STIRLING APPROXIMATION FORMULA

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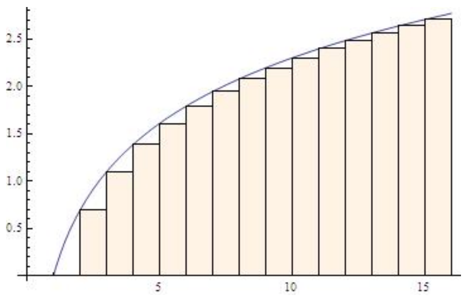
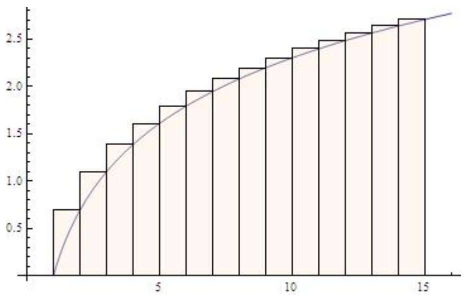
ABSTRACT. This note constains aa elementary and complete proof of the Stirling approximation formula $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$ of the factorial function.

1. INTRODUCTION

It is quite easy to get an approximation of the number $n!$ which gives an information about its ratio of growth. Namely, let us consider the sequence $S(n) = \ln(n!)$. Then $S(n) = \sum_{k=1}^{n} \ln k$. So we see that $\ln n \leq S(n) \leq n \ln n$, so the sequence $S(n)$ grows rather slowly. Look at this picture:



I placed at this picture the plot of the fuction $x \mapsto \ln x$. The area of the figure below the plot is equal to $S(15)$. This observation, generalized to arbitrary n gives us the bound $S(n) \leq \int_1^{n+1} \ln x dx$. Look now at the following picture:



We read from this picture that $S(n) \geq \int_1^n \ln x dx$ So we have derived the following bounds:

$$\int_1^n \ln x dx \leq S(n) \leq \int_1^{n+1} \ln x dx.$$

All what we need now is the formula $\int \ln x dx = x \ln x - x + C$. You can derive this formula using the integration by parts ($\int \ln x dx = \int (x)' \ln x dx = \ldots$). Let us note that $x \ln x - x = x(\ln x - 1) = x(\ln x - \ln e) = x \ln(\frac{x}{e})$. Using this formula we get

$$\int_1^n \ln x dx = [x \ln(\frac{x}{e})]_{x=1}^n = n \ln \frac{n}{e} - \ln \frac{1}{e} = \ln(\frac{n}{e})^n + \ln e = \ln \frac{n^n}{e^{n-1}}$$

and

$$\int_1^{n+1} \ln x dx = \ldots = \ln \frac{(n+1)^{n+1}}{e^n}$$

Hence

$$\ln \frac{n^n}{e^{n-1}} \leq S(n) \leq \ln \frac{(n+1)^{n+1}}{e^n}$$

Finally, we observe that $n! = \exp(S(n))$ and we transform this formula info the form

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{(n+1)^{n+1}}{e^n}$$

or

$$e(\frac{n}{e})^n \leq n! \leq (\frac{n}{e})^n (n+1)(1+\frac{1}{n})^n < e(n+1)(\frac{n}{e})^n$$

Therefore using a very elementary tools we derived the following formula

$$n! = \alpha(n) \left(\frac{n}{e}\right)^n$$

where α is some function such that $e \leq \alpha(n) \leq (n+1)e$. This is a quite precise result and is sufficient for many applications.

2. More precise result: Stirling's formula

We are going to prove in this section the Strirling approximation formula:

$$\boxed{n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n}$$

i.e. that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = 1 \ .$$

Let

$$a_n = \frac{n!}{\sqrt{2n} \left(\frac{n}{e}\right)^n}$$

Our plan is following:

(1) First we show that $\lim_{n\to\infty} a_n = C$ for some constant C. This will imply that

$$\lim_{n\to\infty} \frac{n!}{C\sqrt{2n}(\frac{n}{e})^n} = 1$$

(2) Next we derive Wallis formula which gives a precise asymptotic result involving $n!$

(3) Finally we put in Wallis formula the approximation $C\sqrt{2n}(\frac{n}{e})^n$ and this will give us the precise value of the constant C.

2.1. **Part 1.** Let

$$b_n = \ln a_n \ .$$

After easy transformations we get the following equality

$$b_n - b_{n+1} = \frac{1}{2}(n+1)\ln\frac{n+1}{n} - 1$$

We are going to use an expansion of the function $\ln$ into the Taylor series at point 1. However the most obvious approach

$$\ln\frac{n}{n+1} = \ln\frac{1}{1+\frac{1}{n}} = - \cdot \frac{1}{n} + \frac{1}{2}\cdot\frac{1}{n^2} - \dots$$

gives us a series with alternating terms which are usually difficult to handle. So we try to be more ingenious. Observe that that for $|t| < 1$ we have

$$\begin{aligned} \ln(1+t) &= t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5 + \dots \\ -\ln(1-t) &= t + \frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{4}t^4 + \frac{1}{5}t^5 + \dots \end{aligned}$$

hence

$$\ln\frac{1+t}{1-t} = \ln(1+t) - \ln(1-t) = 2\sum_{k=0}^{\infty}\frac{1}{2k+1}t^{2k+1}$$

The only solution of the equation $\frac{n+1}{n} = \frac{1+t}{1-t}$ is equal to $t = \frac{1}{2n+1}$ so we get

$$\ln\frac{n+1}{n} = 2\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{1}{2n+1}\right)^{2k+1} \ .$$

The first term of this series is equal to $\frac{2}{2n+1}$, hence

$$b_n - b_{n+1} = \frac{1}{2}(2n+1)\ln\frac{n+1}{n} - 1 = \sum_{k=1}^{\infty}\frac{1}{2k+1}\left(\frac{1}{2n+1}\right)^{2k}$$

so **the sequence** (b_n) **is decreasing**. Next we have

$$b_n - b_{n+1} < \sum_{k=1}^{\infty}\left(\frac{1}{(2n+1)^2}\right)^k = \frac{1}{(2n+1)^2}\frac{1}{1-\frac{1}{(2n+1)^2}} = \frac{1}{4}\frac{1}{n(n+1)}$$

Observe that (a telescoping sum)

$$b_1 - b_n = (b_1 - b_2) + (b_2 - b_3) + \dots (b_{n-1} - b_n)$$

therefore

$$b_1 - b_n < \frac{1}{4}\sum_{m=1}^{n-1} \frac{1}{m(m+1)} < \frac{1}{4}\sum_{m=1}^{\infty} \frac{1}{m(m+1)} = \frac{1}{4}$$

hence

$$b_n > b_1 - \frac{1}{4} = \frac{e}{\sqrt{2}} - \frac{1}{4} \approx 1.67212$$

so $(b_n)_{n\geq 1}$ is bounded from below, hence is convergent to some constant D. This implies that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{b_1} = e^{\lim_{n\to\infty} a_n} = e^D.$$

2.2. **Part 2.** We will prove in this part the Wallis product formula

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1}\frac{2n}{2n+1} = \frac{\pi}{2} \tag{1}$$

This formula can be easily derived immediately from the Euler formula $\sin(x) = x\prod_{n=1}^{\infty}(1-(\frac{x}{\pi n})^2)$, but for completness of our arguments we shall give its elementary proof.

Let us start from the interval $\int \sin^n x dx$. Intergrating by parts we get

$$\int \sin^n x dx = \int \sin^{n-1} x \sin x dx = -\int \sin^{n-1} x (\cos x)' dx =$$

$$-\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x dx =$$

$$-\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x(1-\sin^2 x)dx = \dots$$

and after easy calculus we get

$$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x dx\ . \tag{2}$$

Hence

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx\ .$$

Notice that $\int_0^{\pi/2} 1 dx = \pi/2$ and $\int_0^{\pi/2} \sin x dx = 1$. Hence we are able to calculate the integral $\int_0^{\pi/2} \sin^n x dx$ for arbitrary n. After a while we get

$$S_n = \int_0^{\pi/2} \sin^{2n} x dx \quad = \quad \frac{\pi}{2}\prod_{k=1}^{n} \frac{2k-1}{2k}$$

$$C_n = \int_0^{\pi/2} \sin^{2n+1} x dx \quad = \quad \prod_{k=1}^{n} \frac{2k}{2k+1}$$

Hence, finally we get

$$\frac{\pi}{2} = S_n \prod_{k=1}^{n} \frac{2k}{2k-1} =$$

$$= S_n \prod_{k=1}^{n} \frac{2k}{2k-1} \left(\prod_{k=1}^{n} \frac{2k}{2k+1} \right) \left(\prod_{k=1}^{n} \frac{2k}{2k+1} \right)^{-1} =$$

$$\frac{S_n}{C_n} \prod_{k=1}^{n} \frac{2k}{2k-1} \frac{2k}{2k+1} .$$

Therefore the Wallis formula will be proved if we show that $\lim_{n\to\infty} \frac{S_n}{C_n} = 1$. Fortunately this step is easy. Namely for $x \in (0, \frac{\pi}{2})$ we have

$$0 < \sin^{2n+2} x < \sin^{2n+1} x < \sin^{2n} x$$

hence

$$0 < S_{n+1} < C_n < S_n$$

so

$$1 > \frac{S_n}{S_n} > \frac{S_n}{C_n} > \frac{S_{n+1}}{S_n} = \frac{n}{n+1} .$$

Hence the Wallis formula is proved.

2.3. **Part 3.** The Wallis formula 1 may be written in a more compact way as

$$\lim_{n\to\infty} \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} = \frac{\pi}{2} \tag{3}$$

In Part 1 we proved that $n! \sim C\sqrt{2n}(\frac{n}{e})^n$. for come constant C. If we put this approximation into the formula 3 then we get

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^{4n}C^4(2n)^2(\frac{n}{e})^{4n}}{C^2 4n(\frac{2n}{e})^{2n}(2n+1)} =$$

$$C^2 \lim_{n\to\infty} \frac{2^{4n}4n^2n^{4n}}{4n(2n+1)(2n)^{4n}} = \lim_{n\to\infty} C^2 \frac{n^2}{n(2n+1)} = \frac{C^2}{2}$$

Therefore $C = \sqrt{\pi}$ and the Stirling formula is proved.

3. Much more precise results

The Strirlin approximation formula can be extended to the following inequality

$$\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n}}$$

A more precise version of the Stirling formula is given by

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + O(\frac{1}{n^4})\right)$$

4. Remarks

You should do yourself the following:

(1) complete derivation of Equation 3 from Equation 1
(2) complete derivation of Equation 2.