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Dualization of Van Douwen Diagram

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Introduction. This paper is a continuation of [1] where we present some results concerning the dualization of certain combinatorial properties of ω obtained by considering partitions of ω instead of its subsets. In the meantime the area of the investigations has enlarged. In this work we present the dualized version of the van Douwen diagram, i.e. the set of the relations between six cardinals related to simple properties of almost disjointness and almost containedness.

1. Preliminaries. We use standard set theoretic conventions and notation. By (ω) we denote the set of all disjoint partitions of ω. We will distinguish between infinite partitions denoted by $(\omega)^\omega$ and finite partitions denoted by $(\omega)^{<\omega}$. If X and Y are partitions of ω then we say that Y is contained in X, and we write $Y \leqq X$, if each piece of Y is a union of pieces of X. In [2], P. Matet noticed that $\langle(\omega), \leqq\rangle$ is a lattice with the least element $\mathbb{0}$ – the partition of ω into one piece and the greatest element $\mathbb{1}$ – the partition into singletons. With any partition $X \in (\omega)$ we can consider some finite modification of X, denoted by X_*, which is a partition obtained from X by gluing together a finite number of pieces of X. Of course we always have $X_* \leqq X$.

This concept allows to introduce main notions of relations of *almost containedness* and *almost orthogonality* defined as follows:

$$X \perp_* Y \text{ (} X \text{ is almost orthogonal to } Y \text{) iff } X \wedge Y \in (\omega)^{<\omega};$$

$$X \leqq_* Y \text{ (} X \text{ is almost contained in } Y \text{) iff for some finite modification } X_* \text{ of } X \text{, we have } X_* \leqq Y.$$

If X and Y are not almost orthogonal then we say that they are *compatible*.

Now we are able to define all cardinal numbers that occur in the dual van Douwen diagram.

Definition 1. (1). We say that $\mathscr{A} \subseteq (\omega)^\omega$ is a *maximal almost orthogonal* family

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of partitions (*mao*) if $\mathscr{A}$ is a maximal family of pairwise almost orthogonal partitions;

$$\mathfrak{a} = \min\{|\mathscr{A}| : \mathscr{A} \text{ is a } mao \text{ in } (\omega)^\omega\}\,.$$

(2). For $X, Y \in (\omega)^\omega$, we say that X *splits* Y, if there are $Z, T \in (\omega)^\omega$, $Z, T \leqq Y$, such that $Z \leqq X$ and $T \perp X$. We say that $\mathscr{S} \subseteq (\omega)^\omega$ *splits* $\mathscr{T} \subseteq (\omega)^\omega$ if for each $A \in \mathscr{T}$ there is some $S \in \mathscr{S}$ that splits A. $\mathscr{S} \subseteq (\omega)^\omega$ is a *splitting* family if $\mathscr{S}$ splits $(\omega)^\omega$

$$\mathfrak{s} = \min\{|\mathscr{S}| : \mathscr{S} \text{ is a splitting family}\}\,.$$

(3). We say that $\mathscr{R} \subseteq (\omega)^\omega$ is a *refining* family if for each $A \in (\omega)^\omega$ there is some $R \in \mathscr{R}$ such that either $R \leqq_* A$ or $R \perp_* A$;

$$\mathfrak{r} = \min\{|\mathscr{R}| : \mathscr{R} \text{ is a refining family}\}\,.$$

Notice that if $\mathscr{R}$ is not a refining family then there is a partition X that splits $\mathscr{R}$.

(4). A family $\mathbb{F}$ of *mao* families of partitions *shatters* a partition $A \in (\omega)^\omega$ if there are $\mathscr{F} \in \mathbb{F}$, and two distinct partitions $X, Y \in \mathscr{F}$ such that A is compatible with both X and Y. A family $\mathbb{F}$ of *mao* families of partitions is *shattering* if for each $A \in (\omega)^\omega$, $\mathbb{F}$ shatters A.

$$\mathfrak{h} = \min\{|\mathbb{F}| : \mathbb{F} \text{ is a shattering family of } maos\}\,.$$

In the classical van Douwen diagram there are two more cardinals, namely the coefficient $\mathfrak{p}$ which is the minimal cardinal number being the cardinality of a centered family without lower bounds and the *tower* coefficient $\mathfrak{t}$, i.e. the minimal cardinal number being the cardinality of a family well-ordered by the inverse almost containedness without lower bounds. By the strange Carlson's result (see [4]) both these coefficients for partitions are equal ω_1 in ZFC.

Proposition 1. *All the relations collected in the diagram can be proved in ZFC.*

$$\begin{array}{ccccccc}
 & & \omega_1 & \to & \mathfrak{a} & & \\
 & & & & & \searrow & \\
\omega_1 & \to & \mathfrak{h} & \to & \mathfrak{s} & \to & 2^\omega \\
 & & & & & \nearrow & \\
 & & \omega_1 & \to & \mathfrak{r} & &
\end{array}$$

Proof: The proof is a modification of the proof for the classical case.

2. The diagram. Let us list now some more facts which are not included in the diagram. We shall start with information concerning $\mathfrak{a}$.

(1). **MA** $\vdash \mathfrak{a} = 2^\omega$ (see [1]).

(2). $\mathfrak{a} \geqq \operatorname{cov}(\mathbb{K})$.

(3). $\min\{|\mathscr{A}| :$ *$\mathscr{A}$ is a mao family consisted of partitions having infinite pieces only*$\} = 2^\omega$.

Below we will assume **AMA** (Anti-Martins' Axiom). We are not going to give here precise formulation of it. One can think about it that for a class of some good compact Hausdorff spaces satysfying ccc there exist Lusin sets. There are few versions of the axiom (Cichoń, Fleissner) but the main idea is the same: to make all reasonably defined cardinals equal ω_1 independent on the size of continuum. In particular assuming **AMA** all the cardinals occuring in the classical van Douwen diagram are equal ω_1. In our case we have the following

(4). **AMA** $\vdash$ '*there is an* ω_1*-family* $\mathscr{A}$ *of pairwise orthogonal partitions such that every* $X \in (\omega)^\omega$ *having infinitely many finite pieces is compatible with some* $A \in \mathscr{A}$'.

So far we do not know any model in which $\mathfrak{a} < 2^\omega$ Having the fact (3), we may ask whether ZFC $\vdash \mathfrak{a} = 2^\omega$. On the other hand, the last fact may suggest negative answer.

Now let us have a closer look at numbers $\mathfrak{s}$ and $\mathfrak{r}$ which are actually dual to each other.

(5). **MA** $\vdash \mathfrak{r} = \mathfrak{s} = 2^\omega$.

(6). Con (ZFC + $\mathfrak{r} = \omega_1$ & $\mathfrak{s} \geqq \omega_2$.

(7). *For every* $\kappa \leqq 2^\omega$ *with* $\operatorname{cof}(\kappa) > \omega$, *we have* Con (ZFC + $\mathfrak{r} = \mathfrak{s} = \kappa$) *and* Con (ZFC + $\mathfrak{r} \geqq \kappa$ & $\mathfrak{s} \leqq \kappa$).

We still do not know any model with $\mathfrak{r} < \mathfrak{s}$. So there is a question: is this inequality consistent with **ZFC**?

We shall close the list with two facts concerning $\mathfrak{h}$.

(8). $\mathfrak{h}$ *is a regular cardinal* (*see* [1]).

(9). $\mathfrak{h} \leqq b = \min\{|B| : B$ *is an unbounded family in* $\omega^\omega\}$.

Since we do not know any model with $\mathfrak{h} \neq \omega_1$, we may ask if **ZFC** $\vdash \mathfrak{h} = \omega_1$.

References

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[2] MATET P., „Partitions and filters'', Journal of Symbolic Logic, 51 (1986), p. 12—21.