Hamel-isomorphic images of the unit ball

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Abstract. In this article we consider linear isomorphisms over the field of rational numbers between linear spaces $\mathbb{R}^2$ and $\mathbb{R}$. We prove that if f is such an isomorphism, then the image by f of the unit disk is a strictly non-measurable subset of the real line, which is strongly non-similar to any classical non-measurable subset of reals. We also show the consistency and independence of the proposition that all images of bounded measurable subsets of the plane via a such mapping are non-measurable.

1. Introduction

Let us recall a well known theorem essentially due to S. Banach. *Let* $(G, +)$ *and* $(H, +)$ *be locally compact Polish groups (not necessarily abelian). If* $f : G \longrightarrow H$ *is a homomorphism, which is Haar measurable or has the Baire property, then* f *is continuous*. The proof follows immediately from well known theorem of H. Steinhaus. Indeed, if f has one of the above properties, then there exists a "massive" set $A \subseteq G$ such that $f \restriction A$ is continuous. Then $f \restriction A - A$ is also continuous and by that theorem of Steinhaus $A - A$ contains a neighborhood of unity. Thus f is continuous everywhere.

Let $\mathbb{R}$ denotes the real line. Let us say that $X \subseteq \mathbb{R}^m$ is *strictly non-measurable* if the inner measures of X and of $\mathbb{R}^m \setminus X$ both vanish. And X is *strictly non-Baire* if all Borel sets included in X or in $\mathbb{R}^m \setminus X$ are meager (i.e. of the first category).

Consider the case when $G = \mathbb{R}^m$, $H = \mathbb{R}$ and f is discontinuous. Then if I is a non-degenerated interval, then $f^{-1}[I]$ is strictly non-measurable and strictly non-Baire. This was shown by A. Ostrowski and M. Kuczma (see [**13**], [**8**]).

All that suggested to us a study of images of sufficiently regular subsets of $\mathbb{R}^2$ in the case when f is an isomorphism of $\mathbb{R}^2$ onto $\mathbb{R}$.

Recall that all the spaces $\mathbb{R}^m$ $(m > 0)$ viewed as linear spaces over the field $\mathbb{Q}$ of rational numbers are isomorphic (all have Hamel bases of the same power $\mathfrak{c}$). Let f be an isomorphism of $\mathbb{R}^2$ onto $\mathbb{R}$. We shall prove that if D is a disk of positive radius, then $f[D]$ is non-measurable and lacks the Baire property. Moreover, 1. *The image* $f[D]$ *is strictly non-measurable and strictly non-Baire*; 2. *The following proposition is consistent in* ZFC*:* $(\star)$ *For all isomorphisms* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *and all bounded measurable sets* $A \subseteq \mathbb{R}^2$ *of positive measure (or containing a non-meager Borel set),* $f[A]$ *is not measurable (or non-Baire).* I. Recław has proved that if

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we assume the Continuum Hypothesis (CH), then one can prove the negation of ($\star$). This follows from his theorem (a proof will be included below with his kind permission): 3. *(Assuming CH). Let $A \cup B = \mathbb{R}^2$ be such that A is meager and B is of measure zero (thus both A and B are Lebesgue measurable and have the Baire property). Then there exist isomorphisms f, $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $f[A]$ is of measure zero and $g[B]$ is meager.*

Remarks: 1. *There exist isomorphisms $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that both sets $f[\mathbb{R} \times \{0\}]$ and $f[\{0\} \times \mathbb{R}]$ are measurable.* This follows from Theorem 5 of P. Erdös, K. Kunen, and R. Mauldin [**6**]. 2. *There exist isomorphisms $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such that no image of a bounded set in $\mathbb{R}^3$ is strictly non-mesurable nor strictly non-Baire in $\mathbb{R}^2$.* Indeed, if $f(x, y, z) = (g(x, y), z)$, where $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is an isomorphism, then images of bounded sets are not everywhere dense in $\mathbb{R}^2$.

2. Concepts and Notations

We use the standard set theoretical notations. The set of natural numbers is denoted by ω. We identify the set ω with the first infinite cardinal number. We shall denote by $\mathfrak{c}$ the cardinal number continuum. The cardinality of a set A is denoted by $|A|$. If $f : A \longrightarrow B$ and $X \subseteq A$, then we shall denote by $f[X]$ the image of X by f. The complement of a set A to a fixed space is denoted by A^c. The power set of a set A is denoted by $\mathcal{P}(A)$. We denote by ZFC the Zermelo-Fraenkel set theory with the Axiom of Choice.

Let $(G, +)$ be a group. For the time being, we shall not assume that $+$ is an abelian operation. If A, B are subsets of G, then we mean by the algebraic sum $A + B$ the set $\{a + b : a \in A \,\&\, b \in B\}$ and by $A - B$ the set $\{a - b : a \in A \,\&\, b \in B\}$. We write $A + b$ instead of $A + \{b\}$ if $b \in G$.

If $A \subseteq \mathbb{R}^n$, then we denote by $Span(A)$ the set of all elements of the form $q_0 \cdot a_0 + \ldots + q_n \cdot a_n$, where n is an arbitrary natural number, $q_0, \ldots, q_n \in \mathbb{Q}$ and $a_0, \ldots, a_n \in A$.

Let E be a topological space. Let $Int(A)$ denote the interior of a set $A \subseteq E$. We denote the family of all Borel subsets of the space E by $Bor(E)$, the σ-ideal of first category subsets of E by $\mathcal{B}_0(E)$, and the family of all subsets of E with the Baire propety by $\mathcal{B}(E)$. If (X, d) is a metric space, $a \in X$ and $\varepsilon > 0$, then we denote the open ball with center a and radius ε by $B(a, \varepsilon)$. We denote the n-dimensional Lebesgue measure on the space $\mathbb{R}^n$ by λ^n, the family of all Lebesgue measurable subsets of $\mathbb{R}^n$ by $\mathcal{M}(\mathbb{R}^n)$, and the σ-ideal of Lebesgue measure zero subsets of $\mathbb{R}^n$ by $\mathcal{M}_0(\mathbb{R}^n)$. We simplify notations in the case of the real line. For example, λ denotes λ^1, $\mathcal{M}_0$ denotes the ideal $\mathcal{M}_0(\mathbb{R})$, $\mathcal{B}_0$ denotes $\mathcal{B}_0(\mathbb{R})$, Bor denotes $Bor(\mathbb{R})$, and so on.

In 1920, H. Steinhaus showed (see [**18**]) that if $A, B \subseteq \mathbb{R}$ are Lebesgue measurable and $\lambda(A) > 0$, $\lambda(B) > 0$, then $Int(A - B) \neq \emptyset$. A similar fact for the Baire property is also well known (see e.g. [**14**]). We shall use the following two generalisations of these results:

Theorem 2.1. *(McShane, see* [**11**]*) Let $(G, +)$ be a topological group and let $A, B \subseteq G$ be sets of second category such that one of them has the Baire property. Then $Int(A - B) \neq \emptyset$.*

THEOREM 2.2. *(Beck, Corson, Simon, see* [**1**]*) Let* $(G, +)$ *be a locally compact topological group with completed Haar measure and let* $A, B \subseteq G$ *have positive outer measures and one of them is measurable. Then* $Int(A - B) \neq \emptyset$.

Let $\mathcal{J}$ be an ideal of subsets of a topological space E. We say that the ideal $\mathcal{J}$ *has a Borel base* if for each $A \in \mathcal{J}$ there exists $B \in Bor(E) \cap \mathcal{J}$ such that $A \subseteq B$. Notice that both ideals $\mathcal{B}_0$ and $\mathcal{M}_0$ have Borel bases.

A field $\mathcal{S}$ of subsets of a set X is called *nontrivial* if $\mathcal{S} \neq \{\emptyset, X\}$ and $\mathcal{S} \neq \mathcal{P}(X)$. An ideal $\mathcal{J}$ of subsets of a set X is called *nontrivial* if $\mathcal{J} \neq \{\emptyset\}$ and $\bigcup \mathcal{J} \notin \mathcal{J}$.

Suppose that $\mathcal{S}$ is a σ-field of subsets of a set X and that $\mathcal{J}$ is a σ-ideal of subsets of X. Then we denote the smallest σ-field containing $\mathcal{S} \cup \mathcal{J}$ by $\mathcal{S}(\mathcal{J})$. It is well known that $\mathcal{S}(\mathcal{J}) = \{A \triangle B : A \in \mathcal{S} \,\&\, B \in \mathcal{J}\}$. Let us notice that $Bor(\mathcal{M}_0) = \mathcal{M}$ and $Bor(\mathcal{B}_0) = \mathcal{B}$.

Let $\mathcal{S}$ be a σ-field of subsets of X. We define $\mathcal{S}^- = \{A \subseteq X : \mathcal{P}(A) \subseteq \mathcal{S}\}$. Then $\mathcal{S}^-$ is a σ-ideal and $\mathcal{S}^- \subseteq \mathcal{S}$. It follows easily from [**3**] that if $\mathcal{J}$ is a σ-ideal of subsets of a Polish topological space E with a Borel base, then $Bor(E)(\mathcal{J})^- = \mathcal{J}$. Therefore $\mathcal{M}^- = \mathcal{M}_0$ and $\mathcal{B}^- = \mathcal{B}_0$.

From now on, $(G, +)$ means an abelian group. If $\mathcal{F}$ is a family of subsets of a group $(G, +)$, then we say that $\mathcal{F}$ is *invariant* if $A + g \in \mathcal{F}$ for each $A \in \mathcal{F}$ and $g \in G$. Notice that if $\mathcal{S}$ is an invariant σ-field of subsets of G, then S^- is an invariant σ-ideal of subsets of the set G.

Let $\mathcal{S}$ be a field of subsets of a set X and let $A \subseteq X$. Then we say that A is *$\mathcal{S}$-non-measurable* if $A \notin \mathcal{S}$. The set A is *downward $\mathcal{S}$-null* if $\mathcal{P}(A) \cap \mathcal{S} \subseteq \mathcal{S}^-$. We say that A is *upward $\mathcal{S}$-full* if $\mathcal{P}(A^c) \cap \mathcal{S} \subseteq \mathcal{S}^-$. Finally, we say that A is *strictly $\mathcal{S}$-non-measurable* if A is downward $\mathcal{S}$-null and upward $\mathcal{S}$-full.

DEFINITION 2.1. *Let $\mathcal{S}$ be a field of subsets of a topological group $(G, +)$. Then we say that $\mathcal{S}$ has the strong Steinhaus property if $Int(A - B) \neq \emptyset$ for each $A, B \subseteq G$ such that $A \in \mathcal{S} \setminus \mathcal{S}^-$ and $B \notin \mathcal{S}^-$.*

Therefore, by Theorems 2.1 and 2.2, the σ-fields $\mathcal{M}$ and $\mathcal{B}$ have the strong Steinhaus property.

3. Main results

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be *additive* if it satisfies the Cauchy equation

$$f(x + y) = f(x) + f(y)$$

for each $x, y \in \mathbb{R}^n$. We begin our considerations with some preliminary well known facts. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an additive function, then the following three properties of the function f are equivalent: 1. f is continuous; 2. f is continuous in zero; 3. there exists a nonempty open ball $B \subseteq \mathbb{R}^n$ such that $f[B]$ is a bounded set.

We shall consider the finite dimensional spaces $\mathbb{R}^n$ as linear spaces over the field of rational numbers. Notice that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is additive if and only if f is linear over $\mathbb{Q}$. Recall that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an isomorphism and $n \neq m$, then f is not continuous. Finally, it is well known and easy to show that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a discontinuous additive function, then the graph of f is a dense subset of the topological space $\mathbb{R}^{n+1}$. Therefore $f[U]$ is a dense subset of $\mathbb{R}$ for each nonempty open subset U of the space $\mathbb{R}^n$.

We shall call a function briefly *isomorphism* if it is a linear isomorphism over the field of rational numbers.

THEOREM 3.1. *Let $\mathcal{S}$ be a nontrivial, invariant σ-field of subsets of the group $(\mathbb{R},+)$ with strong Steinhaus property containing all finite sets. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an isomorphism and let A be a bounded subset of $\mathbb{R}^2$ such that countably many translations of A cover $\mathbb{R}^2$. Then the image $f[A]$ is $\mathcal{S}$-non-measurable.*

PROOF. Suppose that $f[A] \in \mathcal{S}$ and that T is a countable subset of $\mathbb{R}^2$ such that $A + T = \mathbb{R}^2$. Notice that $f[A+T] = f[A] + f[T]$ and that $\mathcal{S}^-$ is a nontrivial invariant σ-ideal. Therefore $f[A] \notin \mathcal{S}^-$. By the strong Steinhaus property of the σ-field $\mathcal{S}$ there exists a nonempty open interval $I \subseteq f[A] - f[A]$. Hence $I \subseteq f[A-A]$, so $f^{-1}[I] \subseteq A - A$. But f^{-1} is an isomorphism and $A - A$ is a bounded subset of $\mathbb{R}^2$, so f^{-1} is a continuous additive function, which is impossible. □

We may apply Theorem 3.1 to the σ-fields $\mathcal{M}$, $\mathcal{B}$ and to a nonempty open ball B. Hence we deduce that if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is an arbitrary isomorphism, then the image $f[B]$ is non-measurable and does not have the Baire property.

THEOREM 3.2. *Let $\mathcal{S}$ be a nontrivial, invariant σ-field of subsets of the group $(\mathbb{R},+)$ with the strong Steinhaus property containing all finite sets. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an isomorphism and let $A \subseteq \mathbb{R}^2$ be such that $Int(A) \neq \emptyset$ and $Int(A^c) \neq \emptyset$. Then the image $f[A]$ is strictly $\mathcal{S}$-non-measurable.*

PROOF. It is sufficient to show that the image of any nonempty open ball $B(a,\varepsilon)$ is upward $\mathcal{S}$-full, because this implies that both $f[A]$ and its complement are upward $\mathcal{S}$-full, which gives strictly $\mathcal{S}$-non-measurability of $f[A]$.

Let $B^\star = f[B(a,\frac{\varepsilon}{2})]$, $D = f[B((0,0),\frac{\varepsilon}{2})]$. Notice that $B(a,\frac{\varepsilon}{2}) + B((0,0),\frac{\varepsilon}{2}) = B(a,\varepsilon)$. Therefore we have $B^\star + D = f[B(a,\varepsilon)]$.

Suppose now that there exists a set $E \in \mathcal{S} \setminus \mathcal{S}^-$ such that $f[B(a,\varepsilon)] \cap E = \emptyset$. Then $(B^\star + D) \cap E = \emptyset$. Theorem 3.1 implies that $B^\star \notin \mathcal{S}$, so the strong Steinhaus property of $\mathcal{S}$ implies that $Int(E - B^\star) \neq \emptyset$. But D is a dense subset of the real line, so let $d \in (E - B^\star) \cap D$. Let $e \in E$ and $b \in B^\star$ be such that $d = e - b$. Then $e = b + d$, so $(B^\star + D) \cap E \neq \emptyset$. This contradiction finishes the proof. □

Theorem 3.1 can be generalized to the class of all discontinuous isomorphisms between spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ for each $n, m > 0$. Theorem 3.2 cannot be similarly generalized because of remark 2 in the introduction. However, 3.2 can be generalized to the class of all discontinuous isomorphisms between $\mathbb{R}^n$ and $\mathbb{R}$.

4. Additive images of measurable sets

We shall now show that the flattening of any set of positive Lebesgue measure may be non-measurable. We restrict our attention to sets of positive Lebesgue measure because of remark 1 in the introduction.

Before the formulation of the next result we shall introduce some notions. Let us recall (see [**7**]) the following two cardinal numbers connected with ideals:

DEFINITION 4.1. *Let $\mathcal{J}$ be an ideal of sets. Then $cov(\mathcal{J}) = min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J}$ & $\bigcup \mathcal{A} = \bigcup \mathcal{J}\}$ and $non(\mathcal{J}) = min\{|A| : A \subseteq \bigcup \mathcal{J}$ & $A \notin \mathcal{J}\}$.*

Obviously, $\aleph_1 \leq non(\mathcal{M}_0) \leq \mathfrak{c}$, $\aleph_1 \leq cov(\mathcal{M}_0) \leq \mathfrak{c}$, $\aleph_1 \leq non(\mathcal{B}_0) \leq \mathfrak{c}$, and $\aleph_1 \leq cov(\mathcal{B}_0) \leq \mathfrak{c}$. It is well known that both theories $ZFC \cup \{non(\mathcal{M}_0) < cov(\mathcal{M}_0)\}$ and $ZFC \cup \{non(\mathcal{B}_0) < cov(\mathcal{B}_0)\}$ are relatively consistent (see [**4**]).

THEOREM 4.1. *Suppose that* $non(\mathcal{M}_0) < cov(\mathcal{M}_0)$. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be an isomorphism and let* $A \subseteq \mathbb{R}^2$ *be a bounded Lebesgue measurable set of positive measure. Then the image* $f[A]$ *is non-measurable.*

PROOF. Suppose that the image $f[A]$ is Lebesgue measurable. We first show that $f[A] \in \mathcal{M}_0$. Suppose that $f[A] \notin \mathcal{M}_0$. The strong Steinhaus property of the σ-field $\mathcal{M}$ implies that $Int(f[A]-f[A]) \neq \emptyset$. But $A-A$ is bounded, so there exists a nonempty ball $B \subseteq \mathbb{R}^2$ such that $B \cap (A-A) = \emptyset$. But then $f[B] \cap (f[A]-f[A]) = \emptyset$, which contradicts the density of $f[B]$.

Let $D = A + \mathbb{Q}^2$. Then $f[D] = f[A] + f[\mathbb{Q}^2]$, so $f[D] \in \mathcal{M}_0$. It is clear that $\lambda^2(\mathbb{R}^2 \setminus D) = 0$. Let $T \subseteq \mathbb{R}^2$ be such that $|T| = non(\mathcal{M}_0)$ and $T \notin \mathcal{M}_0(\mathbb{R}^2)$.

We claim that $\mathbb{R}^2 = D - T$. Indeed, suppose that $D - T \neq \mathbb{R}^2$ and let us fix an arbitrary element $u \in \mathbb{R}^2 \setminus (D - T)$. Then $(u + T) \cap D = \emptyset$, so $u + T \subseteq D^c \in \mathcal{M}_0$.

From the equality $\mathbb{R}^2 = D-T$ we deduce that $\bigcup_{t \in T}(f[D]-f(t)) = f[D]-f[T] = \mathbb{R}$. Hence $cov(\mathcal{M}_0) \leq |T| = non(\mathcal{M}_0)$, which contradicts our assumption. $\square$

An analogous result is true for the Baire property. Namely, if $non(\mathcal{B}_0) < cov(\mathcal{B}_0)$, $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is an isomorphism, and $A \subseteq \mathbb{R}^2$ is a bounded set of second category with the Baire property, then the image $f[A]$ does not have the Baire property.

In Theorem 4.1 we used the assumption $non(\mathcal{M}_0) < cov(\mathcal{M}_0)$. We shall show that some kinds of set theoretical assumptions are necessary for the validity of the conclusion of Theorem 4.1. Before we formulate the next result we introduce one technical notion. Namely, for a given set $A \subseteq \mathbb{R}^n$ we define

$$tc(A) = min\{|T| : T \subseteq \mathbb{R}^n \,\&\, \mathbb{Q} \cdot A + T = \mathbb{R}^n\}.$$

LEMMA 4.2. *If* A, $T \subseteq \mathbb{R}$, $|T| < \mathfrak{c}$ *and* $tc(A) = \mathfrak{c}$, *then* $|(\mathbb{Q} \cdot A + T)^c| = \mathfrak{c}$.

PROOF. Suppose on the contrary that $|(\mathbb{Q} \cdot A + T)^c| < \mathfrak{c}$. Let us fix an $a \in A$ and put $T^\star = T \cup ((\mathbb{Q} \cdot A+T)^c - a)$. Then $\mathbb{R} \neq \mathbb{Q} \cdot A + T^\star = (\mathbb{Q} \cdot A+T) \cup (\mathbb{Q} \cdot A + (\mathbb{Q} \cdot A + T)^c - a) \supseteq (\mathbb{Q} \cdot A + T) \cup (a + (\mathbb{Q} \cdot A + T)^c - a) = \mathbb{R}$, a contradiction. $\square$

Let us remark that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an isomorphism then $tc(f[A]) = tc(A)$ for each $A \subseteq \mathbb{R}^n$.

LEMMA 4.3. *If* $A, B \subseteq \mathbb{R}$ *and* $tc(A) = tc(B^c) = \mathfrak{c}$, *then there exists an isomorphism* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *over* $\mathbb{Q}$ *such that* $f[A] \subseteq B$.

PROOF. Let $\preceq$ be a well ordering of $\mathbb{R}$. We define by transfinite recursion of length $\mathfrak{c}$ a sequence $((x_\alpha, y_\alpha))_{\alpha<\mathfrak{c}}$ such that $\{x_\alpha : \alpha < \mathfrak{c}\}$ and $\{y_\alpha : \alpha < \mathfrak{c}\}$ are Hamel bases. Then the unique additive extension f of $((x_\alpha, y_\alpha))_{\alpha<\mathfrak{c}}$ will be the required function. By f_η we denote the unique additive extension of $((x_\xi, y_\xi))_{\xi<\eta}$. Suppose that $\alpha < \mathfrak{c}$ and that the sequence $((x_\xi, y_\xi))_{\xi<\alpha}$ is defined.

Let us consider first the case when α is even. Let x_α be the $\preceq$-minimal element of $\mathbb{R} \setminus Span(\{x_\xi : \xi < \alpha\})$. We choose any element y_α from $\bigcap\{qB - r : q \in \mathbb{Q} \setminus \{0\} \wedge r \in Span(\{y_\xi : \xi < \alpha\})\} \setminus Span(\{y_\xi : \xi < \alpha\})$ (this is possible by Lemma 4.2). This choice of y_α guarantees us that $f_{\alpha+1}[A \cap Span(\{x_\xi : \xi \leq \alpha\})] \subseteq B$. Suppose now that α is even. Let y_α be the $\preceq$-minimal element of $\mathbb{R} \setminus Span(\{y_\xi : \xi < \alpha\})$. Then we put x_α to be any element from $\bigcap\{qA^c - r : q \in \mathbb{Q} \setminus \{0\} \wedge r \in Span(\{x_\xi : \xi < \alpha\})\} \setminus Span(\{x_\xi : \xi < \alpha\})$. This guarantees us, as before, that $f_{\alpha+1}[A \cap Span(\{x_\xi : \xi \leq \alpha\})] \subseteq B$. $\square$

THEOREM 4.4 (Recław). *Suppose that* $cov(\mathcal{B}_0) = \mathfrak{c}$. *Then there exists a Lebesgue measurable set* $A \subseteq \mathbb{R}^2$ *such that* $A^c \in \mathcal{M}_0(\mathbb{R}^2)$ *and an isomorphism* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *such that* $f[A] \in \mathcal{M}_0$.

PROOF. Let us fix an isomorphism $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Let $A \subseteq \mathbb{R}^2$ and $B \subseteq \mathbb{R}$ be such that $A^c \in \mathcal{M}_0(\mathbb{R}^2)$, $A \in \mathcal{B}_0(\mathbb{R}^2)$, $B \in \mathcal{M}_0$, $B^c \in \mathcal{B}_0$. The assumption $cov(\mathcal{B}_0) = \mathfrak{c}$ implies that $tc(A) = tc(B^c) = \mathfrak{c}$. By Lemma 4.3, there exists an isomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $g[\varphi[A]] \subseteq B$. So $g \circ \varphi$ is the required function. $\square$

It is well known that Martin's Axiom implies that $cov(\mathcal{B}_0) = cov(\mathcal{M}_0) = \mathfrak{c}$ (see [**10**]). Therefore, applying Theorems 4.1 and 4.4, we immediately obtain the following result:

COROLLARY 4.5. *The sentence "for every isomorphism* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *and every bounded Lebesgue measurable set* $A \subseteq \mathbb{R}^2$ *of positive measure the image* $f[A]$ *is non-measurable" is independent from the set theory ZFC.*

An analogous result also holds for the Baire property.

5. Properties of images of additive functions

We shall show in this section that images of isomorphisms like in Theorem 3.1 have different properties than some other classical pathological subset of the real line.

Let $(G, +)$ be a subgroup of the group $(\mathbb{R}^n, +)$ and let $\sim_G$ be the equivalence relation on $\mathbb{R}^n$ defined by the formula: $x \sim_G y \longleftrightarrow x - y \in G$. A selector of the family $\mathbb{R}^n / \sim_G$ is called *a set of Vitali's type* for the group $(G, +)$ (see [**5**]). It is well known that if $(G, +)$ is a dense and countable subgroup of the real line $(\mathbb{R}, +)$, then every set of Vitali's type is not Lebesgue measurable and has no the Baire property. Let us consider the subgroup $(\mathbb{Z}^2, +)$ of the group $(\mathbb{R}^2, +)$ and notice that the square $[0, 1)^2$ is a set of Vitali's type for this group. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an isomorphism. Therefore the image of the set $[0, 1)^2$ by f is a set of Vitali's type for the dense subgroup of the real line generated by $\{f(0, 1), f(1, 0)\}$. But if we take instead of $[0, 1)^2$ any nonempty open ball B, then the image $f[B]$ is not a set of Vitali's type for any subgroup of the group $(\mathbb{R}, +)$ because there is no countable $T \subseteq \mathbb{R}^2$ such that the family $\{B + t : t \in T\}$ is a partition of $\mathbb{R}^2$. Therefore we see that images $f[A]$ of regular subsets A of $\mathbb{R}^2$ may be and may not be Vitali's type subsets of $\mathbb{R}$.

Let us now denote by B the unit ball in $\mathbb{R}^2$ at center $(0, 0)$. Let $H \subseteq B$ be a Hamel base of $\mathbb{R}^2$, P be a perfect set of algebraic independent members (see [**12**]), and $P^\star \supseteq P$ be a Hamel base of $\mathbb{R}$. Let us fix a bijection $f_0 : H \longrightarrow P^\star$ and let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the isomorphism extending the function f_0. Then $P \subseteq f[B]$, so the image of B by f is not a Bernstein set. On the other hand there exists an isomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $f[B]$ is a Bernstein set. Such an isomorphism can be easy defined by the transfinite recursion. Therefore, we see that images $f[A]$ of regular subsets A of $\mathbb{R}^2$ may be and may not be Bernstein subsets of $\mathbb{R}$.

Let us recall that a set $X \subseteq \mathbb{R}$ is called a *Lusin set* if $|X| = \mathfrak{c}$ and for each $L \in \mathcal{M}_0$ we have $|X \cap L| < \mathfrak{c}$. A set $X \subseteq \mathbb{R}$ is called a *Sierpiński set* if $|X| = \mathfrak{c}$ and for each $K \in \mathcal{B}_0$ we have $|X \cap K| < \mathfrak{c}$. Let us fix an isomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and let $A \subseteq \mathbb{R}^2$ be such that $A + T = \mathbb{R}^2$ for some countable $T \subseteq \mathbb{R}^2$. Then $f[A]$ is neither a Lusin set nor a Sierpiński set. Namely, suppose that $f[A]$ is a Lusin set. Let $\mathcal{C}$ denote the classical ternary Cantor set. Then

$$|\mathcal{C}| = |\mathcal{C} \cap (f[A] + f[T])| \leq \sum_{t \in T} |(\mathcal{C} - f(t)) \cap f[A])| < \mathfrak{c}.$$

A similar argument shows that $f[A]$ is not a Sierpiński set.

Let us consider another construction. Let $\preceq$ be any well ordering of the real line $\mathbb{R}$ of type continuum and let $S=\{(x,y) \in \mathbb{R}^2 : x \preceq y\}$. Sierpiński observed (see [**17**]) that S is not Lebesgue measurable. Let us call a subset $S \subseteq \mathbb{R}^2$ a *Sierpiński half-plane* if for each $x \in \mathbb{R}$ we have $|\{t : (x,t) \in S\}| < \mathfrak{c}$ and $|\{t : (t,x) \notin S\}| < \mathfrak{c}$. It is easy to see that each Sierpiński half-plane is not Lebesgue measurable. A countable union of translates of Sierpiński half-plane is also a Sierpiński half-plane. Therefore additive images of sets A such that $A+T = \mathbb{R}$ for some countable $T \subseteq \mathbb{R}^2$ have different algebraic properties than Sierpiński half-planes.

Let H be a fixed Hamel base of $\mathbb{R}$. Sierpiński observed (see [**16**]) that there are $q_1, ..., q_k \in \mathbb{Q}$ such that the set $q_1 H + ... + q_k H$ is not Lebesgue measurable. Notice that $\bigcup_{n \in \omega} n \cdot (q_1 H + ... + q_k H) \neq \mathbb{R}$. But if A is a subset of the plane such that $0 \in Int(A) \neq \emptyset$, then $\bigcup_{n \in \omega} n \cdot f[A] = \mathbb{R}$ for any additive function from $\mathbb{R}^2$ onto $\mathbb{R}$.

Let μ be the completion of the probability Haar measure on the compact group $(\{0,1\}^\omega, +)$ and let U be a non-principal ultrafilter on ω. Let $U^\star = \{x \in \{0,1\}^\omega : x^{-1}[\{1\}] \in U\}$. Sierpiński proved that $U^\star$ is a non-measurable set with respect to the measure μ (see [**15**]). There are two steps in the classical proof of this fact. In the first one the equality $U^\star + \mathbf{1} = \{0,1\}^\omega \setminus U^\star$ is shown, where $\mathbf{1}$ denotes the constant function with value 1. In the second step it is shown that $U^\star + D = U^\star$, where $D = \{x \in \{0,1\}^\omega : (\exists n)(\forall m > n)(x_m = 0)\}$. Observe that if $A \subseteq \mathbb{R}^2$ is a bounded set, $C \neq \{0\}$ is a nonempty subset of $\mathbb{R}$, and f is an isomorphism between $\mathbb{R}^2$ and $\mathbb{R}$, then $f[A] + C \neq f[A]$. Therefore the set $f[A]$ has different algebraic properties than the set $U^\star$.

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