the first Baire class. Hence, we deduce from these observations that the function  $\phi_X$  is strictly of the second class. At the same time let us notice that the function  $\phi_X$  is equivalent (with respect to the  $\sigma$ -ideal of first category subsets of  $\mathbb{R}$ ) to a function identically equal to zero. It is worth reminding that every Lebesgue measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is equivalent (with respect to the  $\sigma$ -ideal of  $\lambda$ -measure zero sets) to some function  $f^* : \mathbb{R} \rightarrow \mathbb{R}$  from the second Baire class. Let us also emphasize that for the classical mathematical analysis an important role is played by various subclasses of the class  $B_1(\mathbb{R}, \mathbb{R})$ . Such subclasses are, for example, the class of all semicontinuous functions (upper and lower), the class of all derivatives of continuous functions, the class of functions with the period  $2\pi$  which can be represented as a sum of a pointwise convergent (on the whole real line) trigonometric series and so on.

From now on we shall consider such subsets of the real line  $\mathbb{R}$  the existence of which is implied by more serious set-theoretical tools. One of the most famous subsets of  $\mathbb{R}$  is the so called Vitali set, constructed by Vitali in 1905.

Let us consider in  $\mathbb{R}$  the following equivalence relation:

$$x \in \mathbb{R}, \quad y \in \mathbb{R}, \quad x - y \in \mathbb{Q},$$

where  $\mathbb{Q}$  denotes, as usual, the subgroup of all rationals of the real line  $\mathbb{R}$ . This equivalence relation canonically determines a partition of  $\mathbb{R}$  into continuum many equivalence classes. Let  $X$  be any selector of this partition, i.e.  $X$  is such a subset of  $\mathbb{R}$  which intersects each equivalence class exactly at one point. The existence of such a set is implied by the Axiom of Choice (notice that we need here an uncountable version of this axiom). Any set of this form is called a **Vitali set**. This set satisfies the following two relations:

$$1) \quad \bigcup_{q \in \mathbb{Q}} (X + q) = \mathbb{R},$$

$$2) \quad (\forall q \in \mathbb{Q})(q_1 \neq 0 \rightarrow (X + q_1) \cap X = \emptyset).$$

From these relations it is easy to deduce the next classical result due to Vitali, which gives us the first example of Lebesgue non-measurable set and also the first example of a set without the Baire property.

**Theorem 8.2 (Vitali)** *Any Vitali set is Lebesgue non-measurable and has no Baire property.*

**Proof.** Let  $X$  be a Vitali set and suppose that  $X$  is a Lebesgue measurable set (respectively, has the Baire property). Then relation 1) shows us that a countable union of translations of the set  $X$  covers the whole real line  $\mathbb{R}$ . Hence,  $X$  cannot have the Lebesgue measure zero (respectively, cannot be a first category set). Hence,  $\lambda(X) > 0$  (respectively,  $X \in \bar{B}(\mathbb{R}) \setminus \mathbb{K}$ ). So, by the Steinhaus property for measure or by an analogous property for category (see Chapter 7), there exists a real number  $\epsilon > 0$  such that

$$] - \epsilon, \epsilon[ \subseteq X - X,$$

where

$$X - X = \{x - y : x, y \in X\}.$$

Now if  $q \in \mathbb{Q} \setminus \{0\}$  is such that  $|q| < \epsilon$ , then

$$(X + q) \cap X \neq \emptyset,$$

which contradicts relation 2). Thus, the Vitali theorem is proved.

**Example 2.** As we can see the proof of the last theorem was based on a very special characteristic of the Lebesgue measure and the Baire property on the real line  $\mathbb{R}$ . The reader should know that for some Vitali sets  $X$  it is possible to construct a non-zero  $\sigma$ -finite complete and invariant (under all translations of  $\mathbb{R}$ ) measure  $\nu$  such that  $X \in \text{dom}(\nu)$ . Moreover, if we consider the von Neumann topology on the real line  $\mathbb{R}$  (see Appendix B) associated with the measure  $\nu$ , then the set  $X$  has the Baire property for this topology. So, we see that Vitali sets are pathological objects only with respect to the Lebesgue measure and with respect to the Baire property for the Euclidean topology. These sets may have "regular" and "nice" properties with respect to other measures and topologies on the real line  $\mathbb{R}$ . The same can be said about other pathological objects with which we shall deal in our further considerations.

It is a good moment to notice here that the Axiom of Choice in its uncountable form is necessary for the construction of Lebesgue non-measurable sets or sets without the Baire property. This follows from a famous result of Solovay stating that it is possible to construct some models of theory **(ZF) & (DC)** in which all subsets of the real line  $\mathbb{R}$  are Lebesgue measurable and have the Baire property with respect to the Euclidean topology on  $\mathbb{R}$ .

We shall consider now the second important class of subsets of the real line  $\mathbb{R}$ . Namely, we shall discuss some properties of Hamel bases of  $\mathbb{R}$ . The attention to these sets was drawn by Hamel also in 1905. The construction of these sets is not difficult, either.

We shall use a general theorem of the theory of vector spaces over arbitrary fields which states that in every vector space there exists a base, i.e. a maximal linearly independent set. This theorem follows almost immediately from the Zorn Lemma (i.e. from the Axiom of Choice). We shall consider the real line  $\mathbb{R}$  as a vector space over the field  $\mathbb{Q}$  of all rational numbers. Hence, applying the mentioned theorem we get that there are bases of  $\mathbb{R}$  over  $\mathbb{Q}$ . Any such base is called a **Hamel base**.

Let  $(e_i)_{i \in I}$  be an arbitrary Hamel base. It is clear that  $\text{card}(I) = 2^\omega$ . As we know every element  $x$  of  $\mathbb{R}$  may be uniquely represented in the form

$$x = \sum_{i \in I} q_i(x) \cdot e_i,$$

where  $(q_i(x))_{i \in I}$  is some indexed family of rational numbers such that

$$\text{card}(\{i \in I : q_i(x) \neq 0\}) < \omega.$$

For each index  $i \in I$  the rational number  $q_i(x)$  is called the coordinate of  $x$  corresponding to this index.

Hamel bases were found not accidentally but as a tool for a solution of the concrete question of the mathematical analysis. We shall formulate this question now. Let us consider the class of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  which satisfy the following functional equation:

$$f(x + y) = f(x) + f(y), \quad (x, y \in \mathbb{R}).$$

This equation is called the Cauchy functional equation. Notice that this equation simply says that  $f$  is a homomorphism from the additive group  $\mathbb{R}$  into itself. The task is to find all solutions of the Cauchy functional equation. It is clear that there are natural solutions of this equation. Namely, every function of the form

$$f(x) = a \cdot x,$$

where  $a$  is a fixed real number is a solution of this equation. Such solutions we shall call trivial solutions. Hamel bases allow us to construct non-trivial solutions of the Cauchy functional equation.

**Theorem 8.3 (Hamel)** *There are non-trivial solutions of the Cauchy functional equation.*

**Proof.** Let  $(e_i)_{i \in I}$  be an arbitrary Hamel base in  $\mathbb{R}$ . As we noticed above any  $x \in \mathbb{R}$  can be uniquely represented in the form

$$x = \sum_{i \in I} q_i(x) \cdot e_i.$$

Let us fix  $i_0 \in I$  and define a function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  by the following formula:

$$\phi(x) = q_{i_0}(x) \quad (x \in \mathbb{R}).$$

It is clear that  $\phi$  satisfies the Cauchy functional equation. Moreover, the range of this function is contained in  $\mathbb{Q}$ . This function is not constant:

$$\phi(0) = 0, \quad \phi(e_{i_0}) = 1.$$

So  $\phi$  is not a continuous function. We shall see below that  $\phi$  even is not Lebesgue measurable and does not have the Baire property.

Suppose now that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies the Cauchy functional equation. It is clear that for every  $x \in \mathbb{R}$  and for every  $q \in \mathbb{Q}$  we have

$$f(q \cdot x) = q \cdot f(x).$$

From this fact we immediately deduce that if  $f$  is a continuous function at least at one point, then

$$f(x) = f(1) \cdot x$$

for any  $x \in \mathbb{R}$ . This fact was first proved by Cauchy. It is also easy to prove that if  $f$  is a solution of the Cauchy functional equation and is upward bounded on some non-empty open interval then  $f$  is a trivial solution. Hence, we see that non-trivial solutions of the Cauchy functional equation are discontinuous at each point. Neither is it difficult to see that the graph of an arbitrary non-trivial solution of the Cauchy functional equation is a dense subset of the plane.  $\mathbb{R}^2$ .

**Theorem 8.4 (Frechet)** *Any non-trivial solution of the Cauchy functional equation is not Lebesgue measurable and does not have the Baire property.*

**Proof.** There are lot of proofs of this theorem. We shall show a very simple one, also based on the Steinhaus property (or on its analogue for category). Suppose that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies the Cauchy functional equation and that  $f$  is Lebesgue measurable (respectively, has the Baire property). Let us consider the sets of the form

$$f^{-1}([-n, n]) \quad (n \in \mathbb{N}).$$

All these sets are Lebesgue measurable (respectively, have the Baire property) and

$$\bigcup_{n \in \mathbb{N}} f^{-1}([-n, n]) = \mathbb{R}.$$

Hence, there exists  $n \in \mathbb{N}$  such that

$$\lambda(f^{-1}([-n, n])) > 0,$$

respectively,

$$f^{-1}([-n, n]) \in \bar{B}(\mathbb{R}) \setminus \mathbb{K}.$$

Then the Steinhaus property (or its analogue for category) implies that the set

$$f^{-1}([-n, n]) - f^{-1}([-n, n])$$

is a neighbourhood of the zero element of  $\mathbb{R}$ . Let  $V$  be any non-degenerated segment in  $\mathbb{R}$  such that

$$V \subseteq f^{-1}([-n, n]) - f^{-1}([-n, n])$$

and  $0 \in V$ . Then

$$f(V) \subseteq [-2n, 2n],$$

so the function  $f$  is bounded on  $V$ . From this fact it immediately follows that  $f$  is continuous at the point 0. Hence,  $f$  is a trivial solution of the Cauchy functional equation.

Observing the previous two theorems we can see that the existence of a Hamel base in  $\mathbb{R}$  implies an existence of a Lebesgue non-measurable subset of  $\mathbb{R}$  and an existence of a subset of  $\mathbb{R}$  without the Baire property. More exactly, the proof of this fact can be done in theory **(ZF) & (DC)**.

**Example 3.** One can prove that there are Lebesgue measurable and Lebesgue non-measurable Hamel bases (see exercises after this Chapter). This fact shows us an essential difference between Hamel bases and Vitali sets from the point of view of the Lebesgue measurability. The same is true for the Baire property. The following question arises in a natural way: if  $\mu$  is a non-zero  $\sigma$ -finite measure on the real line  $\mathbb{R}$  invariant under all translations of  $\mathbb{R}$ , then does there exist a Hamel base in  $\mathbb{R}$  non-measurable with respect to  $\mu$ ? It turns out that this question is undecidable in theory **ZFC** (see exercises to this Chapter).

Hamel bases have a lot of interesting and important applications in many branches of mathematics. One of the most beautiful applications may be found in the theory of polyhedra. More precisely, in the part of this theory which is connected with the third Hilbert problem about the equivalence (by finite decomposition) of a three-dimensional cube and a regular three-dimensional simplex with the same volume (the third Hilbert problem is shortly discussed in one of exercises after this Chapter). Let us recall to the reader that Hamel was one of Hilbert's students and worked in geometry, too.

We shall consider now the so called Bernstein subsets of the real line  $\mathbb{R}$ . These sets were constructed by the method of transfinite recursion by Bernstein in 1908.



The existence of these sets, similarly as the existence of Vitali sets or Hamel bases, cannot be proved in theory **(ZF)** & **(DC)**. We give now a precise definition of Bernstein sets.

Let  $X$  be a subset of the real line  $\mathbb{R}$ . We say that  $X$  is a **Bernstein set** in  $\mathbb{R}$  if for every non-empty perfect set  $P \subseteq \mathbb{R}$  both sets

$$P \cap X, \quad P \cap (\mathbb{R} \setminus X)$$

are non-empty. In other words, a set  $X$  is a Bernstein subset of  $\mathbb{R}$  if no non-empty perfect subset of  $\mathbb{R}$  is contained in  $X$  or in  $\mathbb{R} \setminus X$ .

**Theorem 8.5 (Bernstein)** *There are Bernstein subsets of the real line  $\mathbb{R}$ .*

**Proof.** Let  $(P_i)_{i \in I}$  be the family of all non-empty perfect subsets of the real line  $\mathbb{R}$ , where  $I$  is some set of the cardinality continuum. So we have

$$\text{card}(I) = \mathbf{c},$$

$$(\forall i \in I)(\text{card}(P_i) = \mathbf{c}),$$

where  $\mathbf{c}$  denotes, as usual, the cardinality continuum. Thus, we may apply here Theorem 3 from Chapter 1. According to this theorem, there exists a family  $(Y_i)_{i \in I}$  of pairwise disjoint sets such that

$$Y_i \subseteq P_i \quad \& \quad \text{card}(Y_i) = \text{card}(P_i) = \mathbf{c}$$

for each  $i \in I$ . Let  $X$  be any selector of the family  $(Y_i)_{i \in I}$ . It is easy to see that  $X$  is a Bernstein subset of  $\mathbb{R}$ .

Let us note that through a slight change of the above arguments we obtain a partition  $(X_\alpha)_{\alpha < \mathbf{c}}$  of the real line  $\mathbb{R}$  into Bernstein sets.

It is almost obvious that no Bernstein set is Lebesgue measurable or has the Baire property. Indeed, it is clear, for instance, that

$$\lambda_*(X) = 0, \quad \lambda_*(\mathbb{R} \setminus X) = 0$$

for every Bernstein set  $X \subseteq \mathbb{R}$  (recall that  $\lambda_*$  denotes the inner Lebesgue measure on  $\mathbb{R}$ ). Moreover, from the point of view of topological measure theory the Bernstein sets are extremely bad. This opinion is legitimized by the next theorem.

**Theorem 8.6** *Let  $\mu$  be an arbitrary non-zero  $\sigma$ -finite diffused Borel measure on  $\mathbb{R}$  and let  $\bar{\mu}$  be its completion. If  $X$  is a Bernstein subset of  $\mathbb{R}$ , then we have  $X \notin \text{dom}(\bar{\mu})$ .*

**Proof.** Suppose that  $X$  is a Bernstein set and that  $X \in \text{dom}(\bar{\mu})$ . Then it is clear that we also have  $\mathbb{R} \setminus X \in \text{dom}(\bar{\mu})$ . Since the measure  $\mu$  is non-zero, the following disjunction holds:

$$\bar{\mu}(X) > 0 \quad \vee \quad \bar{\mu}(\mathbb{R} \setminus X) > 0.$$

Without loss of generality we may assume that  $\bar{\mu}(X) > 0$ . As we know every  $\sigma$ -finite Borel measure defined on a Polish topological space is a Radon measure. Hence, the given measure  $\mu$  is a Radon measure, too. Thus, there exists a compact set  $K \subseteq \mathbb{R}$  such that

$$K \subseteq X, \quad \mu(K) \geq \frac{1}{2}\mu(X) > 0.$$

But the measure  $\mu$  is diffused and  $\mu(K) > 0$ , so the compact set  $K$  is uncountable. Therefore, there exists a non-empty perfect (in  $\mathbb{R}$ ) subset of  $K$ . Thus, we get that  $X$  is not a Bernstein set.

Other properties of the Bernstein sets are considered in exercises after this Chapter.

We change the subject of discussion to the so called Luzin subsets of the real line  $\mathbb{R}$ . These sets were constructed by Luzin in 1914 under the assumption of the Continuum Hypothesis. The same sets were constructed by Mahlo one year before Luzin. In the mathematical literature the name *Luzin sets* is used, probably, because Luzin investigated these sets deeply and showed a number of their important properties. Now we define these sets precisely.

Let  $X$  be a subset of the real line  $\mathbb{R}$ . We say that  $X$  is a **Luzin set** if

- 1)  $X$  is uncountable;
- 2) for every first category subset  $Y$  of  $\mathbb{R}$  the intersection  $X \cap Y$  is at most countable.

It is obvious that the family of all Luzin subsets of the real line is a  $\sigma$ -ideal invariant under all transformations of the real line which preserve the  $\sigma$ -ideal  $\mathbb{K}$  (in particular, this family is invariant under all translations of  $\mathbb{R}$ ).

We remark at once that it is impossible to prove in theory **ZFC** the existence of a Luzin set. Namely, assume that

$$(\text{MA}) \ \& \ (2^\omega > \omega_1)$$

holds and take an arbitrary set  $X \subseteq \mathbb{R}$ . If  $X$  is countable, then it is not a Luzin set. Suppose now that  $X$  is uncountable. We know that Martin's Axiom implies the equality  $\text{add}(\mathbb{K}) = 2^\omega$ . Let  $Y$  be any subset of  $X$  of cardinality  $\omega_1$ . Then we have  $Y \in \mathbb{K}$  and  $\text{card}(X \cap Y) > \omega$ , so we obtain again that  $X$  is not a Luzin set.

However, if we assume the Continuum Hypothesis, then Luzin sets exist on the real line  $\mathbb{R}$ .

**Theorem 8.7 (Luzin-Mahlo)** *If the Continuum Hypothesis holds, then there are Luzin subsets of the real line  $\mathbb{R}$ .*

**Proof.** Let us recall that the Continuum Hypothesis means that  $\mathfrak{c} = \omega_1$ . Let  $(X_\xi)_{\xi < \omega_1}$  be a family of all first category Borel subsets of  $\mathbb{R}$ . We define by the transfinite recursion a family  $(x_\xi)_{\xi < \omega_1}$  of points from  $\mathbb{R}$ . Suppose that  $\xi < \omega_1$  and that a partial family  $(x_\zeta)_{\zeta < \xi}$  has already been defined. Let us consider the set

$$Z_\xi = \left( \bigcup_{\zeta < \xi} X_\zeta \right) \cup \{x_\zeta : \zeta < \xi\}.$$

It is clear that  $Z_\xi$  is a first category subset of the real line. Hence, by the Baire theorem, there exists a point

$$x_\xi \in \mathbb{R} \setminus Z_\xi.$$

This ends the construction of the family  $(x_\xi)_{\xi < \omega_1}$ . Now we put

$$X = \{x_\xi : \xi < \omega_1\}.$$

Notice that if  $\zeta < \xi < \omega_1$ , then  $x_\zeta \neq x_\xi$ . Therefore,  $\text{card}(X) = \omega_1$ . Suppose now that  $Z$  is an arbitrary first category subset of the real line. Then there exists an ordinal  $\xi < \omega_1$  such that  $Z \subseteq X_\xi$ . Obviously, we have

$$X \cap Z \subseteq X \cap X_\xi \subseteq \{x_\zeta : \zeta \leq \xi\},$$

and thus we see that  $X$  is a Luzin set.

Notice that if  $X$  is a Luzin set on  $\mathbb{R}$ , then the set  $X \cup \mathbb{Q}$  is also a Luzin set. Hence, the Continuum Hypothesis implies that there are Luzin sets dense in  $\mathbb{R}$ . We can easily get a much stronger result. Namely, a slight modification of the above proof gives us such a Luzin set  $X$  that for any set  $Y \subseteq \mathbb{R}$  with the Baire property we have

$$\text{card}(Y \cap B) \leq \omega \iff Y \in \mathbb{K}.$$

Luzin sets have a lot of interesting properties. The next theorem was proved by Luzin.

**Theorem 8.8** *Suppose that  $X$  is a Luzin set. Then  $X$  does not have the Baire property in  $\mathbb{R}$  (and any uncountable subset of  $X$  does not have the Baire property). Moreover, in the topological space  $X \cup \mathbb{Q}$  with the topology induced by the topology of  $\mathbb{R}$  every first category set is at most countable and, conversely, every at most countable subset of the space  $X \cup \mathbb{Q}$  is a first category set in  $X \cup \mathbb{Q}$ .*

**Proof.** Let  $X$  be a Luzin set and suppose that  $X$  has the Baire property. Since  $X$  is uncountable and for every first category set  $Y \subseteq \mathbb{R}$  we have  $\text{card}(X \cap Y) \leq \omega$ , we see that  $X$  is not a first category subset of  $\mathbb{R}$ . But then  $X$  contains some uncountable  $G_\delta$ -subset  $Z$ . Let  $Y$  be any subset of  $Z$  homeomorphic with the Cantor discontinuum. Then  $Y$  is a nowhere dense set in  $\mathbb{R}$  and  $Y \subseteq X$ . We also have  $\text{card}(X \cap Y) \geq \omega_1$ , which contradicts the definition of a Luzin set. So  $X$  does not have the Baire property.

The second part of this theorem follows from the fact that  $X \cup \mathbb{Q}$  is a Luzin set, too, and  $X \cup \mathbb{Q}$  is dense in  $\mathbb{R}$ .

The theorem proved above shows us that if the Continuum Hypothesis holds, then there exists an uncountable everywhere dense topological space  $X \subseteq \mathbb{R}$  such that

$$K(X) = [X]^{\leq \omega},$$

where  $K(X)$  denotes the  $\sigma$ -ideal of first category subsets of the space  $X$ . This equality implies also that

$$\bar{B}(X) = B(X),$$

where  $\bar{B}(X)$  denotes the class of all subsets of  $X$  with the Baire property and  $B(X)$  denotes the Borel  $\sigma$ -algebra of the space  $X$ .

A Hausdorff topological space  $E$  without isolated points is called a **Luzin space** if the equality

$$K(E) = [E]^{\leq \omega}$$

holds. Hence, we see that there exists a Luzin set on the real line  $\mathbb{R}$  which is an example of a topological Luzin space.

The next theorem, also essentially due to Luzin, shows us an interesting connection between Luzin sets on  $\mathbb{R}$  and the topological measure theory.

**Theorem 8.9** *Let  $X$  be an arbitrary Luzin subset of the real line  $\mathbb{R}$ . Then the following two sentences hold:*

- 1) *if  $\mu$  is a  $\sigma$ -finite diffused Borel measure on  $\mathbb{R}$ , then  $\mu^*(X) = 0$ , where  $\mu^*$  is the outer measure associated with  $\mu$ ;*
- 2) *if  $\mu$  is a  $\sigma$ -finite diffused Borel measure on the topological space  $X$ , then  $\mu$  is identically equal to zero.*

**Proof.** It is almost obvious that sentences 1) and 2) are equivalent. Hence, we shall prove only the second one. Without loss of generality we may assume that  $X$  is a dense subset of  $\mathbb{R}$ . Let  $\mu$  be any  $\sigma$ -finite diffused Borel measure defined on the topological space  $X$ . Since  $X$  is a separable metric space, we may apply the theorem about supports of Borel measures (see Chapter 4) and find a first category subset of  $X$  on which  $\mu$  is concentrated. But any first category subset of  $X$  is countable and  $\mu$  is diffused, so  $\mu$  is identically equal to zero.

This theorem shows us that from the topological point of view Luzin sets are extremely pathological (any uncountable subset of a Luzin set does not have the Baire property) but from the point of view of topological measure theory Luzin sets are very small, since they have a measure zero with respect to any  $\sigma$ -finite diffused Borel measure on  $\mathbb{R}$ .

A dual (in a certain sense) object to Luzin set is the so called Sierpiński set, which was constructed by Sierpiński also under the assumption of the Continuum Hypothesis in 1924. We give a definition of this set now.

Let  $X$  be a subset of  $\mathbb{R}$ . We say that  $X$  is a **Sierpiński set** if

- 1)  $X$  is uncountable;
- 2) for every Lebesgue measure zero subset  $Y$  of  $\mathbb{R}$  the intersection  $X \cap Y$  is at most countable.

A lot of facts about Sierpiński sets are similar to the facts about Luzin sets. For example:

- a) the family of all Sierpiński sets is a  $\sigma$ -ideal invariant under all transformations of the real line which preserve the  $\sigma$ -ideal  $\mathbb{L}$  (in particular, this family is invariant under all translations of  $\mathbb{R}$ );
- b) the assumption (MA) & ( $2^\omega > \omega_1$ ) implies that there are no Sierpiński sets.

Analogously, the following theorem holds:

**Theorem 8.10 (Sierpiński)** *Assume the Continuum Hypothesis. Then there are Sierpiński subsets of the real line  $\mathbb{R}$ .*

The proof of this theorem is very similar to the proof of Theorem 7. The only one change that must be done is the replacement of the family  $(X_\xi)_{\xi < \omega_1}$  of all first category Borel subsets of  $\mathbb{R}$  by the family  $(Y_\xi)_{\xi < \omega_1}$  of all  $\lambda$ -measure zero Borel subsets of  $\mathbb{R}$  (notice also that one can deduce Theorem 10 from Theorem 7 and, conversely, using the Erdős-Sierpiński duality principle).

Now we want to show the next similarity between Luzin and Sierpiński sets.

**Theorem 8.11** *Every Sierpiński set is a first category subset of the real line  $\mathbb{R}$ . No uncountable subset of a Sierpiński set is Lebesgue measurable.*

**Proof.** Let  $X$  be a Sierpiński set. As we know the  $\sigma$ -ideals  $\mathbb{K}$  and  $\mathbb{L}$  are orthogonal, i.e. there exists a partition  $\{K, L\}$  of  $\mathbb{R}$  such that  $K \in \mathbb{K}$  and  $L \in \mathbb{L}$ . But we have

$$\text{card}(X \cap L) \leq \omega$$

and the inclusion

$$X \subseteq K \cup (X \cap L),$$

holds, so we get  $X \in \mathbb{K}$ .

Suppose now that  $Y$  is an uncountable subset of a Sierpiński set  $X$ . Since  $X \cap Y$  is uncountable, we see that  $Y \notin \mathbb{L}$ . Suppose, however, that  $Y$  is Lebesgue

measurable. Then  $\lambda(Y) > 0$  and we can find an uncountable set  $Z \subseteq Y$  of Lebesgue measure zero. But then the set  $X \cap Z$  is uncountable, so we get a contradiction with the definition of the set  $X$ .

Other interesting properties of Luzin sets and Sierpiński sets are presented in exercises to this Chapter.

It is easy to see that if we replace the Continuum Hypothesis by Martin's Axiom (which is a much weaker assertion than CH), then we can prove the existence of some analogs of Luzin and Sierpiński sets. Namely, if Martin's Axiom holds, then there exists a set  $A \subseteq \mathbb{R}$  such that  $\text{card}(A) = \mathfrak{c}$  and for each set  $X \in \mathbb{K}$  we have

$$\text{card}(A \cap X) < \mathfrak{c}.$$

A set  $A$  with the above property is called a **generalized Luzin subset** of  $\mathbb{R}$ . Similarly, if Martin's Axiom holds, then there exists a set  $B \subseteq \mathbb{R}$  such that  $\text{card}(B) = \mathfrak{c}$  and for each set  $Y \in \mathbb{L}$  we have

$$\text{card}(B \cap Y) < \mathfrak{c}.$$

A set  $B$  with the above property is called a **generalized Sierpiński subset** of  $\mathbb{R}$ . Let us remark that for the existence of generalized Luzin sets or generalized Sierpiński sets we do not need the full power of Martin's Axiom. In fact, the existence of a generalized Luzin set is implied by the equalities

$$\text{cov}(\mathbb{K}) = \text{cof}(\mathbb{K}) = \mathfrak{c}.$$

In an analogous way the equalities

$$\text{cov}(\mathbb{L}) = \text{cof}(\mathbb{L}) = \mathfrak{c}$$

imply the existence of a generalized Sierpiński subset of the real line  $\mathbb{R}$ .

In our further considerations we shall meet other generalizations of Luzin sets and Sierpiński sets. But now we shall use once more Martin's Axiom and construct a generalized Sierpiński set with the Baire property in the restricted sense.

**Theorem 8.12** *Suppose that Martin's Axiom holds. Then there exists a set  $X \subseteq \mathbb{R}$  such that:*

- 1) *for every non-empty perfect set  $P \subseteq \mathbb{R}$  the intersection  $X \cap P$  is a first category set in  $P$ ;*
- 2) *for every Lebesgue measurable set  $Y \subseteq \mathbb{R}$  with  $\lambda(Y) > 0$  the intersection  $X \cap Y$  is non-empty.*

*In particular, the set  $X$  has the Baire property in the restricted sense and is not Lebesgue measurable.*

**Proof.** Let, as usual,  $\mathfrak{c}$  denote the cardinality continuum. Let  $(Z_\xi)_{\xi < \mathfrak{c}}$  denote the family of all Borel subsets of the real line of a strictly positive Lebesgue measure, i.e.

$$\{Z_\xi : \xi < \mathfrak{c}\} = B(\mathbb{R}) \setminus \mathbb{L},$$

and let  $(T_\xi)_{\xi < \mathfrak{c}}$  denote the family of all Borel subsets of the real line of Lebesgue measure zero, i.e.

$$\{T_\xi : \xi < \mathfrak{c}\} = B(\mathbb{R}) \cap \mathbb{L}.$$

For each ordinal  $\xi < \mathfrak{c}$  we fix a partition  $\{Z_\xi^0, Z_\xi^1\}$  of  $Z_\xi$  such that  $Z_\xi^0$  is a Lebesgue measure zero set and  $Z_\xi^1$  is a first category subset of  $Z_\xi$ . Now we define an injective

$\mathfrak{c}$ -sequence  $(x_\xi)_{\xi < \mathfrak{c}}$  of real numbers. Suppose that  $\xi < \mathfrak{c}$  and that a partial sequence  $(x_\zeta)_{\zeta < \xi}$  has already been defined. Let us consider the set

$$D_\xi = \left( \bigcup_{\zeta \leq \xi} Z_\zeta^0 \right) \cup \{x_\zeta : \zeta < \xi\} \cup \left( \bigcup_{\zeta \leq \xi} T_\zeta \right).$$

Martin's Axiom implies that the set  $D_\xi$  is also of Lebesgue measure zero. Hence, we have

$$Z_\xi \setminus D_\xi \neq \emptyset.$$

So we can choose a point  $x_\xi$  as any element from the last non-empty difference of sets. In this way we shall define a whole  $\mathfrak{c}$ -sequence  $(x_\xi)_{\xi < \mathfrak{c}}$ . Now we put

$$X = \{x_\xi : \xi < \mathfrak{c}\}$$

and we will show that the set  $X$  is a required one. Let  $P$  be any non-empty perfect subset of  $\mathbb{R}$ . If its Lebesgue measure is equal to zero, then for some  $\xi < \mathfrak{c}$  we have  $P = T_\xi$ . Hence, from the method of construction of the set  $X$  we immediately have

$$\text{card}(P \cap X) < \mathfrak{c}.$$

Using Martin's Axiom again we see that the intersection  $P \cap X$  is a first category set in  $P$ . Now suppose that  $\lambda(P) > 0$ . Then for some ordinal  $\xi < \mathfrak{c}$  we have  $P = Z_\xi$ . Therefore,

$$P \cap X \subseteq Z_\xi^1 \cup \{x_\zeta : \zeta < \xi\}.$$

Taking into account the fact that the set  $P$  does not have isolated points, we obtain from the last inclusion that  $P \cap X$  is a first category set in  $P$ . Thus, condition 1) is satisfied for our set  $X$ . Since we have

$$x_\xi \in Z_\xi$$

for each ordinal  $\xi < \mathfrak{c}$ , we get that condition 2) holds for the set  $X$ , too.

It is easy to see that the set  $X$  constructed in the proof of the above theorem is a generalized Sierpiński set. Hence, we can deduce that Martin's Axiom implies the existence of a massive (with respect to the Lebesgue measure) generalized Sierpiński set which has the Baire property in the restricted sense.

Now we will discuss another method of a construction of Lebesgue non-measurable sets (or sets without the Baire property). This method is essentially different from the previous constructions and is based on the notion of an ultrafilter in the set  $\omega$  of all natural numbers. This construction was done by Sierpiński in 1938.

Let us recall that if  $E$  is an arbitrary basic set and  $\Phi$  is a proper filter of subsets of  $E$ , then there exists an ultrafilter of subsets of  $E$  which extends  $\Phi$ . If the set  $E$  is infinite, then for the proof of this fact we need strong versions of the Axiom of Choice. We shall consider here only the case  $E = \omega$  where  $\Phi$  is the dual filter to the ideal

$$I = [\omega]^{<\omega}.$$

Any ultrafilter which extends this filter is non-principal. Notice now that any family  $S$  of subsets of the power set  $P(\omega)$  can be considered as some subset of the Cantor discontinuum  $\{0, 1\}^\omega$ . Namely, it is enough to take the set

$$\{1_X : X \in S\},$$

where  $1_X$  denotes the characteristic function of the set  $X$ . Let us also consider the probability Haar measure  $\mu$  on the compact topological group  $\{0, 1\}^\omega$  and let  $\bar{\mu}$  denote the standard completion of the measure  $\mu$ . The following theorem shows us that every non-principal ultrafilter on  $\omega$  is a non-measurable subset of the space  $\{0, 1\}^\omega$ .

**Theorem 8.13 (Sierpiński)** *Every non-principal ultrafilter on  $\omega$  is non-measurable with respect to the measure  $\bar{\mu}$ .*

**Proof.** Let  $U$  be a non-principal ultrafilter in  $\omega$  and let

$$Z = \{1_X : X \in U\}.$$

It is clear that  $Z \subseteq \{0, 1\}^\omega$ . Suppose that the set  $Z$  is measurable with respect to the measure  $\bar{\mu}$ . Let  $\Gamma$  denote the family of all elements  $x = (x_n)_{n \in \omega}$  from the space  $\{0, 1\}^\omega$  which are eventually equal to zero, i.e.

$$\text{card}(\{n \in \omega : x_n = 1\}) < \omega.$$

It is obvious that  $\Gamma$  is a dense subgroup of the Cantor discontinuum  $\{0, 1\}^\omega$ . The ultrafilter  $U$  is non-principal, so for every element  $x \in \Gamma$  we have

$$Z + x = Z,$$

where  $+$  denotes the group operation in  $\{0, 1\}^\omega$ . Hence, we see that the set  $Z$  is invariant under the subgroup  $\Gamma$ . Then the metric transitivity of the Haar measure  $\mu$  implies that

$$\bar{\mu}(Z) = 0 \vee \bar{\mu}(Z) = 1.$$

Let us put

$$e = (1, 1, \dots, 1, \dots).$$

Since  $U$  is an ultrafilter, we get

$$Z + e = \{0, 1\}^\omega \setminus Z.$$

Therefore, we have

$$(Z + e) \cap Z = \emptyset, \quad (Z + e) \cup Z = \{0, 1\}^\omega.$$

Since the measure  $\mu$  is invariant under translations of  $\{0, 1\}^\omega$ , the same is true for the measure  $\bar{\mu}$ . So we see that none of the equalities  $\bar{\mu}(Z) = 0$  and  $\bar{\mu}(Z) = 1$  can hold. Indeed, the first equality gives us  $0 + 0 = 1$  and the second gives us  $1 + 1 = 1$ . Hence, the set  $Z$  is not  $\bar{\mu}$ -measurable.

As we know (see Chapter 4) the measure  $\bar{\mu}$  is isomorphic with the restriction of the Lebesgue measure  $\lambda$  to the unit segment  $[0, 1]$ , i.e. there exists a Borel bijection

$$g : \{0, 1\}^\omega \rightarrow [0, 1]$$

such that for every set  $X \subseteq \{0, 1\}^\omega$  we have

$$X \in \text{dom}(\bar{\mu}) \iff g(X) \in \text{dom}(\lambda)$$

and if  $X \in \text{dom}(\bar{\mu})$ , then

$$\bar{\mu}(X) = \lambda(g(X)).$$

Hence, in theory **(ZF)** & **(DC)** the existence of a Lebesgue non-measurable subset of the real line  $\mathbb{R}$  is equivalent with the existence of a  $\bar{\mu}$ -non-measurable subset of the Cantor discontinuum.

Summarizing all these remarks we see that in theory **(ZF)** & **(DC)** the following implication holds:

there exists a non-principal ultrafilter on  $\omega \rightarrow$

there exists a Lebesgue non-measurable subset of the real line  $\mathbb{R}$ .

Actually, a similar result can be established for the Baire property. Namely, any non-principal ultrafilter on  $\omega$  considered as a subset of the Cantor discontinuum does not have the Baire property. Moreover, in theory **(ZF)** & **(DC)** the existence of a non-principal ultrafilter on  $\omega$  implies the existence of a subset of the real line without the Baire property.

At this place we finish our short review of some classical subsets of the real line  $\mathbb{R}$ . Other interesting subsets of  $\mathbb{R}$  and interesting families of subsets of  $\mathbb{R}$  will be considered in Part 2 of this book.

## Exercises

**Exercise 8.1** Let  $E$  be an infinite compact topological space with the weight  $w(E) = \kappa$ . Show that  $E$  is a continuous image of some closed subset of the generalized Cantor discontinuum  $0, 1^\kappa$ .

**Exercise 8.2** Give an example of a  $\sigma$ -ideal  $I$  of subsets of  $\mathbb{R}$  such that

- 1)  $I$  is invariant under the group of all isometric transformations of  $\mathbb{R}$ ;
- 2) there exists a set  $X \in I$  such that  $\text{card}(X) = \mathfrak{c}$ ;
- 3)  $(\forall Y, Z \in I)(Y + Z \in I)$ .

**Exercise 8.3** Prove Theorem 1 from this Chapter.

**Exercise 8.4** Let us consider the complete separable metric space

$$(\text{dom}(\lambda), \rho_\lambda)$$

which is canonically associated with the family of all Lebesgue measurable subsets of the unit segment  $[0, 1]$ . Let us recall that the metric  $\rho_\lambda$  is defined by the formula:

$$\rho_\lambda(X, Y) = \lambda(X \Delta Y) \quad (X, Y \in \text{dom}(\lambda)),$$

where two sets  $X, Y$  are identified if  $\rho_\lambda(X, Y) = 0$ . Let  $M$  be the family of all sets  $Z \in \text{dom}(\lambda)$  such that for every non-empty open interval  $V \subseteq [0, 1]$  the inequalities

$$\lambda(Z \cap V) > 0, \quad \lambda([0, 1] \setminus Z \cap V) > 0$$

hold. Show that  $M$  is a residual subset of the space  $(\text{dom}(\lambda), \rho_\lambda)$ , i.e.  $M$  is the complement of a first category set in this space.

**Exercise 8.5** Let  $\Phi$  be some class of functions from  $\mathbb{R}$  into  $\mathbb{R}$ . We say that a function

$$g : \mathbb{R}^2 \rightarrow \mathbb{R}$$

is universal for the class  $\Phi$  if for every function  $\phi \in \Phi$  there exists  $y \in \mathbb{R}$  (certainly, depending on  $\phi$ ) such that

$$(\forall x \in \mathbb{R})(\phi(x) = g(x, y)).$$

Prove, using the method of transfinite induction, that for every ordinal  $\alpha < \omega_1$  there exists a Borel function universal for the class  $B_\alpha(\mathbb{R}, \mathbb{R})$ . Deduce from this fact the Lebesgue theorem which says that for every  $\alpha < \omega_1$  there exists a function strictly of the class  $B_\alpha(\mathbb{R}, \mathbb{R})$ . Formulate and prove an analogous result for an arbitrary uncountable Polish topological space  $E$ . Formulate and prove an analogous result for the classes  $B_\alpha^*(E)$  ( $\alpha < \omega_1$ ) of Borel subsets of an uncountable Polish topological space  $E$ . Prove also that there does not exist a Borel function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  universal for all Borel functions from  $\mathbb{R}$  into  $\mathbb{R}$ .



**Exercise 8.6** Show that for an arbitrary function  $f : \mathbb{R} \rightarrow \mathbb{R}$  the following two sentences are equivalent:

- a)  $f$  is of the first Baire class;
- b) for every non-empty perfect set  $P \subseteq \mathbb{R}$  the restriction  $f|_P$  is continuous at least at one point of  $P$ .

This result is due to Baire and is known as the Baire characterization of functions of the first Baire class.

**Exercise 8.7** Prove that the function  $\phi_X$  from Example 1 from this Chapter is not equivalent (with respect to the  $\sigma$ -ideal of all  $\lambda$ -measure zero sets) to any function of the first Baire class. Using this fact show that  $\phi_X$  is strictly of the second Baire class.

**Exercise 8.8** Prove that there exists a measure  $\mu$  on the real line  $\mathbb{R}$  satisfying the following conditions:

- a)  $\mu$  is a non-zero complete  $\sigma$ -finite measure invariant under the group of all isometric transformations of the real line;
- b)  $\text{dom}(\lambda) \subseteq \text{dom}(\mu)$  where  $\lambda$  denotes the Lebesgue measure on  $\mathbb{R}$ ;
- c)  $(\forall Y \subseteq \mathbb{R})(\lambda(Y) = 0 \rightarrow \mu(Y) = 0)$ ;
- d) there is a Vitali set  $X$  such that  $X \in \text{dom}(\mu)$ .

Show also that for any measure  $\nu$  on the real line invariant under the group  $\mathbb{Q}$  and extending the Lebesgue measure  $\lambda$  no Vitali set is  $\nu$ -measurable.

**Exercise 8.9** Let  $E$  be an arbitrary vector space. Show that any two bases of  $E$  have the same cardinality. Their common cardinality is called the algebraic dimension of the space  $E$ . Notice here that this fact is a very particular case of a general theorem about the cardinality of any system of free generators of a free universal algebra.

**Exercise 8.10** Let  $E$  be an arbitrary normed vector space over the field of real numbers (or over the field of complex numbers) and let  $E$  have an infinite algebraic dimension. Show that there exists a linear functional defined on  $E$  and discontinuous at each point of  $E$ . Compare this result to the fact that every linear functional defined on a finite-dimensional normed vector space is continuous.

**Exercise 8.11** Using a Hamel base describe all solutions of the Cauchy functional equation.

**Exercise 8.12** Find all continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  which satisfy the following functional equation:

$$f(x + y) = f(x) \cdot f(y) \quad (x, y \in \mathbb{R}).$$

Show that there are discontinuous (and Lebesgue non-measurable) solutions of this functional equation, too.

**Exercise 8.13** Prove that there are two sets  $A, B \subseteq \mathbb{R}$  both of the Lebesgue measure zero and of the first category such that

$$A + B = \mathbb{R}.$$

Conclude from this fact that there exists a Hamel base in  $\mathbb{R}$  which is contained in the set  $A \cup B$  and, consequently, is of the first category and of the Lebesgue measure

zero. Show also that for every  $\sigma$ -finite measure  $\mu$  on the real line which is invariant (or, more generally, quasi-invariant) under the group of all translations of  $\mathbb{R}$  and for every Hamel base  $(e_i)_{i \in I}$  we have the implication

$$\{e_i : i \in I\} \in \text{dom}(\mu) \rightarrow \mu(\{e_i : i \in I\}) = 0.$$

**Exercise 8.14** Using the method of transfinite recursion construct a  $\lambda$ -massive Hamel base in  $\mathbb{R}$ , where  $\lambda$  denotes the Lebesgue measure on  $\mathbb{R}$ . Deduce from this fact that there exists a Hamel base in  $\mathbb{R}$  non-measurable with respect to the measure  $\lambda$ .

**Exercise 8.15** Prove that the following two sentences are equivalent:

- a) the Continuum Hypothesis;
- b) for every non-zero  $\sigma$ -finite measure  $\mu$  on the real line  $\mathbb{R}$  invariant under all translations of  $\mathbb{R}$  there exists a Hamel base non-measurable with respect to  $\mu$ .

**Exercise 8.16** Let us recall that a polyhedron in the three-dimensional Euclidean space  $\mathbb{R}^3$  is an arbitrary subset of this space which can be represented as the union of a finite family of closed three-dimensional simplexes. For any two polyhedra  $X \subseteq \mathbb{R}^3$  and  $Y \subseteq \mathbb{R}^3$  we say that they are equivalent by a finite decomposition if there exist two finite families

$$(X_i)_{i \in I}, \quad (Y_i)_{i \in I}$$

of polyhedra such that:

- a)  $X = \bigcup_{i \in I} X_i, \quad Y = \bigcup_{i \in I} Y_i$ ;
- b)  $(\forall i, j \in I)(i \neq j \rightarrow \text{int}(X_i) \cap \text{int}(X_j) = \text{int}(Y_i) \cap \text{int}(Y_j) = \emptyset)$ , where the symbol  $\text{int}$  denotes, as usual, the interior of a set;
- c) for each  $i \in I$  the polyhedra  $X_i$  and  $Y_i$  are congruent with respect to the group of all motions of the space  $\mathbb{R}^3$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be any solution of the Cauchy functional equation such that  $f(\pi) = 0$ . Let  $P_3$  denote the class of all polyhedra in  $\mathbb{R}^3$  and let us define a functional

$$\Phi_f : P_3 \rightarrow \mathbb{R}$$

by the formula

$$\Phi_f(X) = \sum_{j \in J} f(\alpha_j) \cdot |b_j| \quad (X \in P_3),$$

where  $(b_j)_{j \in J}$  is the injective family of all edges of the polyhedron  $X$  and  $\alpha_j$  is the value of the angle of  $X$  corresponding to the edge  $b_j$  and, finally,  $|b_j|$  denotes the length of the edge  $b_j$ . Show that the functional  $\Phi_f$  is invariant under the group of all motions of the space  $\mathbb{R}^3$  and has equal values for any two polyhedra equivalent by finite decomposition.

We say that  $\Phi_f$  is a Dehn functional on  $P_3$  associated with the solution  $f$  of the Cauchy functional equation.

Let  $\alpha$  be the value of an angle corresponding to an edge of the regular three-dimensional simplex. Show that the set  $\{\alpha, \pi\}$  can be extended to a Hamel base in  $\mathbb{R}$ . Conclude from this fact that there exists a solution  $f : \mathbb{R} \rightarrow \mathbb{R}$  of the Cauchy functional equation such that  $f(\alpha) = 1$  and  $f(\pi) = 0$ . Show that the functional  $\Phi_f$  assigns a strictly positive value on the three-dimensional regular simplex of the volume 1 and that  $\Phi_f$  assigns the value zero on the unit closed cube in  $\mathbb{R}^3$ . Hence, these two polyhedra, having the same volume, are not equivalent by finite decomposition.

This gives the solution of the **third Hilbert problem**, obtained first by his disciple Dehn.

This fact also explains us why during school-lectures on elementary geometry we are needed to use some infinite procedures or limit processes for a calculation of the volume of a three-dimensional simplex.

**Exercise 8.17** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be any non-trivial solution of the Cauchy functional equation. Show that the graph of the function  $f$  is an everywhere dense subset of the Euclidean plane  $\mathbb{R}^2$ .

**Exercise 8.18** Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a Lebesgue measurable function satisfying the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad (x, y \in \mathbb{R}).$$

Show that the function  $f$  is continuous and, consequently,  $f$  is a convex function in the usual sense.

This result was obtained by Sierpiński. Formulate and prove an analogous result for the Baire property.

**Exercise 8.19** Let  $E$  be a topological space. We say that a subset  $X$  of  $E$  is **totally imperfect** in  $E$  if  $X$  contains no non-empty perfect (in  $E$ ) subset. Moreover, we say that  $X$  is a **Bernstein subset** of  $E$  if both sets  $X$  and  $E \setminus X$  are totally imperfect.

- a) Prove, in theory **(ZF)** & **(DC)**, that if there exists a totally imperfect subset of the real line of cardinality  $\mathfrak{c}$ , then there exists a Lebesgue non-measurable subset of the real line (prove also the same for the Baire property);
- b) Show that in every complete metric space  $E$  of the cardinality continuum (hence, in every uncountable Polish space) there exists a Bernstein set. Moreover, show that if the space  $E$  has no isolated points, then any Bernstein subset of  $E$  does not have the Baire property in  $E$ .

**Exercise 8.20** Let  $E$  be an uncountable Polish space and let  $X$  be any subset of  $E$ . Show that the following two sentences are equivalent:

- a)  $X$  is a Bernstein subset of  $E$ ;
- b) for every non-zero  $\sigma$ -finite diffused Borel measure  $\mu$  defined on  $E$  the set  $X$  is  $\bar{\mu}$ -non-measurable, where  $\bar{\mu}$  denotes the standard completion of the measure  $\mu$ .

**Exercise 8.21** Let  $n$  be a natural number greater or equal to 2 and let  $X$  be a totally imperfect subset of the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . Prove that the set  $\mathbb{R}^n \setminus X$  is connected (in the usual topological sense). In particular, any Bernstein subset of  $\mathbb{R}^n$  is connected.

**Exercise 8.22** Let us consider the first uncountable ordinal  $\omega_1$  equipped with its order topology and let  $I$  be the  $\sigma$ -ideal of all non-stationary subsets of  $\omega_1$ . Let us put

$$S = I \cup I',$$

where  $I'$  is the dual filter to the ideal  $I$ . Observe that  $S$  is the  $\sigma$ -algebra generated by  $I$ . Show that for any set  $X \subseteq \omega_1$  the following two relations are equivalent:

- a) the sets  $X$  and  $\omega_1 \setminus X$  are stationary in  $\omega_1$ ;

b) for every non-zero  $\sigma$ -finite diffused measure  $\mu$  defined on  $S$  we have

$$X \notin \text{dom}(\bar{\mu}),$$

where  $\bar{\mu}$  denotes the usual completion of the measure  $\mu$ .

Any set  $X$  with the above properties can be considered as a certain analog (for the topological space  $\omega_1$ ) of a Bernstein subset of  $\mathbb{R}$ .

**Exercise 8.23** Let  $P(\mathbb{R})$  denote the complete Boolean algebra of all subsets of the real line  $\mathbb{R}$ . Let us consider the quotient algebras  $P(\mathbb{R})/\mathbb{L}$  and  $P(\mathbb{R})/\mathbb{K}$ . Show that these Boolean algebras are not complete.

**Exercise 8.24** Prove that every Luzin subset of the real line  $\mathbb{R}$  is a strongly measure zero set. Does the converse implication hold?

**Exercise 8.25** Let  $X$  be any Luzin subset of the real line  $\mathbb{R}$  and let  $\mu$  be any  $\sigma$ -finite diffused Borel measure on  $\mathbb{R}$ . Suppose also that  $f : X \rightarrow \mathbb{R}$  is a mapping which has the Baire property. Prove that

$$\mu^*(f(X)) = 0,$$

where  $\mu^*$  is the outer measure associated with the measure  $\mu$  (in particular,  $f(X)$  is a totally imperfect subset of  $\mathbb{R}$ ).

**Exercise 8.26** Suppose that the Continuum Hypothesis holds and let  $X$  be an uncountable subset of the real line  $\mathbb{R}$  such that

$$\bar{B}(X) = B(X).$$

Show that  $X$  is a Luzin set.

**Exercise 8.27** Let  $X$  be any Lebesgue measurable subset of the real line  $\mathbb{R}$  and let  $x$  be any point from  $\mathbb{R}$ . We recall that the point  $x$  is a density point of the set  $X$  if

$$\lim_{\lambda(V(x)) \rightarrow 0} \frac{\lambda(V(x) \cap X)}{\lambda(V(x))} = 1,$$

where  $V(x)$  denotes an arbitrary open interval containing the point  $x$ . Let us recall also that the Lebesgue theorem about density points states that almost every point of the set  $X$  is the density point of  $X$ . Let us consider the family  $T$  of all Lebesgue measurable subsets  $Y$  of  $\mathbb{R}$  such that each point of  $Y$  is a density point of  $Y$ . Show that

- a)  $T$  is a topology on  $\mathbb{R}$  which strictly extends the standard Euclidean topology on  $\mathbb{R}$ ;
- b) the topological space  $(\mathbb{R}, T)$  is a Baire space and satisfies the Suslin condition;
- c) every first category set in the space  $(\mathbb{R}, T)$  is nowhere dense;
- d) a set  $X \subseteq \mathbb{R}$  is Lebesgue measurable if and only if  $X$  has the Baire property in the space  $(\mathbb{R}, T)$ .
- e) a set  $X \subseteq \mathbb{R}$  is of Lebesgue measure zero if and only if  $X$  is a first category set in the space  $(\mathbb{R}, T)$ .

The topology  $T$  is called the **density topology** on  $\mathbb{R}$ . Show that a set  $Z \subseteq \mathbb{R}$  is a Sierpiński set in  $\mathbb{R}$  if and only if  $Z$  is a Luzin set in the space  $(\mathbb{R}, T)$  (this means that  $Z$  is uncountable and for every first category set  $Y$  in  $(\mathbb{R}, T)$  the intersection  $X \cap Y$  is at most countable).

Let us remark that the density topology is a very particular case of the von Neumann topology (see Appendix B).

**Exercise 8.28** Assume that the Continuum Hypothesis holds. Prove that there exists a set  $X \subseteq \mathbb{R}$  satisfying the following conditions:

- a)  $X$  is a vector space over the field  $\mathbb{Q}$ ;
- b)  $X$  is an everywhere dense Luzin subset of  $\mathbb{R}$ .

Show also that there exists a set  $Y \subseteq \mathbb{R}$  satisfying the following conditions:

- a)  $Y$  is a vector space over the field  $\mathbb{Q}$ ;
- b)  $Y$  is an everywhere dense Sierpiński subset of  $\mathbb{R}$ .

Moreover, assuming Martin's Axiom, formulate and prove analogous results for generalized Luzin sets and for generalized Sierpiński sets. Deduce from these results, assuming Martin's Axiom, that there exist an isomorphism  $f$  of the additive group of  $\mathbb{R}$  onto itself and a generalized Luzin set  $X$  in  $\mathbb{R}$  such that  $f(X)$  is a generalized Sierpiński set in  $\mathbb{R}$ .

**Exercise 8.29** Let  $X$  be a Sierpiński set on the real line  $\mathbb{R}$ . Equip  $X$  with the topology induced by the density topology of  $\mathbb{R}$ . Prove that the topological space  $X$  is non-separable and hereditarily Lindelof.

**Exercise 8.30** Assume that the Continuum Hypothesis holds. Let  $X$  be a Sierpiński set on the real line  $\mathbb{R}$ . Equip  $X$  with the topology induced by the Euclidean topology of  $\mathbb{R}$ . Prove that

$$A(X) = B(X)$$

where  $A(X)$  denotes the class of all analytic subsets of  $X$  and  $B(X)$  denotes the class of all Borel subsets of  $X$ .

**Exercise 8.31** Suppose that Martin's Axiom holds. Applying a generalized Luzin set on the real line  $\mathbb{R}$  show that there exists a  $\sigma$ -algebra  $S$  of subsets of  $\mathbb{R}$  such that

- a) for each point  $x \in \mathbb{R}$  we have  $\{x\} \in S$ ;
- b)  $S$  is a countably generated  $\sigma$ -algebra, i.e. there exists a countable subfamily of  $S$  which generates  $S$ ;
- c) there is no non-zero  $\sigma$ -finite diffused measure defined on  $S$ ;

A similar result can be proved in theory **ZFC** if we replace  $\mathbb{R}$  by a certain uncountable subspace  $E$  of  $\mathbb{R}$ . We shall consider this result in Part 2 of the book.

**Exercise 8.32** Let us consider the set  $E = \omega_1 \times \omega_1$ , where, as usual,  $\omega_1$  is the first uncountable ordinal. Prove that there exists a partition  $\{X, Y\}$  of the set  $E$  such that

- a) for any ordinal  $\xi < \omega_1$  we have

$$\text{card}(X \cap (\{\xi\} \times \omega_1)) \leq \omega;$$

b) for any ordinal  $\zeta < \omega_1$  we have

$$\text{card}(Y \cap (\omega_1 \times \{\zeta\})) \leq \omega.$$

The partition  $\{X, Y\}$  is called a **Sierpiński partition** of the set  $E$ .

Conversely, suppose that  $\kappa$  is an uncountable cardinal and let  $E = \kappa \times \kappa$ . Assume that there exists a partition  $\{X, Y\}$  of the set  $E$  such that

c) for any ordinal  $\xi < \kappa$  we have

$$\text{card}(X \cap (\{\xi\} \times \kappa)) \leq \omega;$$

d) for any ordinal  $\zeta < \kappa$  we have

$$\text{card}(Y \cap (\kappa \times \{\zeta\})) \leq \omega.$$

Prove that the equality  $\kappa = \omega_1$  holds.

**Exercise 8.33** Let us consider the family  $P(\omega_1)$  of all subsets of  $\omega_1$  as a  $\sigma$ -algebra. Let  $P(\omega_1) \otimes P(\omega_1)$  denote the product  $\sigma$ -algebra. Show that

$$P(\omega_1) \otimes P(\omega_1) = P(\omega_1 \times \omega_1).$$

Deduce from this fact, using the Fubini theorem, the following classical result of Ulam: there is no non-zero  $\sigma$ -finite diffused measure defined on the  $\sigma$ -algebra  $P(\omega_1)$ .

**Exercise 8.34** Let  $\preceq$  be well-ordering of the real line  $\mathbb{R}$ . Prove, using the Kuratowski-Ulam theorem, that this well-ordering considered as a subset of  $\mathbb{R}^2$  does not have the Baire property. Similarly prove, using the Fubini theorem, that  $\preceq$  is not measurable with respect to the two-dimensional Lebesgue  $\lambda^2$ .

Deduce from these facts that

a) neither  $\preceq$  nor  $\mathbb{R}^2 \setminus \preceq$  is an analytic subset of the plane  $\mathbb{R}^2$ ;

b) in theory **(ZF)** & **(DC)** the following implication holds:

there exists a well-ordering on  $\mathbb{R} \rightarrow$

there exists a subset of  $\mathbb{R}$  without the Baire property

and non-measurable with respect to the Lebesgue measure.

**Exercise 8.35** Let  $Z$  be the subset of the Cantor discontinuum defined in the proof of Theorem 13 starting with a non-principal ultrafilter on  $\omega$ . Show that the set  $Z$  does not have the Baire property in the product space  $\{0, 1\}^\omega$ . Deduce from this that in theory **(ZF)** & **(DC)** the following implication holds:

there exists a non-principal ultrafilter on  $\omega \rightarrow$

there exists a subset of the real line  $\mathbb{R}$  without the Baire property.

**Exercise 8.36** Let  $\Phi$  be any non-principal filter in  $\omega$  and let

$$Z = \{1_X : X \in \Phi\},$$

where  $1_X$  denotes the characteristic function of the set  $X$ . Prove that in the Cantor discontinuum  $\{0, 1\}^\omega$  we have:

a) if the set  $Z$  has the Baire property, then it is a first category subset of  $\{0, 1\}^\omega$ ;

b) if the set  $Z$  is measurable, then it is a measure zero subset of  $\{0,1\}^\omega$ .

**Exercise 8.37** Assume that Martin's Axiom holds. Show that there are two  $\sigma$ -algebras  $S_1$  and  $S_2$  of subsets of the real line  $\mathbb{R}$  satisfying the following relations:

- a)  $(\forall x \in \mathbb{R})(\{x\} \in S_1 \cap S_2)$ ;
- b) the  $\sigma$ -algebras  $S_1$  and  $S_2$  are countably generated;
- c) there exists a probability diffused measure on  $S_1$  and a probability diffused measure on  $S_2$ ;
- d) there is no probability diffused measure on the  $\sigma$ -algebra generated by  $S_1 \cup S_2$ .

We shall return to the result of this exercise in Part 2 of the book.

**Exercise 8.38** Prove that the order type of an arbitrary totally imperfect subset of  $\mathbb{R}$  is strictly less than the order type of  $\mathbb{R}$ . Apply this result to a Bernstein set (respectively, to a Luzin set and to a Sierpiński set).

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