

ON IDEALS WITH PROJECTIVE BASES

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Abstract. A theorem concerning some descriptive properties of σ -ideals and generalizing the main result of [1] is proved. Various applications of this theorem are also presented.

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The article is concerned with those descriptive properties of a σ -ideal of sets, which are implied by the existence of a projective base for this ideal and are closely connected with the existence of nonmeasurable sets. The main result of the article (Theorem 1) generalizes the result of [1] and can successfully be applied to some questions of measure theory and set-theoretic topology (in this connection, cf. the references below and, especially, [2]).

The notation used throughout the paper is fairly standard.

As usual, we denote by ω the set of all natural numbers.

If X is any set, then $[X]^{\leq\omega}$ is the family of all countable subsets of X .

The cardinality of a set X is denoted by $\text{card}(X)$.

The set of all real numbers is denoted by R .

Recall that if X is an arbitrary set, then $P(X)$ is the family of all subsets of X .

If X is an arbitrary topological space, then $B(X)$ is the family of all Borel subsets of X . Respectively, the well-known classes of projective subsets of X are denoted by

$$\Sigma_1(X), \Pi_1(X), \Sigma_2(X), \Pi_2(X), \dots$$

It is also convenient to put $\Sigma_0(X) = \Pi_0(X) = B(X)$. As usual, we define

$$\Delta_n(X) = \Sigma_n(X) \cap \Pi_n(X).$$

A Σ_n -space is any Σ_n -subset of a Polish space X , equipped with the topology induced by the topology of X . In particular, according to this definition, a Suslin space is any Σ_1 -space.

If \mathcal{S} and \mathcal{T} are any two families of subsets of a given set X , then we denote by $\mathcal{S}\Delta\mathcal{T}$ the family of sets

$$\{Y\Delta Z : Y \in \mathcal{S} \ \& \ Z \in \mathcal{T}\},$$

where, as usual,

$$Y \Delta Z = (Y \setminus Z) \cup (Z \setminus Y).$$

It is clear that if \mathcal{S} is a σ -algebra of sets and \mathcal{I} is a σ -ideal of sets, then $\mathcal{S} \Delta \mathcal{I}$ is a σ -algebra of sets, too.

If X is a metric space and $x \in X$, then $B(x, \varepsilon)$ denotes the open ball with center x and radius ε .

Let X be a topological space and let \mathcal{I} be an ideal of subsets of X . We recall that \mathcal{I} has a Borel base if, for every set $Y \in \mathcal{I}$, there exists a set $Z \in \mathcal{I} \cap B(X)$ such that $Y \subset Z$. In an analogous manner, we say that \mathcal{I} has a Π_n -base if, for every set $Y \in \mathcal{I}$, there exists a set $Z \in \mathcal{I} \cap \Pi_n(X)$ such that $Y \subset Z$.

In this article we investigate those σ -ideals of subsets of Σ_n -spaces, which have Π_n -bases. We are especially interested in situations where such σ -ideals produce nonmeasurable (e.g., in the Lebesgue sense) subsets of an original space.

Notice first that if A is a subset of a Polish space X and

$$A \in \Pi_n(X) \setminus \Sigma_n(X),$$

then the σ -ideal generated by the family of sets $P(A) \cup [X \setminus A]^{\leq \omega}$ is a σ -ideal with some Π_n -base and, obviously, this ideal has no Σ_n -base. We now give a slightly more elaborated example.

Example 1. Let P be a nonempty perfect subset of the real line \mathbf{R} , consisting of linearly independent elements (over the field \mathbf{Q} of all rational numbers). It is well known that such a set P exists (in this connection, see, e.g., [3]). Let us fix a natural number $n > 0$ and consider an arbitrary subset A of P belonging to the class $\Pi_n(\mathbf{R}) \setminus \Sigma_n(\mathbf{R})$. It is also well known (see, for instance, [4]) that, for any two different real numbers t and q , we have

$$\text{card}((P + t) \cap (P + q)) \leq 1.$$

Now, let us consider the σ -ideal \mathcal{J} of subsets of the real line, generated by the family of all translates of the set A . Then \mathcal{J} is a σ -ideal invariant under the group of all translations of the real line. Evidently, \mathcal{J} has a Π_n -base. On the other hand, \mathcal{J} does not possess a Σ_n -base. To see this, suppose to the contrary that \mathcal{J} has a Σ_n -base. Then there exist a set $B \in \Sigma_n(\mathbf{R})$ and a sequence $\{t_n : n \in \omega\}$ of reals, such that

$$A \subset B \subset \cup\{A + t_n : n \in \omega\}.$$

From these inclusions we get

$$\text{card}((B \cap P) \setminus A) \leq \omega$$

and therefore we obtain $A \in \Sigma_n(\mathbf{R})$, which is impossible.

Moreover, let us remark that if a σ -ideal \mathcal{I} has a projective base, then, for some $n \in \omega$, it has also a Π_n -base. This fact is an immediate consequence of the following simple set-theoretical proposition:

Lemma 1. *Suppose that (X, \leq) is an upward σ -centered partially ordered set, B is a cofinal subset of X and suppose that $B = \cup\{B_n : n \in \omega\}$. Then, for some $n \in \omega$, the set B_n is also cofinal in X .*

We omit an easy proof of this proposition.

We shall say that a class of subsets of a Polish space has the perfect subset property if every uncountable set from this class contains a nonempty perfect subset. Let us recall that, in the theory **ZFC**, the classes Σ_0 and Σ_1 have the perfect subset property. Recall also that the statement

the class Π_1 has the perfect subset property

is independent of the theory **ZFC**. Moreover, it is known that, for each natural number $n > 0$, the theory

ZFC & Σ_n has the perfect subset property &
 Π_n has not the perfect subset property

is relatively consistent.

Let \mathcal{A} and \mathcal{S} be any two families of sets. We say that \mathcal{A} is \mathcal{S} -summable if, for every $\mathcal{A}' \subset \mathcal{A}$, we have $\cup \mathcal{A}' \in \mathcal{S}$ (cf. [2]).

A family \mathcal{A} of sets is called point-finite if $\{A \in \mathcal{A} : a \in A\}$ is finite for each point $a \in \cup \mathcal{A}$.

The following result can be regarded as a stronger version of the main theorem from [1].

Theorem 1. *Suppose that the class Σ_n has the perfect subset property. Let X be an arbitrary Σ_n -space and let \mathcal{I} be a σ -ideal of subsets of X with a Π_n -base. Suppose also that:*

- 1) \mathcal{A} is a point-finite family of sets;
- 2) \mathcal{A} is a $(\Sigma_n(X) \Delta \mathcal{I})$ -summable family.

Then there exists a family $\mathcal{B} \in [\mathcal{A}]^{\leq \omega}$ such that $(\cup \mathcal{A} \setminus \cup \mathcal{B}) \in \mathcal{I}$.

Proof. Without loss of generality we may assume that X is an uncountable Σ_n -space and the cardinality of the given family \mathcal{A} is less than or equal to the cardinality continuum. Let T be a subset of X which contains no perfect subset and satisfies the equality $\text{card}(T) = \text{card}(\mathcal{A})$ (note that the set T can be realized as a subset of some Bernstein set in the space X). Obviously, we may identify the set T with the set of indices of the given family \mathcal{A} . In other words, we can write $\mathcal{A} = \{A_t : t \in T\}$. Furthermore, we put

$$\Gamma = \left\{ (x, t) \in X \times T : x \in A_t \right\}.$$

Let D be a countable dense subset of X . It is easy to check that

$$\Gamma = \bigcap_{k \in \omega} \bigcup_{d \in D} \left(\Gamma^{-1} \left(B \left(d, \frac{1}{k+1} \right) \right) \times B \left(d, \frac{1}{k+1} \right) \right).$$

For any $k \in \omega$ and for any $d \in D$, let $S_{k,d} \in \Sigma_n(X)$ and $A_{k,d} \in \mathcal{I}$ be subsets of X such that

$$\Gamma^{-1}\left(B\left(d, \frac{1}{k+1}\right)\right) = S_{k,d} \Delta A_{k,d}.$$

Define

$$A = \bigcup_{k \in \omega} \bigcup_{d \in D} A_{k,d}$$

and, taking into account that $A \in \mathcal{I}$, fix some Π_n -set $A' \in \mathcal{I}$ such that $A \subset A'$. Then the set

$$\Gamma' = \Gamma \cap ((X \setminus A') \times X)$$

is a Σ_n -subset of the product space $X \times X$. Hence, the set

$$T_1 = \{t \in T : (\exists x)((x, t) \in \Gamma')\}$$

is a Σ_n -subset of the set T , too. Consequently, T_1 must be countable. Thus we obtain

$$\bigcup_{t \in T_1} A_t \supset \bigcup_{t \in T} A_t \setminus A'.$$

In virtue of the relation $A' \in \mathcal{I}$, this completes the proof. \square

Remark 1. Suppose that $\Gamma \subset A \times B$ is a relation with finite vertical sections, i.e.,

$$\text{card}(\{y : (x, y) \in \Gamma\}) < \omega$$

for each $x \in A$. Let $\{V_n : n \in \omega\}$ be a countable family of subsets of B , which separates the points in B , i.e., for any two distinct points $a, b \in B$, there exists $n \in \omega$ such that $\text{card}(V_n \cap \{a, b\}) = 1$. Then we have

$$\Gamma = \bigcup_{f \in 2^\omega} \bigcap_n \Gamma^{-1}(V_0^{f(0)} \cap \dots \cap V_n^{f(n)}) \times (V_0^{f(0)} \cap \dots \cap V_n^{f(n)}),$$

where $V^0 = V$ and $V^1 = B \setminus V$ for each set $V \subset B$. This fact enables us to develop some analogues of the above theorem for a wider class of topological spaces. In this connection, recall that a typical example of a non-separable Banach space with a countable family of Borel sets, separating the points, is the classical space l^∞ consisting of all bounded real-valued sequences.

It is possible to apply directly Theorem 1 to the family of all analytic subsets of a Polish space X (put $\mathcal{I} = \{\emptyset\}$), to the σ -algebra of subsets of X with the Baire property, to the σ -algebra of Lebesgue measurable subsets of the real line, and so on. We shall give now some other applications of this theorem.

Theorem 2. *Suppose that the class Σ_n has the perfect subset property. Let \mathcal{A} be an uncountable family of nonempty pairwise disjoint Σ_n -sets. Then there exists a subfamily \mathcal{C} of \mathcal{A} such that $\cup \mathcal{C}$ is not a Σ_n -set.*

Proof. Indeed, let us put $X = \cup \mathcal{A}$. If X is not a Σ_n -space, then there is nothing to prove. Assume now that X is a Σ_n -space. Then we can consider the given family \mathcal{A} with the σ -ideal

$$\mathcal{I} = [X]^{\leq \omega}.$$

Applying Theorem 1 to \mathcal{A} and \mathcal{I} , we easily get the required result. \square

Theorem 3. *Suppose that X is a Polish space and \mathcal{I} is a σ -ideal of subsets of X with a Borel base. Suppose also that $\mathcal{A} \subset \mathcal{I}$ is a point-finite family of sets, such that $\cup \mathcal{A} = X$. Then there exists a subfamily \mathcal{C} of \mathcal{A} for which*

$$\cup \mathcal{C} \notin B(X) \Delta \mathcal{I}.$$

Proof. Indeed, since the class $B(X) = \Sigma_0(X) = \Pi_0(X)$ has the perfect subset property, we may directly apply Theorem 1 to the family \mathcal{A} and to the ideal \mathcal{I} . \square

Let us consider two immediate consequences of Theorem 3.

Example 2. Let X be a Polish space and let \mathcal{I} denote the σ -ideal of all first category subsets of X . Suppose also that $\mathcal{A} \subset \mathcal{I}$ is a point-finite covering of X . Then, according to Theorem 3, there exists a family $\mathcal{C} \subset \mathcal{A}$ such that the set $\cup \mathcal{C}$ does not have the Baire property.

Example 3. Let X be a Polish space and let μ be a nonzero σ -finite Borel measure on X . Denote by μ' the completion of μ and let \mathcal{I} be the σ -ideal of all μ' -measure zero subsets of X . Suppose also that $\mathcal{A} \subset \mathcal{I}$ is a point-finite covering of X . Then, in view of Theorem 3, there exists a family $\mathcal{C} \subset \mathcal{A}$ such that the set $\cup \mathcal{C}$ is not measurable with respect to μ' .

Now, for a given family \mathcal{S} of sets, we define

$$\mathcal{S}^- = \{Z : (\forall Z' \subset Z)(Z' \in \mathcal{S})\}.$$

Using this notation, we can formulate the following result.

Theorem 4. *If X is a Polish space and \mathcal{I} is a σ -ideal of subsets of X covering X and possessing a Borel base, then*

$$(B(X) \Delta \mathcal{I})^- = \mathcal{I}.$$

This result easily follows from Theorem 3 and seems to be natural.

In order to present some other consequences of Theorem 1, we need to recall the concept of a discrete family of sets (in a topological space). Since we deal here only with metrizable topological spaces, this concept will be introduced for metric spaces (see, e.g., [5]).

A family \mathcal{F} of subsets of a metric space E is called discrete if there exists a nonzero $m \in \omega$ such that

$$(\forall F_1, F_2 \in \mathcal{F}) \left(F_1 \neq F_2 \Rightarrow \text{dist}(F_1, F_2) > \frac{1}{m} \right).$$

Notice that if \mathcal{F} is a discrete family of closed sets and $\mathcal{S} \subset \mathcal{F}$, then $\cup \mathcal{S}$ is a closed set, too. Moreover, if $Z \subset E$ is compact, then the family

$$\{Y \in \mathcal{F} : Y \cap Z \neq \emptyset\}$$

is finite.

In our further considerations we need the following corollary from the well-known Montgomery Lemma (see [6]).

Lemma 2. *Let E be an arbitrary metric space and let \mathcal{S} be any family of open sets in E , such that $\cup \mathcal{S} = E$. Then there exists a sequence $\{\mathcal{F}_n : n \in \omega\}$ of discrete families of closed subsets of E , satisfying the relations:*

- 1) $(\forall n \in \omega)(\forall F \in \mathcal{F}_n)(\exists U \in \mathcal{S})(F \subset U)$;
- 2) $\cup_{n \in \omega}(\cup \mathcal{F}_n) = E$.

The proof of this lemma is presented, e.g., in [6] and [7].

Remark 2. For any infinite cardinal number τ , let us consider the topological sum of the family of spaces $\{[0, 1] \times \{\xi\} : \xi < \tau\}$. Let us identify in this sum all points $(0, \xi)$, where $\xi < \tau$, and denote the obtained space by E_τ . It is well known that any metric space E with weight τ can be embedded into the countable product of copies of E_τ (see, e.g., [5]). From this fact, the previous auxiliary proposition (i.e., Lemma 2) can be deduced without using the Montgomery Lemma.

Let E be a metric space. As usual, we denote by $\text{Comp}(E)$ the family of all nonempty compact subsets of E and we equip this family with the Vietoris topology (see, e.g., [5]). Further, let \mathcal{S} be a family of subsets of a given set X , closed under countable unions and countable intersections.

We shall say that a mapping

$$\Phi : X \rightarrow \text{Comp}(E)$$

is upper \mathcal{S} -measurable if, for every open set $Y \subset E$, the relation

$$\{x \in X : \Phi(x) \cap Y \neq \emptyset\} \in \mathcal{S}$$

is fulfilled. It can easily be checked that, in our case, a mapping Φ is upper \mathcal{S} -measurable if and only if it is lower \mathcal{S} -measurable, i.e., for every closed set $Z \subset E$, we have

$$\{x \in X : \Phi(x) \cap Z \neq \emptyset\} \in \mathcal{S}.$$

Of course, it is essential here that, for each point $x \in X$, the set $\Phi(x)$ is compact and nonempty.

Lemma 3. *Let the class Σ_n have the perfect subset property, let X be an arbitrary Σ_n -space and let \mathcal{I} be a σ -ideal of subsets of X with a Π_n -base. Suppose also that E is a metric space and*

$$\Phi : X \rightarrow \text{Comp}(E)$$

is a lower $(\Sigma_n(X)\Delta\mathcal{I})$ -measurable mapping. Finally, let \mathcal{F} be a discrete family of closed subsets of E . Then there exist a countable subfamily \mathcal{F}' of \mathcal{F} and a set $A \in \mathcal{I}$, such that

$$(\forall x \in X \setminus A)(\Phi(x) \cap (\cup\mathcal{F}) \neq \emptyset \Rightarrow \Phi(x) \cap (\cup\mathcal{F}') \neq \emptyset).$$

Proof. For each set $Z \in \mathcal{F}$, we put

$$A_Z = \{x \in X : \Phi(x) \cap Z \neq \emptyset\}.$$

The compactness of all values of the mapping Φ implies that $\{A_Z : Z \in \mathcal{F}\}$ is a point-finite family. Furthermore, the discreteness of the family \mathcal{F} implies that $\{A_Z : Z \in \mathcal{F}\}$ is a $(\Sigma_n(X)\Delta\mathcal{I})$ -summable family. Hence, by Theorem 1, there exist a countable subfamily \mathcal{F}' of \mathcal{F} and a set $A \in \mathcal{I}$, such that

$$\cup\{A_Z : Z \in \mathcal{F}'\} \cup A = \cup\{A_Z : Z \in \mathcal{F}\}.$$

Suppose now that $x \in X \setminus A$ and that $\Phi(x) \cap (\cup\mathcal{F}) \neq \emptyset$. Then we have

$$x \in \cup\{A_Z : Z \in \mathcal{F}\}.$$

Consequently, $x \in \cup\{A_Z : Z \in \mathcal{F}'\}$ and, therefore,

$$\Phi(x) \cap (\cup\mathcal{F}') \neq \emptyset.$$

This completes the proof. \square

Lemma 4. *Let the class Σ_n have the perfect subset property, let X be an arbitrary Σ_n -space and let \mathcal{I} be a σ -ideal of subsets of X with a Π_n -base. Suppose also that E is a metric space and*

$$\Phi : X \rightarrow \text{Comp}(E)$$

is a lower $(\Sigma_n(X)\Delta\mathcal{I})$ -measurable mapping. Finally, let \mathcal{F} be a discrete family of closed subsets of E . Then there exist a countable subfamily \mathcal{F}' of \mathcal{F} and a set $A \in \mathcal{I}$, such that

$$(\forall x \in X \setminus A)(\forall Z \in \mathcal{F})(\Phi(x) \cap Z \neq \emptyset \Rightarrow Z \in \mathcal{F}').$$

Proof. We use Lemma 3 and define three sequences

$$(A_n)_{n \in \omega}, (B_n)_{n \in \omega}, (\mathcal{F}_n)_{n \in \omega}$$

satisfying the following properties:

- (1) $A_0 = \{x \in X : \Phi(x) \cap (\cup\mathcal{F}) \neq \emptyset\}$;
- (2) $A_0 \supset B_0 \supset A_1 \supset B_1 \supset \dots$;
- (3) $\mathcal{F}_n \cap \mathcal{F}_m = \emptyset$ for all distinct $n, m \in \omega$;
- (4) $\mathcal{F}_n \in [\mathcal{F}]^{\leq \omega}$ for each $n \in \omega$;
- (5) $A_n \setminus B_n \in \mathcal{I}$ for each $n \in \omega$;
- (6) if $x \in B_n$, then $\Phi(x) \cap (\cup\mathcal{F}_n) \neq \emptyset$;
- (7) for any $n \in \omega$, we have

$$A_{n+1} = \{x \in B_n : \Phi(x) \cap (\cup(\mathcal{F} \setminus (\mathcal{F}_0 \cup \dots \cup \mathcal{F}_n))) \neq \emptyset\}.$$

Observe that

$$\bigcap_{n \in \omega} A_n = \bigcap_{n \in \omega} B_n = \emptyset.$$

Indeed, if $x \in \bigcap_{n \in \omega} A_n = \bigcap_{n \in \omega} B_n$, then the set

$$\{Z \in \mathcal{F} : \Phi(x) \cap Z \neq \emptyset\}$$

must be infinite. But this is impossible, since $\Phi(x)$ is a compact set. Hence, $\bigcap_{n \in \omega} A_n = \emptyset$. Let us put

$$\begin{aligned} A &= \bigcup_{n \in \omega} (A_n \setminus B_n), \\ \mathcal{F}' &= \bigcup_{n \in \omega} \mathcal{F}_n. \end{aligned}$$

Notice that $A \in \mathcal{I}$. If $x \in X \setminus A$, $Z \in \mathcal{F}$ and $\Phi(x) \cap Z \neq \emptyset$, then we have

$$x \in (B_0 \setminus A_1) \cup (B_1 \setminus A_2) \cup \dots$$

This yields $x \in B_n \setminus A_{n+1}$ for some $n \in \omega$. Therefore, $x \in B_n$ and $x \notin A_{n+1}$. The last two relations imply at once that if $Z \in \mathcal{F}$ and $\Phi(x) \cap Z \neq \emptyset$, then $Z \in \mathcal{F}'$. Thus, the lemma is proved. \square

Lemma 5. *Let the class Σ_n have the perfect subset property, let X be an arbitrary Σ_n -space and let \mathcal{I} be a σ -ideal of subsets of X with a Π_n -base. Suppose also that E is a metric space and*

$$\Phi : X \rightarrow \text{Comp}(E)$$

is a lower $(\Sigma_n(X) \Delta \mathcal{I})$ -measurable mapping. Then there exist a set $A \in \mathcal{I}$ and a closed separable subset F of E , such that

$$(\forall x \in X \setminus A)(\Phi(x) \subset F).$$

Proof. Applying Lemma 2, we can find a double sequence

$$(\mathcal{F}_{n,m})_{n \in \omega, m \in \omega}$$

of families of closed subsets of the space E , such that:

- 1) $(\forall n \in \omega)(\forall m \in \omega)(\forall Z \in \mathcal{F}_{n,m})(\text{diam}(Z) < \frac{1}{1+m})$;
- 2) $(\forall m \in \omega)(\bigcup_{n \in \omega} (\bigcup \mathcal{F}_{n,m}) = E)$;
- 3) $(\forall n \in \omega)(\forall m \in \omega)(\mathcal{F}_{n,m}$ is discrete).

Next, we use Lemma 4 and, for any $n, m \in \omega$, we can find a set $A_{n,m}$ and a family $\mathcal{H}_{n,m}$, such that:

- a) $\mathcal{H}_{n,m} \in [\mathcal{F}_{n,m}]^{\leq \omega}$;
- b) $A_{n,m} \in \mathcal{I}$;
- c) $(\forall x \in X \setminus A_{n,m})(\forall Z \in \mathcal{F}_{n,m})(\Phi(x) \cap Z \neq \emptyset \Rightarrow Z \in \mathcal{H}_{n,m})$.

Let us consider the subspace

$$F = \text{cl} \left(\bigcap_{m \in \omega} \left(\bigcup_{n \in \omega} (\bigcup \mathcal{H}_{n,m}) \right) \right)$$

of the space E and let us put

$$A = \bigcup_{n,m \in \omega} A_{n,m}.$$

Then it is easy to check that F and A are the required sets, which completes the proof. \square

Now, we are able to formulate and prove several consequences of the preceding results, concerning some kinds of measurable functions and selectors.

Theorem 5. *Suppose again that the class Σ_n has the perfect subset property. Let X be an arbitrary Σ_n -space and let \mathcal{I} be a σ -ideal of subsets of X with a Π_n -base. If E is a metric space and*

$$f : X \rightarrow E$$

is a $(\Sigma_n(X) \Delta \mathcal{I})$ -measurable mapping, then there exists a set $A \in \mathcal{I}$ such that $f(X \setminus A)$ is a separable subspace of E .

Proof. Let us consider the mapping

$$\Phi(x) = \{f(x)\} \quad (x \in X).$$

Then we obviously have

$$\Phi : X \rightarrow \text{Comp}(E)$$

and it is easy to see that Φ is lower $(\Sigma_n(X) \Delta \mathcal{I})$ -measurable. Hence, we may utilize Lemma 5 to the mapping Φ . In this way we get the desired result. \square

Evidently, we can also apply Theorem 5 to the class Σ_1 and to the ideal $\mathcal{I} = \{\emptyset\}$. Then we obtain the following theorem due to Frolik (see [8]).

Theorem 6. *Suppose that X is a Suslin space, E is a metric space and*

$$f : X \rightarrow E$$

is a Borel mapping. Then the range of f is a separable subspace of E .

Let us notice that this result can also be easily deduced from Theorem 2.

Remark 3. Let us consider the topology \mathcal{T} on the real line R generated by the family

$$\{U \setminus T : U \text{ is open \& } T \text{ is countable}\}.$$

Then the function $f : R \rightarrow R$ given by the formula $f(x) = x$ and treated as a mapping from the real line equipped with the standard topology into the real line equipped with the topology \mathcal{T} , is a Borel isomorphism, but the range of f is nonseparable. This simple example shows us that, in Theorem 6, the assumption of metrizability of E is rather essential.

Theorem 7. *Suppose that X is a Polish space, E is a metric space and*

$$f : E \rightarrow X$$

is a mapping satisfying the following conditions:

- a) $(\forall x \in X)(f^{-1}(\{x\}) \in \text{Comp}(E) \cup \{\emptyset\})$;
- b) *for each closed set $Z \subset E$, we have $f(Z) \in \Sigma_1(X)$.*

Then E is a separable space.

Proof. Indeed, let us define

$$\Phi(x) = f^{-1}(x) \quad (x \in f(E))$$

and put $\mathcal{I} = \{\emptyset\}$. Then a straightforward application of Lemma 5 to Φ and \mathcal{I} yields the desired result. \square

Let \mathcal{S} be a family of subsets of a given set X , let E be another set and let $f : X \rightarrow E$. We shall say that f is an \mathcal{S} -step function (with the values in the set E) if there exist a partition $\{A_n : n \in \omega\}$ of X into sets from \mathcal{S} and a sequence $\{e_n : n \in \omega\}$ of elements from E , such that

$$f = \cup_{n \in \omega} (A_n \times \{e_n\}).$$

Of course, here the function f is identified with its graph.

Suppose that the class Σ_n has the perfect subset property. Let X be a Σ_n -space, \mathcal{I} be a σ -ideal of subsets of X with a Π_n -base, E be an arbitrary metric space and let $f : X \rightarrow E$. Then the following two conditions are equivalent:

- a) f is a $(\Delta_n(X)\Delta\mathcal{I})$ -measurable function;
- b) f is \mathcal{I} -almost equal to a pointwise limit of a sequence of $(\Delta_n(X)\Delta\mathcal{I})$ -step functions.

This equivalence directly follows from Theorem 5. The next statement can be obtained analogously.

Theorem 8. *Suppose again that the class Σ_n has the perfect subset property. Let X be a Σ_n -space, \mathcal{I} be a σ -ideal of subsets of X with a Π_n -base, E be any metrizable topological group and let*

$$f : X \rightarrow E, \quad g : X \rightarrow E$$

be any two $(\Delta_n(X)\Delta\mathcal{I})$ -measurable functions. Then the sum $f + g$ is a $(\Delta_n(X)\Delta\mathcal{I})$ -measurable function, too.

Proof. It suffices to reduce Theorem 8 to the case where E is a separable metrizable topological group. But this can easily be done with the aid of Theorem 5 (we can also directly apply the equivalence of the conditions a) and b) above). \square

Theorem 9. *Suppose again that the class Σ_n has the perfect subset property. Let X be a Σ_n -space and let \mathcal{I} be a σ -ideal of subsets of X with a Π_n -base. Suppose also that E is a metric space and let*

$$\Phi : X \rightarrow \text{Comp}(E)$$

be a lower $(\Delta_n(X)\Delta\mathcal{I})$ -measurable mapping. Then there exists a $(\Delta_n(X)\Delta\mathcal{I})$ -measurable selector of Φ .

Proof. According to Lemma 5, there are a Π_n -set $A \in \mathcal{I}$ and a closed separable subspace F of E , such that

$$(\forall x \in X \setminus A)(\Phi(x) \subset F).$$

Let F' denote the completion of the metric space F . Then F' is a Polish space and $\Phi(x)$ is a nonempty compact subset of F' for each $x \in X \setminus A$. Hence, we may apply the classical theorem on measurable selectors (due to Kuratowski and Ryll–Nardzewski [9]) to the restriction $\Phi|(X \setminus A)$ and to the σ -algebra $(\Delta_n(X)\Delta\mathcal{I}) \cap P(X \setminus A)$. In view of this theorem, there exists a $((\Delta_n(X)\Delta\mathcal{I}) \cap P(X \setminus A))$ -measurable selector of $\Phi|(X \setminus A)$. Then it is not hard to see that a suitable extension of this selector gives us a $(\Delta_n(X)\Delta\mathcal{I})$ -measurable selector of the original mapping Φ . \square

Remark 4. A connection between the notion of summability and the existence of measurable selectors is thoroughly investigated in the well-known monograph [2]. In this monograph the paracompactness of any metric space is utilized instead of the Montgomery Lemma, and the further argument is essentially based on the assumption that a given σ -ideal satisfies the countable chain condition. In this respect, it is reasonable to recall here that there are many natural examples of σ -ideals with a Borel base which do not satisfy the countable chain condition.

Note also that some related questions concerning the existence of measurable (or nonmeasurable) selectors for a given partition of the real line are discussed in [10].

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