The Reconstruction of Convex Polyominoes from Approximately Orthogonal Projections

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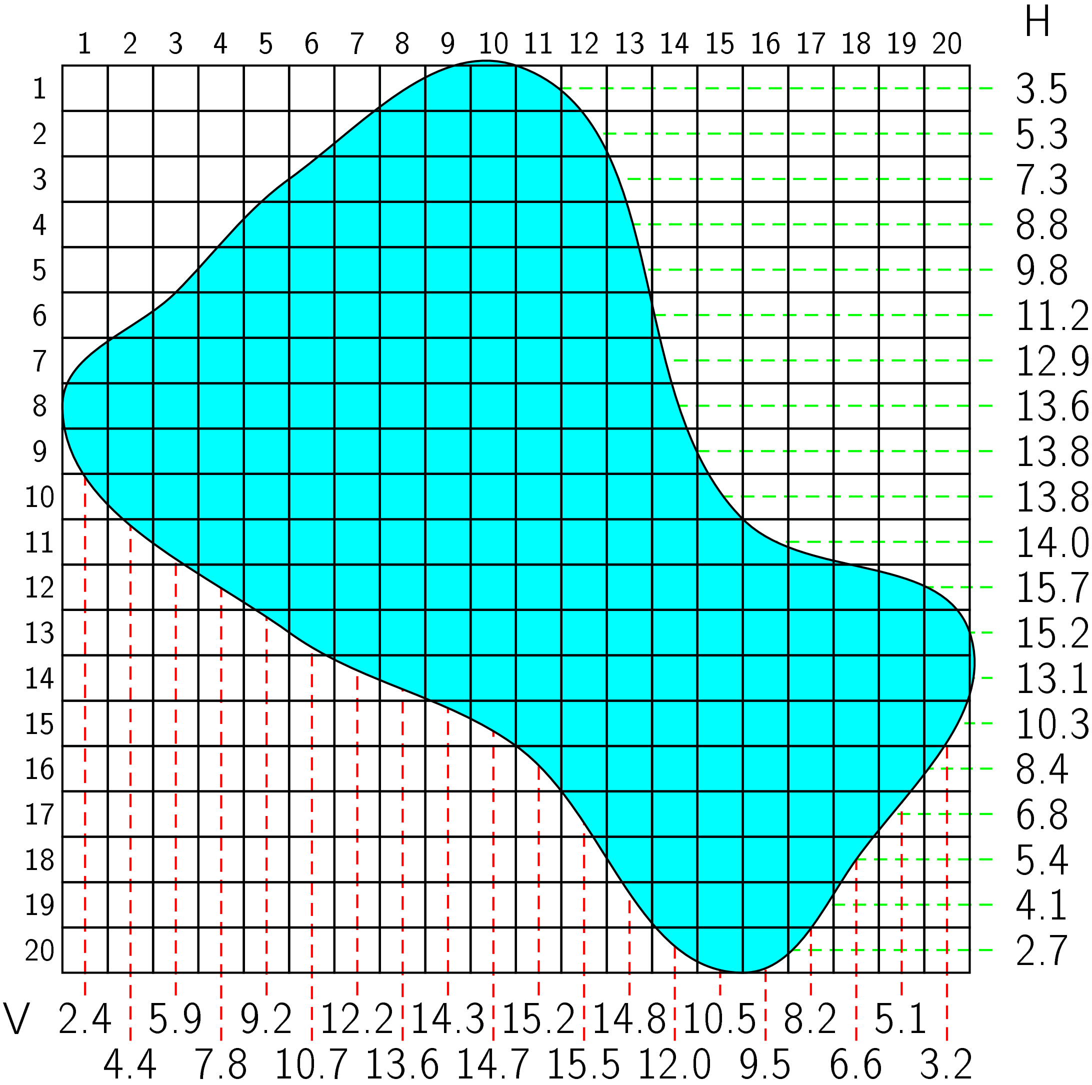
The Polyomino and Orthogonal Projections

A one-color connected picture on the grid $m \times n$.



The projections are real numbers. But when polyomino is reconstructed whole cells are filled and projections are integers.

The filled cell is represented by 1 and not filled by 0.

The Definition of Problem

Does a polynomial time algorithm exist that takes as input

a horizontal projection vector $H \in \mathbb{R}_+^m$ and
a vertical projection vector $V \in \mathbb{R}_+^n$,

and outputs a polyomino with projections

$$H^* \in \{1, \ldots, n\}^m \quad \text{and} \quad V^* \in \{1, \ldots, m\}^n$$

such that (two version of the approximation)

(1) *the approximation with absolute error*

$$|h_i - h_i^*| \leq 1 \qquad |v_j - v_j^*| \leq 1.$$

(2) *the approximation with logarithmic error*

$$|h_i - h_i^*| \leq \log(h_i + 1) \qquad |v_j - v_j^*| \leq \log(v_j + 1).$$

The algorithm outputs "NO" if there does not exist a polyomino with approximately projections V and H.

Our Results

The polyomino P is a hv-convex polyomino if

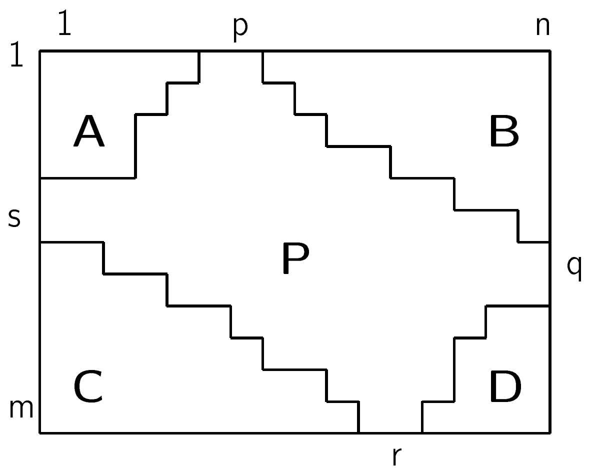
(h) the set P in every row of the grid is connected, and

(v) the set P in every column of the grid is connected.

Theorem 1. *The reconstruction of polyominoes, h-convex polyominoes and v-convex polyominoes from their approximately orthogonal projections is NP-complete.*

Theorem 2. *The problem of reconstruction of hv-convex polyominoes from approximately orthogonal projections can be solved in $O(m^3n^3)$ time.*

Properties of hv-Convex Polyomino



A - *upper-left corner region*

$$(i,j) \in A \;\Rightarrow\; (i-1,j) \in A \;\wedge\; (i,j-1) \in A.$$

B - *upper-right corner region*

$$(i,j) \in B \;\Rightarrow\; (i-1,j) \in B \;\wedge\; (i,j+1) \in B.$$

C - *lower-left corner region*

$$(i,j) \in C \;\Rightarrow\; (i+1,j) \in C \;\wedge\; (i,j-1) \in C.$$

D - *lower-right corner region*

$$(i,j) \in D \;\Rightarrow\; (i+1,j) \in D \;\wedge\; (i,j+1) \in D.$$

Properties of hv-Convex Polyomino

$\overline{P}$ denotes the complement of P.

Lemma 1. [Chrobak and Dürr] *P is an hv-convex polyomino if and only if*

$$P = \overline{A \cup B \cup C \cup D},$$

where A, B, C, D are disjoint corner regions (upper-left, upper-right, lower-left and lower-right, respectively) such that

(i) $(i-1, j-1) \in A$ *implies* $(i, j) \notin D$, *and*

(ii) $(i-1, j+1) \in B$ *implies* $(i, j) \notin C$.

Polyomino P is anchored at (p, q, r, s) iff

$$(1, p), (q, n), (m, r), (s, 1) \in P.$$

Auxiliary Expressions

$f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ - a function of error.

Example:
$f(x) = 1$ - the absolute error
$f(x) = \log(x+1)$ - the logarithmic error.

$$\check{v}_j = \max\{1, \lceil v_j - f(v_j) \rceil\}$$
$$\hat{v}_j = \min\{m, \lfloor v_j + f(v_j) \rfloor\}$$

$\check{v}_j$ - minimal acceptable value of j-th vertical projection.
$\hat{v}_j$ - maximal acceptable value of j-th vertical projection.

$$\check{h}_i = \max\{1, \lceil h_i - f(h_i) \rceil\}$$
$$\hat{h}_i = \min\{n, \lfloor h_i + f(h_i) \rfloor\}$$

$\check{h}_i$ - minimal acceptable value of i-th horizontal projection.
$\hat{h}_i$ - maximal acceptable value of i-th horizontal projection.

The 2SAT Formula - Properties from Lemma 1

$$\begin{array}{lcl} X_{i,j} = 1 \ (true) & \Longleftrightarrow & (i,j) \in X \\ X_{i,j} = 0 \ (false) & \Longleftrightarrow & (i,j) \notin X \end{array}$$

"Corners" (Cor)

$$\bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow A_{i-1,j} & A_{i,j} \Rightarrow A_{i,j-1} \\ B_{i,j} \Rightarrow B_{i-1,j} & B_{i,j} \Rightarrow B_{i,j+1} \\ C_{i,j} \Rightarrow C_{i+1,j} & C_{i,j} \Rightarrow C_{i,j-1} \\ D_{i,j} \Rightarrow D_{i+1,j} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

"Connectivity" (Con)

$$\bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow \overline{D}_{i+1,j+1} & B_{i,j} \Rightarrow \overline{C}_{i+1,j-1} \end{array} \right\}$$

"Anchors" ($Anc_{p,q,r,s}$)

$$\bigwedge \left\{ \begin{array}{llll} \overline{A}_{1,p} & \overline{B}_{1,p} & \overline{C}_{1,p} & \overline{D}_{1,p} \\ \overline{A}_{q,n} & \overline{B}_{q,n} & \overline{C}_{q,n} & \overline{D}_{q,n} \\ \overline{A}_{m,r} & \overline{B}_{m,r} & \overline{C}_{m,r} & \overline{D}_{m,r} \\ \overline{A}_{s,1} & \overline{B}_{s,1} & \overline{C}_{s,1} & \overline{D}_{s,1} \end{array} \right\}$$

The 2SAT Formula - Column Sums

"Lower bound on column sums" (LBC)

$$\bigwedge_{i,j} \left\{ \begin{array}{l} A_{i,j} \Rightarrow \overline{C}_{i+\check{v}_j,j} \\ A_{i,j} \Rightarrow \overline{D}_{i+\check{v}_j,j} \\ B_{i,j} \Rightarrow \overline{C}_{i+\check{v}_j,j} \\ B_{i,j} \Rightarrow \overline{D}_{i+\check{v}_j,j} \end{array} \right\} \wedge \bigwedge_{j} \left\{ \begin{array}{l} \overline{C}_{\check{v}_j,j} \\ \overline{D}_{\check{v}_j,j} \end{array} \right\}$$

"Upper bound on column sums" ($UBC_{p,r}$)

$$\bigwedge_{i} \left\{ \begin{array}{ll} \bigwedge_{j \le \min\{p,r\}} & \overline{A}_{i,j} \Rightarrow C_{i+\hat{v}_j,j} \\ \bigwedge_{p \le j \le r} & \overline{B}_{i,j} \Rightarrow C_{i+\hat{v}_j,j} \\ \bigwedge_{r \le j \le p} & \overline{A}_{i,j} \Rightarrow D_{i+\hat{v}_j,j} \\ \bigwedge_{\max\{p,r\} \le j} & \overline{B}_{i,j} \Rightarrow D_{i+\hat{v}_j,j} \end{array} \right\}$$

The (LBC) assigns to columns the minimal distance between corner regions.

The ($UBC_{p,r}$) assigns to columns the maximal distance between corner regions.

The 2SAT Formula - Row Sums

"Lower bound on row sums" (LBR)

$$\bigwedge_{i,j} \left\{ \begin{array}{l} A_{i,j} \Rightarrow \overline{B}_{i,j+\check{h}_i} \\ A_{i,j} \Rightarrow \overline{D}_{i,j+\check{h}_i} \\ C_{i,j} \Rightarrow \overline{B}_{i,j+\check{h}_i} \\ C_{i,j} \Rightarrow \overline{D}_{i,j+\check{h}_i} \end{array} \right\} \wedge \bigwedge_{i} \left\{ \begin{array}{l} \overline{B}_{i,\check{h}_i} \\ \overline{D}_{\check{h}_i} \end{array} \right\}$$

"Upper bound on row sums" ($UBR_{q,s}$)

$$\bigwedge_{j} \left\{ \begin{array}{ll} \bigwedge_{i \leq \min\{s,q\}} & \overline{A}_{i,j} \Rightarrow B_{i,j+\hat{h}_i} \\ \bigwedge_{s \leq i \leq q} & \overline{C}_{i,j} \Rightarrow B_{i,j+\hat{h}_i} \\ \bigwedge_{q \leq j \leq s} & \overline{A}_{i,j} \Rightarrow D_{i,j+\hat{h}_i} \\ \bigwedge_{\max\{s,q\} \leq j} & \overline{C}_{i,j} \Rightarrow D_{i,j+\hat{h}_i} \end{array} \right\}$$

The (LBR) assigns to rows the minimal distance between corner regions.
The ($UBR_{q,s}$) assigns to rows the maximal distance between corner regions.

The Algorithm

$$F_{p,q,r,s}(H,V) = Cor \wedge Con \wedge Anc_{p,q,r,s} \wedge LBC \wedge UBC_{p,r} \wedge LBR \wedge UBR_{q,s}$$

All literals with indices outside the set $\{1,\ldots,m\} \times \{1,\ldots,n\}$ are assumed to have value 1.

Input: $H \in \mathbb{R}^m_+$, $V \in \mathbb{R}^n_+$

FOR $p, r = 1, \ldots n$ AND $q, s = 1, \ldots m$ DO
  IF $F_{p,q,r,s}(H,V)$ is satisfiable
  THEN RETURN $P = \overline{A \cup B \cup C \cup D}$ AND HALT
RETURN "NO"

Lemma 2. *$F_{p,q,r,s}(H,V)$ is satisfiable if and only if P is an hv-convex polyomino with projections (H^*,V^*) that is anchored at (p,q,r,s) and every component of (H^*,V^*) differs from the correspondent component of (H,V) by at most the value of function f for this component.*