F = O(G)

JACEK CICHOŃ

ABSTRACT. In this note I discuss the basic properties of the notion $f = O(g)$ and explain how in many cases the problem of proving that $f = O(g)$ can be reduced to an easy problem of a calculation of a limit of a sequence, which any student should know from lectures on Mathematical Analysis.

1. BASIC DEFINITIONS

The notion $f = O(g)$ was introduced by L. Landau. It is very useful in mathematics and in Computer Science specially in the analysis of algorithms. Let us recall its definition:

Definition 1. *Let $f, g : \mathbf{N} \to \mathbf{R}$. We say[1] that $f = O(g)$ if*

$$(\exists C)(\exists N)(\forall n > N)(|f(n)| \leq C|g(n)|)\ .$$

We shall translate this definition in this note into the standard language of mathematical Analysis. Namely let us recall the definition of the *upper limit* of a sequence:

Definition 2. *Let $f : \mathbf{N} \to \mathbf{R}$. We say that $\alpha = \limsup_{n\to\infty} f(n)$ if the following two conditions holds:*

(1) $(\forall \varepsilon > 0)(\exists N)(\forall n > N)(f(n) < \alpha + \varepsilon)$,
(2) $(\forall \varepsilon > 0)(\forall N)(\exists n > N)(f(n) > \alpha - \varepsilon)$.

This definition may seems to be sligtly complicated, but the following simply observation should clarify a lot:

Lemma 1. *If the sequence $(f(n))_{n\geq 0}$ is convergent and $\lim_n f(n) = \alpha$ then*

$$\limsup_{n\to\infty} f(n) = \alpha$$

[1] $\mathbf{N}$ denotes the set of natural numbers and $\mathbf{R}$ denotes reals

Proof. Suppose that the sequence $(f(n))_{n\geq 0}$ is convergent and $\lim_n f(n) = \alpha$. This means that for each $\varepsilon > 0$ we can find N such that for all $n > N$ we have $|f(n) - \alpha| < \varepsilon$. Hence each $\varepsilon > 0$ there exists N such that for all $n > N$ we have $f(n) < \alpha + \varepsilon$. This proves the first part of the claim. Once again, let us fix $\varepsilon > 0$ and let us additionally fix N. Let M be such that $(\forall n > M)|f(n) - \alpha| < \varepsilon$. Let $n_0 = \max\{N, M\} + 1$. Then $n_0 > M$, $n_0 > N$, so $|f(n_0) - \alpha| < \varepsilon$, so $f(n_0) > \alpha - \varepsilon$. □

As we know there are a lot of sequences which are not convergent (in fact we can prove in a very precise sense that there are very few convergent sequences). But every bounded sequence[2] of reals has an upper limit. Before proving this fact we need one additional notion from elementary Analysis:

Definition 3. *Let $a \subseteq \mathbf{R}$. We say that te real number α is an infimum of A ($\alpha = \inf(A)$) if the following two conditions holds:*

(1) $(\forall x \in A)(\alpha \leq x)$,
(2) $(\forall \varepsilon > 0)(\exists x \in A)(x < \alpha + \varepsilon)$.

Every subset A of reals bounded from below has an infimum - you can treat this property as an axiom of real numbers.

Lemma 2. *Suppose that $(f(n))_{n\geq 0}$ is bounded by a number $C > 0$. Then the exists an upper limit of this sequence and* $\limsup_{n\to\infty} f(n) \leq C$.

Proof. Assume that $-C \leq f(n) \leq C$ for all $n \geq 0$. Let

$$A = \{x \in (-\infty, C] : (\exists N)(\forall n > N)(f(n) \leq x\} \ .$$

Note that $C \in A$, so $A \neq \emptyset$. Moreover, if $t < -C$ then $t \notin A$. Therefore $A \subseteq [-C, C]$. Hence there exists an infimum of A. Let $\alpha = \inf(A)$.

Let us fix $\varepsilon > 0$. Let us take $a \in [\alpha, \alpha + \varepsilon) \cap A$. The there exists N such that for all $n > N$ we have $f(n) \leq a$. Hence, for all $n > N$ we have $f(n) < \alpha + \varepsilon$.

Let us fix once again $\varepsilon > 0$. Then $\alpha - \varepsilon \notin A$. Hence

$$\neg(\exists N)(\forall n > N)(f(n) \leq \alpha - \varepsilon) \ .$$

Using twice de Morgan laws we get

$$(\forall N)(\exists n > N)(f(n) > \alpha - \varepsilon) \ .$$

[2]A sequence (a_n) of real numbers is bounced if there exists $C \geq 0$ such that $|a_n| \leq C$ for all $n \geq 0$

$\square$

2. Equivalent formulation

We are ready to translate the **big O** notation into the language of mathematical analysis We shall use the following convention $\frac{0}{0} = 1$ which is common in analysis.

Theorem 3. *Let* $f, g : \mathbf{N} \to \mathbf{R}$*. Then*

$$\boxed{(f = O(g)) \iff \limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} < \infty}$$

Proof. Suppose that $f = O(g)$. Let us fix C and N such that $|f(n)| \leq C|g(n)|$ for all $n > N$. So let us consider $n > N$. If $g(n) \neq 0$ then

$$\frac{|f(n)|}{|g(n)|} \leq C\,.$$

If $g(n) = 0$ then also $f(n) = 0$ and in accordance with the convention $\frac{0}{0} = 1$ we have $\frac{|f(n)|}{|g(n|)} = 1$. So in both cases for $n > N$ we have

$$\frac{|f(n)|}{|g(n)|} \leq \max\{C, 1\} = C^*\,.$$

This means that the sequence $\left|\frac{f(n)}{g(n)}\right|$ is bounded from above by the number C^*, hence from Lemma 2 we obtain

$$\limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} \leq C^* < \infty\,.$$

Suppose now that $\limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} \leq C$. We put $\varepsilon = 1$ into the first condition from Definition 2 and find N such that for all $n > N$ we have

$$\frac{|f(n)|}{|g(n)|} < C + 1\,.$$

Hence for all $n > N$ we have

$$|f(n)| \leq (C+1)|g(n)|\,.$$

$\square$

3. Applications

Let us recall that if $\lim a_n = a$ then $\lim |a_n| = |a|$. From Lemma 1 and Theorem 3 we deduce that the following implications holds:

$$\left(\frac{f(n)}{g(n)} \text{ is convergent}\right) \Longrightarrow \left(\frac{|f(n)|}{|g(n)|} \text{ is convergent}\right) \Longrightarrow$$
$$\left(\limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} < \infty\right) \Longrightarrow (f = O(g))$$

This is a very usefull observation: in order to show that $f = O(g)$ we should try to prove that the seguence $(f(n)/g(n))_{n\geq o}$ is convergent and if we succeed then the goal is achieved.

Example 1. Consider the polynomial $w(n) = a_0 + a_1 n + \ldots + a_k n^k$ where $a_k \neq 0$ then

$$\lim_{n\to\infty} \frac{w(n)}{n^k} = \lim_{n\to\infty} \left(\frac{a_0}{n^k} + \frac{a_1}{n^{k-1}} + \ldots + \frac{a_{k-1}}{n} + a_k\right) = a_k \ .$$

Therefore $w = O(n^k)$. It is worth to remark that we also have

$$\lim_{n\to\infty} \frac{w(n)}{n^k} = \frac{1}{a_k} \ ,$$

hence we also have $n^k = O(w)$.

Now let us introduce a notion directly related to the Big O concept:

Definition 4. *Let $f, g : \mathbf{N} \to \mathbf{R}$. We say that $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.*

Example 2. For any polynomial w of rank k we have $w = \Theta(n^k)$. This was shown. in fact, in the previous example.

Example 3. Let $f(n) = n^{\ln n}$ and $g(n) = 2^n$. Then

$$\frac{f(n)}{g(n)} = \frac{n^{\ln n}}{2^n} = \frac{e^{(\ln n)^2}}{e^{n \ln 2}} = \frac{1}{e^{n \ln 2 - (\ln n)^2}}$$

Using twice the l'Hôpital's (at points marked by *) rule we get

$$\lim_{n\to\infty} \frac{n \ln 2}{(\ln n)^2} =^* \lim_{n\to\infty} \frac{\ln 2}{2 \cdot \ln n \cdot \frac{1}{n}} = \lim_{n\to\infty} \frac{n \ln 2}{\ln n} =^*$$
$$\lim_{n\to\infty} \frac{\ln 2}{\frac{1}{n}} = \lim_{n\to\infty} n \ln 2 = \infty$$

so $\lim_{n\to\infty}(n\ln 2 - (\ln n)^2) = \infty$, so $\lim_{n\to\infty}\frac{1}{e^{n\ln 2-(\ln n)^2}} = 0$, so finally we get $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0$, i.e. we have shown that $n^{\ln n} = O(2^n)$.

4. REMARKS

To check how well you understand the concept of Big O notation you should prove yourself a few of its basic property. Prove, for example, that

(1) If $f = O(g)$ and $g = O(h)$ then $f = O(h)$
(2) $f = O(1)$ if and only if f is bounded
(3) If $f_1 = O(g_1)$ and $f_2 = O(g_2)$ then $f_1 + f_2 = O(g_1 + g_2)$.

and

(1) Show that the relation $(f \equiv g) \Leftrightarrow (f = \Theta(g))$ is an equivalence relation.
(2) Compare functions $n^{\sqrt{n}}$ and $(\sqrt{n})^n$

Final remarks:

(1) In mathematical analysis a slightly different version of the Big O notion is widely used: it is defined for an arbitrary point from extended real line. We have considered there the point $+\infty$
(2) There are serious errors in the Polish version of Wikipedia in the article on Big O notation (15.05.2010)
(3) This document may be used without any limitations.