SOME REMARKS ABOUT TWO DEFINITIONS OF CONTINUITY

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ABSTRACT. We formulate some variant of the Axiom of Choice which is neccessary and sufficient for the equivalence of the two definitions of continuity (the Cauchy and the Heine definition) for functions from the real line into any metric space. This result definitely solves the problem of the equivalence of these two classical definitions. We also slightly improve one of Sierpiński's results about global continuity of real functions. We give a negative answer to the problem from Jaegermann's classical paper [2].

We use in this paper standard mathematical notation. We identify the set of all natural numbers with the first infinite ordinal number ω. By $\mathbb{R}$ we denote the set of real numbers and by $\mathbb{Q}$ the set of rational numbers. For two sets A and B we write $A \backsim B$ if they have the same cardinality, i.e. if there exists a bijection $f : A \to B$. The family of all functions from the set X into the set Y we denote by Y^X. If (X, ρ) is a metric space and $A \subseteq X$ then by $Int(A)$ we denote the set of interior points of A. The ball with center $a \in X$ and radius r we denote by $B(a, r)$.

The Zermelo-Fraenkel set theory we denote by ZF. Axiom of Choice we denote by AC. By ZFC we denote the theory $ZF \cup \{AC\}$. A well known result of K. Gödel says that if the set theory ZF is consistent then the theory ZFC is consistent, too. P. Cohen showed that if the theory ZF is consistent, then the theory $ZF + \neg AC$ is consistent.

Let (X, ρ) and (Y, d) be two metric spaces and let $f : X \to Y$ be any function. We define

$$
\begin{aligned}
H(f) &= \{x \in X : (\forall (x_n)_{n\in\omega})(\lim_{n\to\infty} x_n = x \to \lim_{n\to\infty} f(x_n) = f(x))\} \\
C(f) &= \{x \in X : (\forall \varepsilon > 0)(\exists \delta > 0)(\forall t \in X)(\rho(x,t) < \delta \to d(f(x), f(t)) < \varepsilon)\}
\end{aligned}
$$

A well known classical result says that the equality $H(f) = C(f)$ holds in the set theory ZFC. A function $f : X \to Y$ such that $H(f) = X$ is commonly called "sequentially continuous". It is also well known that the inclusion $C(f) \subseteq H(f)$ can be proved without Axiom of Choice, i.e. in the set theory ZF. M. Jaegermann in [2] showed that it is consistent with the theory ZF that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $H(f) \neq C(f)$.

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1. About Local Continuity

In this section we show that the equality $H(f) = C(f)$ holds for any function $f : \mathbb{R} \to \mathbb{R}$ if and only if some version of the Axiom of Choice holds.

Definition 1. *Let $AC_\omega(X)$ denotes the following sentence*

$$(\forall F \in (P(X)\backslash\{\emptyset\})^\omega)(\exists f \in X^\omega)(\forall n \in \omega)(f(n) \in F(n)).$$

Let us observe that $AC_\omega(X)$ holds (in theory ZF) for any set X which can be well-ordered. For example, $AC_\omega(X)$ holds for each countable set X. It is also clear that $ZFC \Vdash (\forall X)AC_\omega(X)$. Moreover, the property $AC_\omega(X)$ is sufficient for the equality $H(f) = C(f)$ for any function $f : X \to Y$. Let AC_ω denotes the sentence $(\forall X)AC_\omega(X)$. N. Brunner in [1] showed in the theory ZF that the sentence AC_ω is equivalent with the sentence "$H(f) = C(f)$ for any metric spaces (X, ρ) and (Y, d) and any function $f : X \to Y$". We shall show a more precise result of this kind for functions $f : \mathbb{R} \to \mathbb{R}$.

Definition 2. *Let $wAC_\omega(X)$ denote the following sentence*

$$(\forall F \in (P(X)\backslash\{\emptyset\})^\omega)(\exists f \in X^\omega)(\forall n \in \omega)(\exists m > n)(f(m) \in F(m)).$$

Lemma 1. *(ZF) Suppose that $X \backsim X^\omega$. Then $AC_\omega(X) \longleftrightarrow wAC_\omega(X)$.*

Proof. Suppose that X is such a set that $X \backsim X^\omega$ and $wAC_\omega(X)$ holds. Let $(X_n)_{n\in\omega}$ be a sequence of non-empty subsets of the set X. Let

$$Y_n = X_0 \times X_1 \times \ldots \times X_n \times X^{\omega\backslash\{0,1,\ldots,n\}}.$$

Then Y_n is a non-empty subset of X^ω for each $n \in \omega$. The assumption $X \backsim X^\omega$ implies that the sentence $wAC_\omega(X^\omega)$ is true. Hence there exists a function $g \in (X^\omega)^\omega$ such that $(\forall n \in \omega)(\exists m > n)(g(m) \in Y_m)$. Let $T = \{m \in \omega : g(m) \in Y_m\}$ and let $\pi_n : X^\omega \to X$ be the canonical projection onto the $n-th$ axis, i.e. $\pi_n(x) = x(n)$ for $x \in X^\omega$. We put

$$f(n) = \pi_n(g(\min\{t \in T : t \geqslant n\}).$$

Then $f(n) \in X_n$ for each $n \in \omega$. □

Notice that the sentence $\mathbb{R} \backsim \mathbb{R}^\omega$ can be proved in the theory ZF. It is also easy observe that the sentence $AC_\omega(\mathbb{R})$ is equivalent with the following one: each countable family of non-empty pairwise disjoint subsets of $\mathbb{R}$ has a selector. The following theorem finally clarify when the two classical definitions of continuity of real functions coincides.

Theorem 1. $ZF \Vdash AC_\omega(\mathbb{R}) \longleftrightarrow (\forall f \in \mathbb{R}^\mathbb{R})(C(f) = H(f))$

Proof. The implication $AC_\omega(\mathbb{R}) \to (\forall f \in \mathbb{R}^\mathbb{R})(C(f) = H(f))$ is clasical and well known. Suppose hence that the sentence $AC_\omega(\mathbb{R})$ is false. Since $\mathbb{R} \backsim \mathbb{R}^\omega$, we may apply Lemma 1, so the sentence $wAC_\omega(\mathbb{R})$ is false, too. Let $(X_n)_{n\in\omega}$ be a such family of non-empty subsets of $\mathbb{R}$ that there is no function $f \in \mathbb{R}^\omega$ that $(\forall n)(\exists m > n)(f(n) \in X_n)$. Let

$$Y_n = \frac{1}{\pi(n+1)(n+2)}(\arctan(X_n) + \frac{\pi}{2}) + \frac{1}{n+2}$$

Then $Y_n \subset (\frac{1}{n+2}, \frac{1}{n+1})$ and there is no function $f \in \mathbb{R}^\omega$ that $(\forall n)(\exists m > n)(f(n) \in Y_n)$. Let $A = \bigcup\{Y_n : n \in \omega\}$ and let

$$h(x) = \begin{cases} 1 & : \quad x \in A \\ 0 & : \quad x \in \mathbb{R} \setminus A \end{cases}$$

Then $0 \in H(h) \setminus C(h)$, so the theorem is proved. $\square$

Remark 1. *Let us notice that we proved, in fact, that the following sentences are equivalent in theory ZF:*

(1) $AC_\omega(\mathbb{R})$,
(2) *for any metric space (Y, d) and any function $f : \mathbb{R} \to \mathbb{Y}$ the equality $C(F) = H(f)$ holds,*
(3) *for any function $f : \mathbb{R} \to \mathbb{R}$ the equality $C(F) = H(f)$ holds,*
(4) *for any function $f : \mathbb{R} \to \{0, 1\}$ the equality $C(F) = H(f)$ holds.*

It is worth to remark, that if the theory ZF is consistent then the theory $ZF + \neg AC_\omega(\mathbb{R})$ is consistent, too. In fact, the negation of the axiom $AC_\omega(\mathbb{R})$ holds in the first Cohen's model for the negation of the Axiom of Choice (see [3], Theorem 53, Lemma 96).

Example 1. *Let us consider a family $(Y_n)_{n\in\omega}$ of non-empty subsets of $\mathbb{R}$ such that $Y_n \subset (\frac{1}{n+2}, \frac{1}{n+1})$ and that there is no function $f \in \mathbb{R}^\omega$ that $(\forall n)(\exists m > n)(f(n) \in Y_n)$. Notice that there are only finitely many n such that $Y_n \cap \mathbb{Q} \neq \emptyset$. Hence we may assume that $\mathbb{Q} \cap \bigcup_n Y_n = \emptyset$. Let $A = \bigcup_n Y_n$. Let us split the set $\mathbb{Q}$ into two dense subsets D_1, D_2 and let*

$$h(x) = \begin{cases} 1 & : \quad x \in A \\ x^2 & : \quad x \in D_1 \\ 0 & : \quad x \in \mathbb{R} \setminus (A \cup D_1) \end{cases}$$

Then $C(h) = \emptyset$ and $H(h) = \{0\}$.

2. About Global Continuity.

In 1918 W. Sierpiński showed (see [4]) in the theory ZF (i.e. in the set theory without the Axiom of Choice) that if $f : (0, 1) \to \mathbb{R}$ and $H(f) = (0, 1)$ then $C(f) = (0, 1)$. We shall slightly improve this result.

Theorem 2. *(ZF) Suppose that (X, ρ) and (Y, d) are metric spaces, and that D is a dense subset of the space X such that $AC_\omega(D)$ holds. Then*

$$Int(H(f)) \subseteq C(f) \subseteq H(f).$$

Proof. Let $U = Int(H(f))$. Then $U \cap D$ is a dense subset of U. Notice that the sentence $AC_\omega(U \cap D)$ holds, since $U \cap D$ is a subset of the set D. Hence we may assume that $H(f) = X$, since we may restrict our considerations to the subspace U. Notice that if x_0 is an arbitrary point then the sentence $AC_\omega(D \cup \{x_0\})$ holds, too. Therefore

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D)(\rho(x, x_0) < \delta \to d(f(x), f(x_0)) < \frac{\varepsilon}{2})$$

for each $x_0 \in X$.

Let us fix now an element $a \in X$ and $\varepsilon > 0$. Let $\delta_a > 0$ be such that

$$(\forall z \in D)(\rho(z, a) < \delta_a \to d(f(z), f(a)) < \frac{\varepsilon}{2}).$$

Let $x \in X$ be such that $\rho(x,a) < \delta_a$. Let $\delta_x > 0$ be such that

$$(\forall z \in D)(\rho(z,x) < \delta_x \to d(f(z), f(x)) < \frac{\varepsilon}{2}).$$

Let $q \in D \cap B(a, \delta_a) \cap B(x, \delta_x)$. Then

$$d(f(x), f(a)) \le d(f(x), f(q)) + d(f(q), f(a)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $a \in C(f)$. $\square$

Corollary 1. *(Sierpiński,* [4]*)(ZF) If* $f : (0,1) \to \mathbb{R}$ *and* $H(f) = (0,1)$ *then* $C(f) = (0,1)$*.*

Proof. It is sufficient to observe that $\mathbb{Q}$ is a dense subset of the real line $\mathbb{R}$ and that $AC_\omega(\mathbb{Q})$ holds, so we may apply directly the previous theorem. $\square$

The next result gives a negative solution of Problem from [2].

Corollary 2. *(ZF) Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is such that the set* $\mathbb{R} \setminus H(f)$ *is countable. Then* $C(f) = H(f)$*.*

Proof. Suppose that the set $A = \mathbb{R} \setminus H(f)$ is countable. Notice that the set $T = \{x + q : x \in A \wedge q \in \mathbb{Q}\}$ is countable, too. Let $d \in \mathbb{R} \setminus T$. Then the set $D = \{q + d : q \in \mathbb{Q}\}$ is a dense subset of $H(f)$ and obvious that the sentence $AC_\omega(D)$ holds. Hence we may apply Theorem 2 to the space $H(f)$ and we deduce that $C(f \restriction H(f)) = H(f)$. Consider now an arbitrary point $a \in H(f)$ and any positive real number ε. Then there exists a real number $\delta_1 > 0$ such that $(\forall x \in H(f))(|x - a| < \delta_1 \to |f(x) - f(a)| < \varepsilon)$. Since the sentence $AC_\omega(A \cup \{a\})$ holds, there exists $\delta_2 > 0$ such that $(\forall x \in A)(|x - a| < \delta_2 \to |f(x) - f(a)| < \varepsilon)$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $(\forall x \in \mathbb{R}))(|x - a| < \delta \to |f(x) - f(a)| < \varepsilon)$. Hence $a \in C(f)$. Therefore $H(f) \setminus C(f) = \emptyset$. $\square$

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