# The Reconstruction of Some 3D Convex Polyominoes from Orthogonal Projections

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# Discrete Tomography

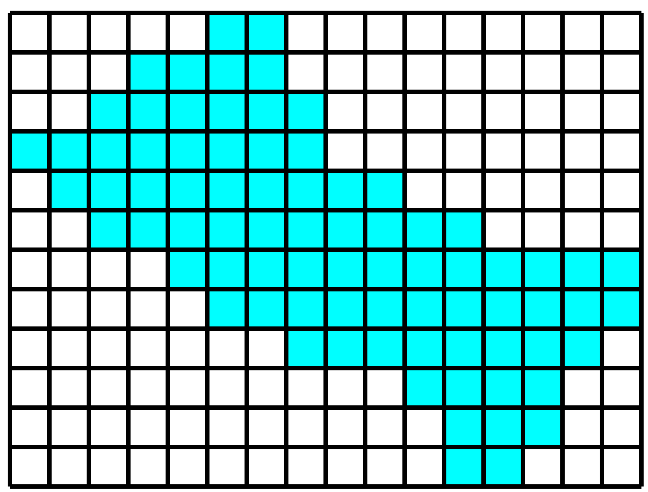
**Basic Problem:**

Reconstruction of finite point sets that are accessible only through some of their discrete X-rays.

# 2D Convex Polyomino

Set of cells $S$ in a $m \times n$ grid with properties:

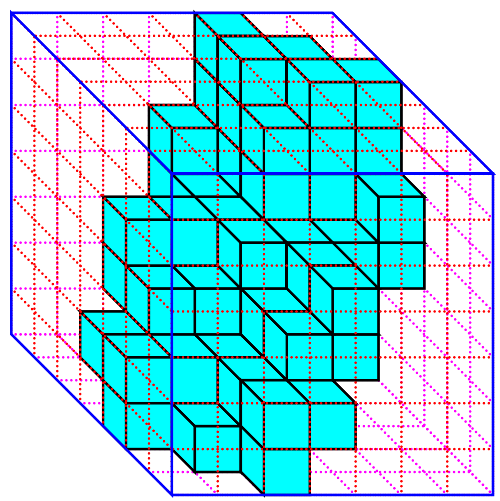
- $S$ is connected,
- $S$ in each row and each column is connected.



# 3D Convex Polyomino

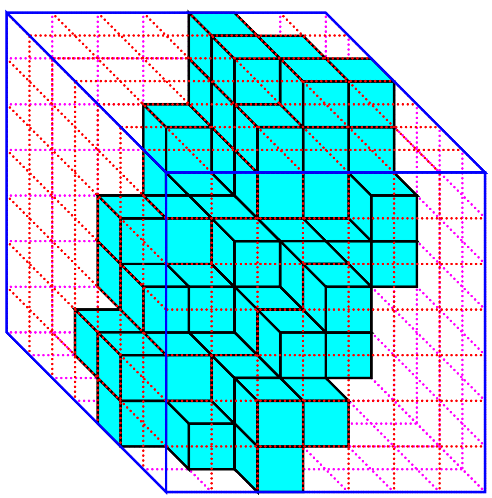
Set of cells $S$ in $n \times n \times n$ grid with properties:

- $S$ is connected,
- each two-dimensional orthogonal section of the grid contains 2D convex polyomino.



# Orthogonal Projections

Orthogonal projections - the number of cells in each bar of the grid that belongs to $S$.



Top projections $P_T$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 3 | 1 | 0 |
| 0 | 0 | 0 | 1 | 4 | 2 | 1 |
| 0 | 2 | 2 | 2 | 3 | 4 | 3 |
| 1 | 2 | 3 | 3 | 2 | 1 | 2 |
| 2 | 2 | 2 | 3 | 2 | 0 | 0 |
| 0 | 1 | 2 | 2 | 2 | 0 | 0 |
| 0 | 0 | 2 | 1 | 0 | 0 | 0 |

Front projections $P_F$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 2 | 2 | 1 |
| 0 | 0 | 0 | 0 | 3 | 3 | 2 |
| 0 | 0 | 0 | 1 | 3 | 1 | 2 |
| 0 | 1 | 1 | 4 | 5 | 2 | 1 |
| 0 | 2 | 2 | 4 | 3 | 0 | 0 |
| 2 | 2 | 4 | 4 | 0 | 0 | 0 |
| 1 | 2 | 4 | 1 | 0 | 0 | 0 |

Side Projections $P_S$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 0 | 0 | 0 | 0 |
| 2 | 3 | 3 | 0 | 0 | 0 | 0 |
| 2 | 1 | 3 | 1 | 0 | 0 | 0 |
| 1 | 2 | 5 | 4 | 1 | 1 | 0 |
| 0 | 0 | 3 | 4 | 2 | 2 | 0 |
| 0 | 0 | 0 | 4 | 4 | 2 | 2 |
| 0 | 0 | 0 | 1 | 4 | 2 | 1 |

# Reconstruction Problem

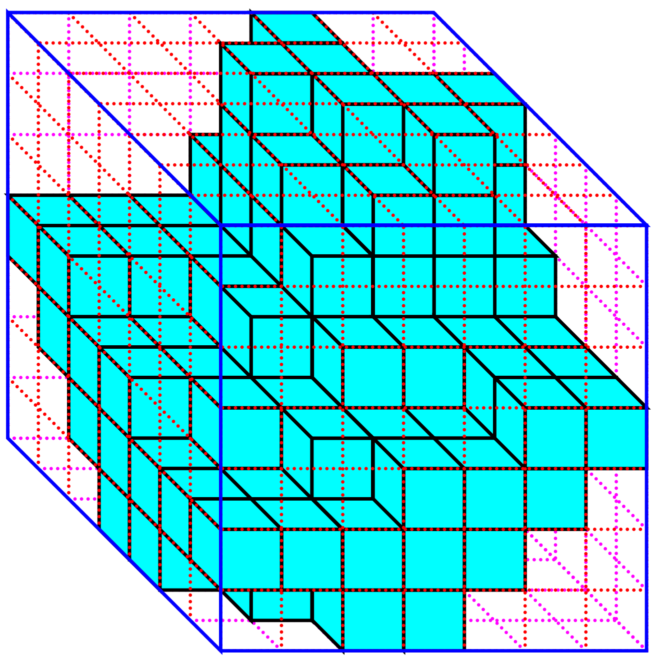
Given three assigned matrices:

$$P_F, P_S, P_T \in \{0, \ldots, n\}^{n \times n},$$

we examine whether there exists at least one convex polyomino $S$ with adequate orthogonal projections.

# Full 3D Convex Polyomino

3D Convex Polyomino with at least one matrix of projections does not contain zeros (*full matrix*).
We determine that $P_T$ is a full matrix.



$P_T$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 | 1 | 1 |
| 2 | 2 | 2 | 4 | 3 | 1 | 1 |
| 2 | 1 | 2 | 3 | 3 | 3 | 3 |
| 3 | 2 | 2 | 3 | 3 | 4 | 2 |
| 2 | 3 | 2 | 2 | 2 | 2 | 1 |
| 1 | 2 | 3 | 2 | 1 | 2 | 1 |
| 1 | 1 | 2 | 3 | 2 | 2 | 1 |

$P_F$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 3 | 2 | 1 |
| 0 | 0 | 0 | 2 | 4 | 4 | 3 |
| 0 | 0 | 0 | 4 | 4 | 2 | 2 |
| 2 | 2 | 4 | 5 | 2 | 4 | 4 |
| 3 | 4 | 5 | 4 | 3 | 3 | 0 |
| 5 | 4 | 3 | 2 | 1 | 0 | 0 |
| 2 | 2 | 2 | 1 | 0 | 0 | 0 |

$P_S$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 0 | 0 | 0 |
| 3 | 4 | 4 | 2 | 0 | 0 | 0 |
| 2 | 2 | 4 | 4 | 0 | 0 | 0 |
| 4 | 4 | 2 | 5 | 4 | 2 | 2 |
| 0 | 3 | 3 | 4 | 5 | 4 | 3 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 1 | 2 | 2 | 2 |

# Our Result

**Theorem.** The problem of the reconstruction of full convex 3D polyominoes from orthogonal projections has the complexity $O(n^7 \log n)$.

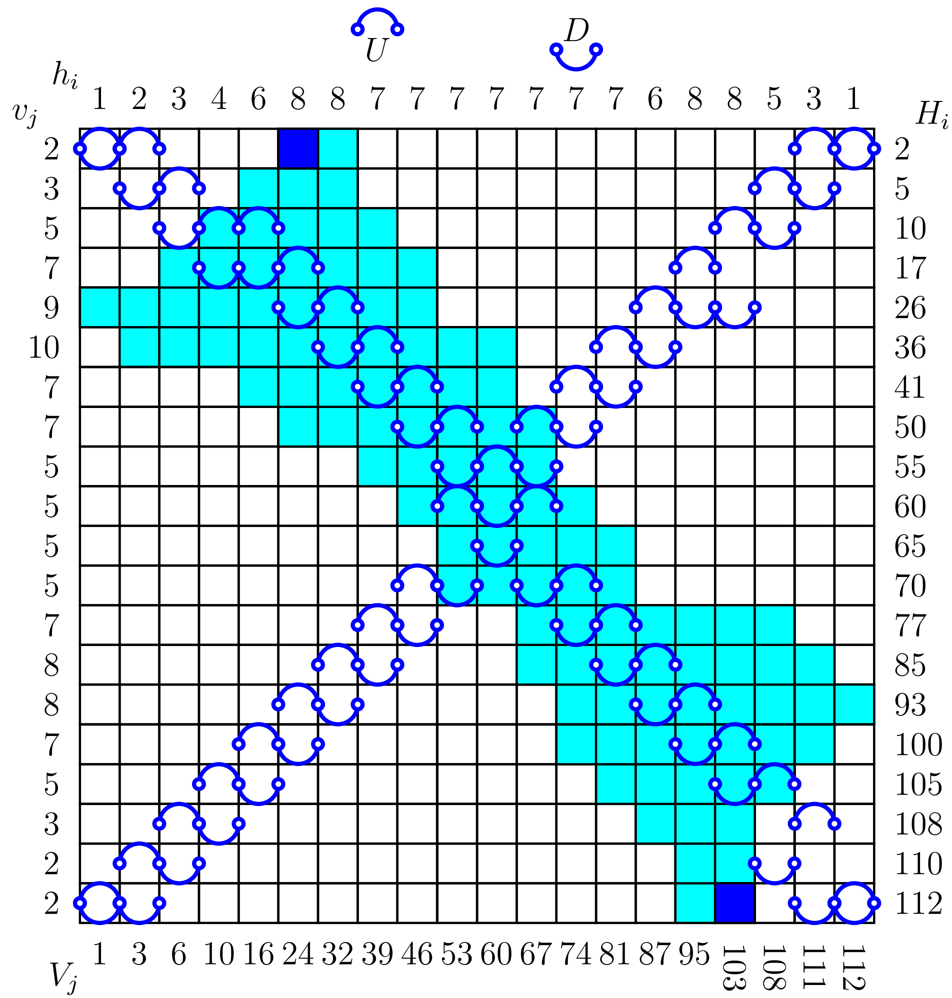
# Properties of 2D Convex Polyom.

- auxiliary values $D_j^{\searrow}$, $U_j^{\searrow}$, $D_j^{\swarrow}$, $U_j^{\swarrow}$ (easy computable)
- $\forall_{j\in\{1,\ldots,n\}}\ U_j^{\searrow} \le D_j^{\searrow} \quad \wedge \quad U_j^{\swarrow} \le D_j^{\swarrow}$.
- $\forall_{j\in\{1,\ldots,n-1\}}\ D_j^{\searrow}+1 \ge U_{j+1}^{\searrow} \quad \wedge \quad D_{j+1}^{\swarrow}+1 \ge U_j^{\swarrow}$.

<u>**Lemma.**</u> Let cells $[1,p_1]$ and $[n,p_2]$ belong to the convex polyomino $S$. Then

1. if $p_1 \le p_2$ then cells $[1,p_1],\ldots,[D_{p_1}^{\searrow},p_1]$, and $[U_j^{\searrow},j],\ldots,[D_j^{\searrow},j]\quad \forall_{j\in\{p_1+1,\ldots,p_2-1\}}$, and $[U_{p_2}^{\searrow},p_2],\ldots,[n,p_2]$ also belong to the $S$, or
2. if $p_1 \ge p_2$ then cells $[U_{p_2}^{\swarrow},p_2],\ldots,[n,p_2]$, and $[U_j^{\swarrow},j],\ldots,[D_j^{\swarrow},j]\quad \forall_{j\in\{p_2+1,\ldots,p_1-1\}}$, and $[D_{p_1}^{\swarrow},p_1],\ldots,[1,p_1]$ also belong to the $S$.

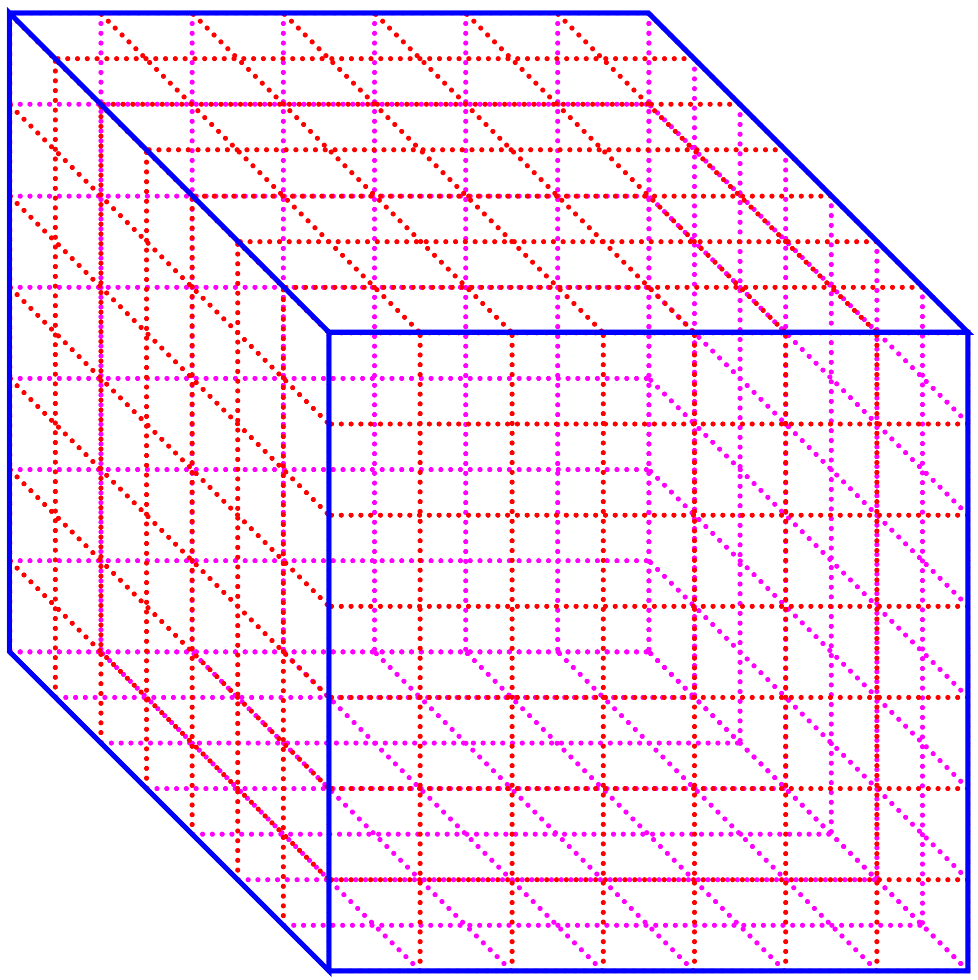
# Example of Properties



# Algorithm

1. Compute start positions
   (a) select one cell in each corner vertical bar
   (b) from Lemma compute positions of cells belonging to $S$ in both lateral slices of the grid
   (c) from Lemma compute positions of cells belonging to $S$ in vertical slices orthogonal to slices from (b)
2. Perform Filling Procedure

# Example of Reconstruction



$P_T$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 | 1 | 1 |
| 2 | 2 | 2 | 4 | 3 | 1 | 1 |
| 2 | 1 | 2 | 3 | 3 | 3 | 3 |
| 3 | 2 | 2 | 3 | 3 | 4 | 2 |
| 2 | 3 | 2 | 2 | 2 | 2 | 1 |
| 1 | 2 | 3 | 2 | 1 | 2 | 1 |
| 1 | 1 | 2 | 3 | 2 | 2 | 1 |

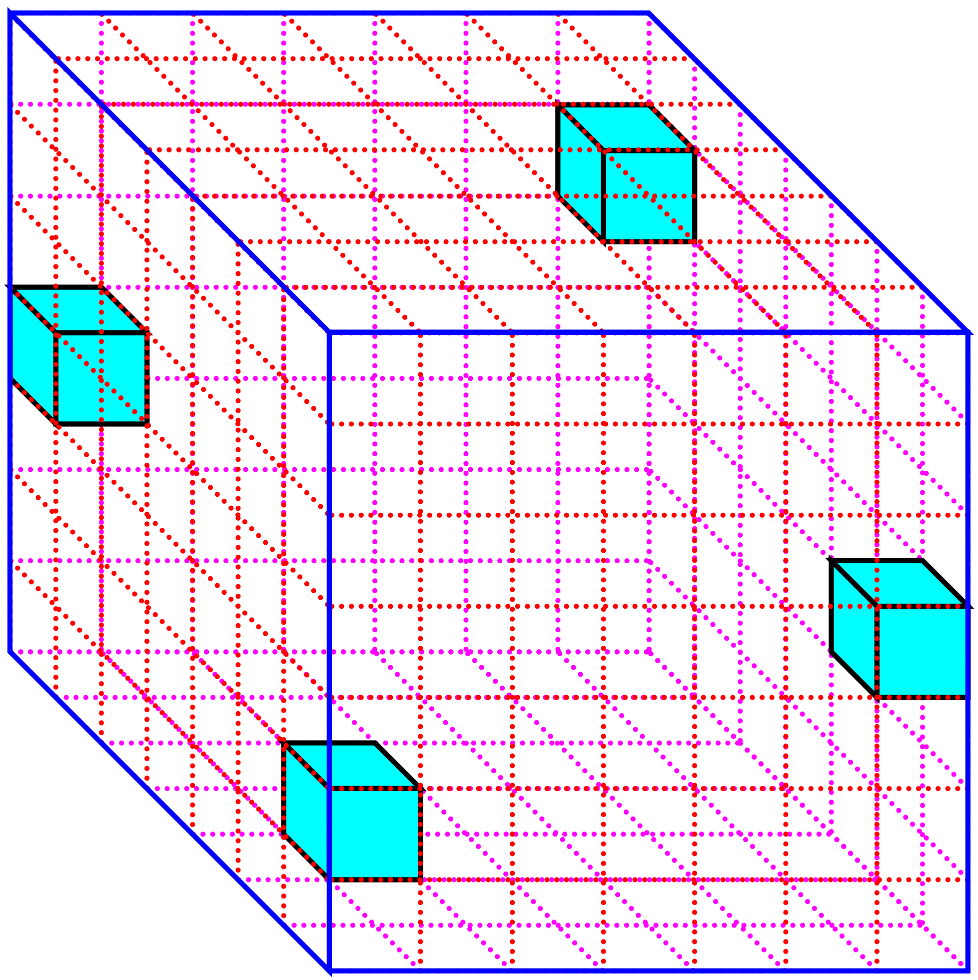
$P_F$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 3 | 2 | 1 |
| 0 | 0 | 0 | 2 | 4 | 4 | 3 |
| 0 | 0 | 0 | 4 | 4 | 2 | 2 |
| 2 | 2 | 4 | 5 | 2 | 4 | 4 |
| 3 | 4 | 5 | 4 | 3 | 3 | 0 |
| 5 | 4 | 3 | 2 | 1 | 0 | 0 |
| 2 | 2 | 2 | 1 | 0 | 0 | 0 |

$P_S$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 0 | 0 | 0 |
| 3 | 4 | 4 | 2 | 0 | 0 | 0 |
| 2 | 2 | 4 | 4 | 0 | 0 | 0 |
| 4 | 4 | 2 | 5 | 4 | 2 | 2 |
| 0 | 3 | 3 | 4 | 5 | 4 | 3 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 1 | 2 | 2 | 2 |

# Example of Initial Positions



$P_T$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 | 1 | 1 |
| 2 | 2 | 2 | 4 | 3 | 1 | 1 |
| 2 | 1 | 2 | 3 | 3 | 3 | 3 |
| 3 | 2 | 2 | 3 | 3 | 4 | 2 |
| 2 | 3 | 2 | 2 | 2 | 2 | 1 |
| 1 | 2 | 3 | 2 | 1 | 2 | 1 |
| 1 | 1 | 2 | 3 | 2 | 2 | 1 |

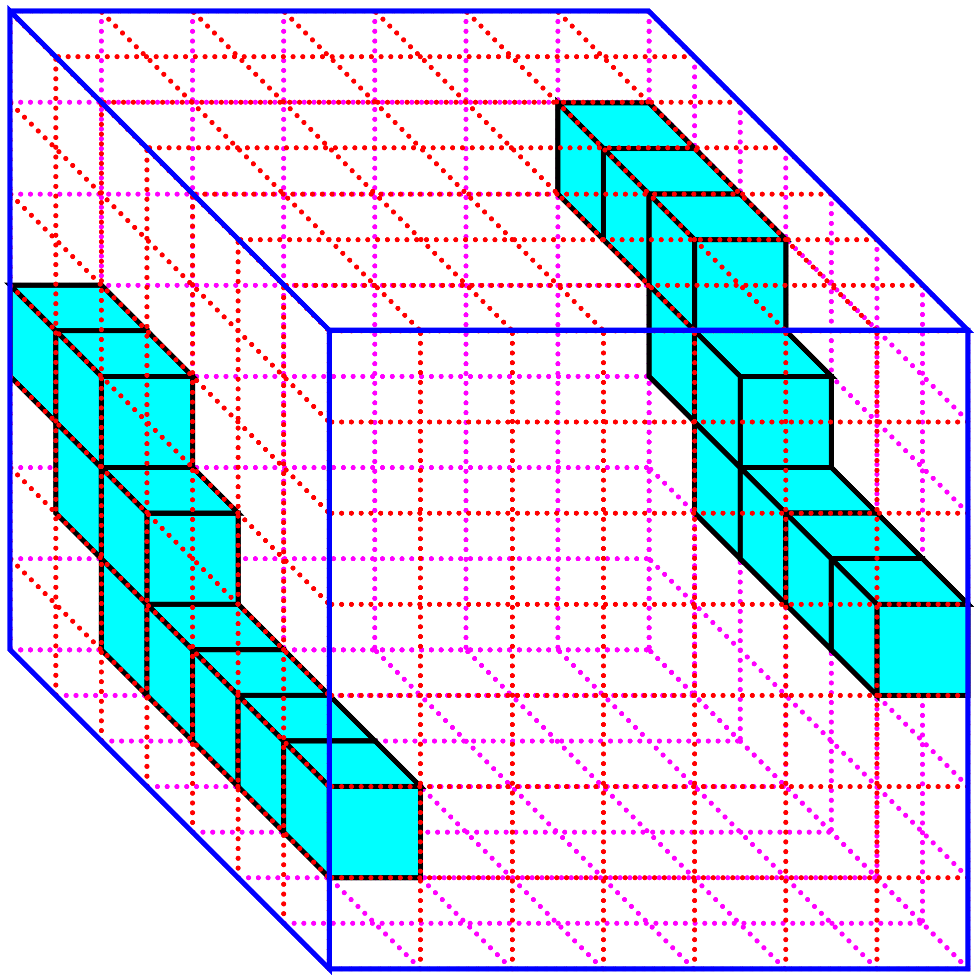
$P_F$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 3 | 2 | 1 |
| 0 | 0 | 0 | 2 | 4 | 4 | 3 |
| 0 | 0 | 0 | 4 | 4 | 2 | 2 |
| 2 | 2 | 4 | 5 | 2 | 4 | 4 |
| 3 | 4 | 5 | 4 | 3 | 3 | 0 |
| 5 | 4 | 3 | 2 | 1 | 0 | 0 |
| 2 | 2 | 2 | 1 | 0 | 0 | 0 |

$P_S$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 0 | 0 | 0 |
| 3 | 4 | 4 | 2 | 0 | 0 | 0 |
| 2 | 2 | 4 | 4 | 0 | 0 | 0 |
| 4 | 4 | 2 | 5 | 4 | 2 | 2 |
| 0 | 3 | 3 | 4 | 5 | 4 | 3 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 1 | 2 | 2 | 2 |

# Example of Using the Lemma



$P_T$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 | 1 | 1 |
| 2 | 2 | 2 | 4 | 3 | 1 | 1 |
| 2 | 1 | 2 | 3 | 3 | 3 | 3 |
| 3 | 2 | 2 | 3 | 3 | 4 | 2 |
| 2 | 3 | 2 | 2 | 2 | 2 | 1 |
| 1 | 2 | 3 | 2 | 1 | 2 | 1 |
| 1 | 1 | 2 | 3 | 2 | 2 | 1 |

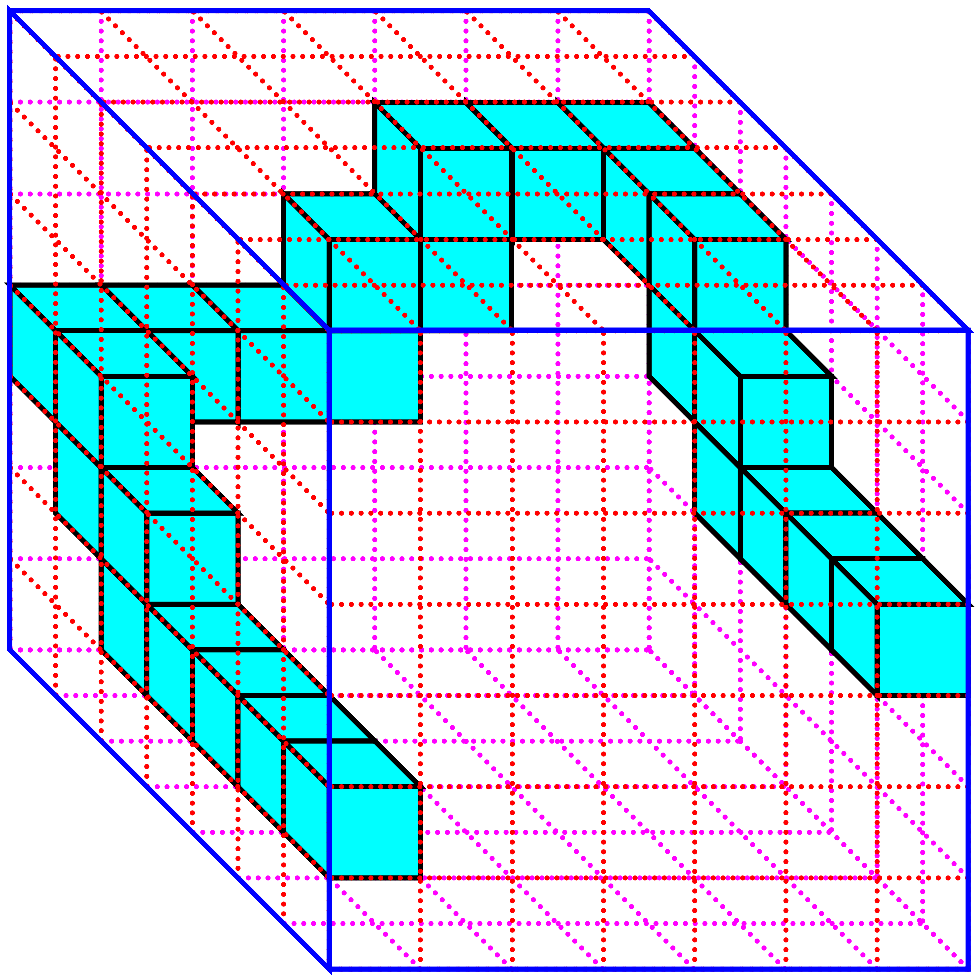
$P_F$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 3 | 2 | 1 |
| 0 | 0 | 0 | 2 | 4 | 4 | 3 |
| 0 | 0 | 0 | 4 | 4 | 2 | 2 |
| 2 | 2 | 4 | 5 | 2 | 4 | 4 |
| 3 | 4 | 5 | 4 | 3 | 3 | 0 |
| 5 | 4 | 3 | 2 | 1 | 0 | 0 |
| 2 | 2 | 2 | 1 | 0 | 0 | 0 |

$P_S$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 0 | 0 | 0 |
| 3 | 4 | 4 | 2 | 0 | 0 | 0 |
| 2 | 2 | 4 | 4 | 0 | 0 | 0 |
| 4 | 4 | 2 | 5 | 4 | 2 | 2 |
| 0 | 3 | 3 | 4 | 5 | 4 | 3 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 1 | 2 | 2 | 2 |

# Example of Using the Lemma



$P_T$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 | 1 | 1 |
| 2 | 2 | 2 | 4 | 3 | 1 | 1 |
| 2 | 1 | 2 | 3 | 3 | 3 | 3 |
| 3 | 2 | 2 | 3 | 3 | 4 | 2 |
| 2 | 3 | 2 | 2 | 2 | 2 | 1 |
| 1 | 2 | 3 | 2 | 1 | 2 | 1 |
| 1 | 1 | 2 | 3 | 2 | 2 | 1 |

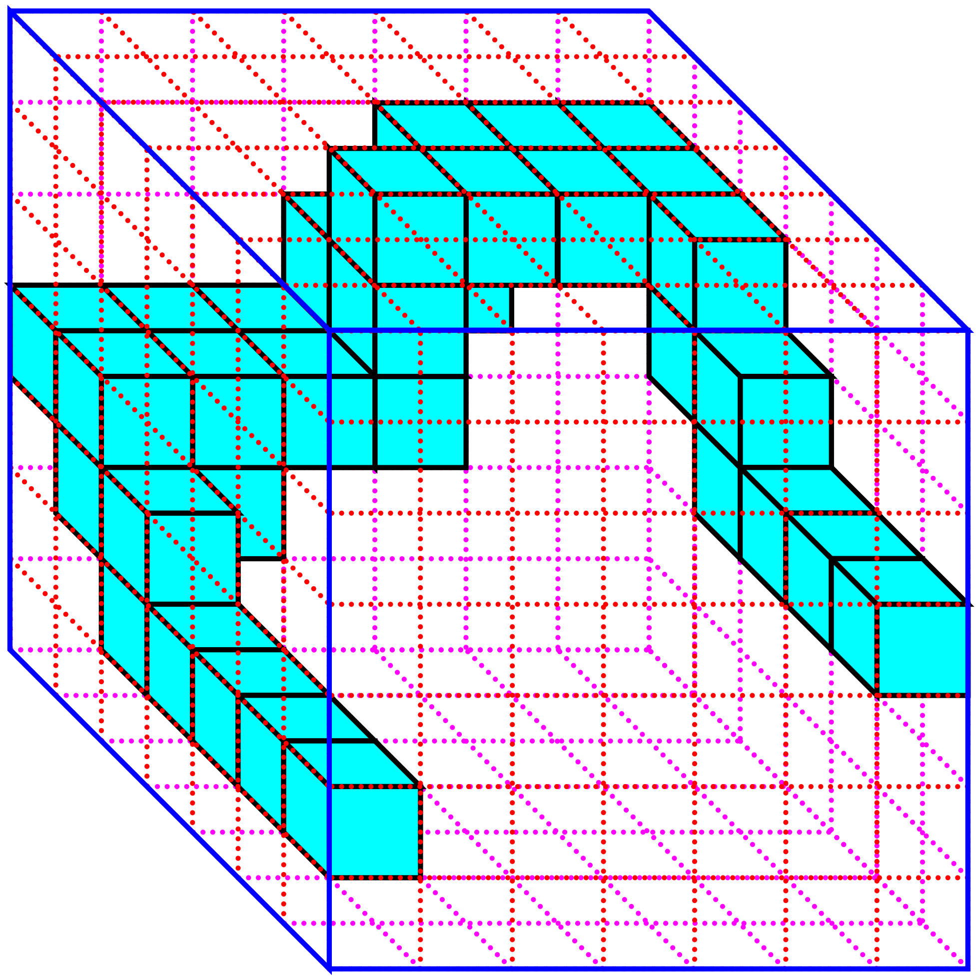
$P_F$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 3 | 2 | 1 |
| 0 | 0 | 0 | 2 | 4 | 4 | 3 |
| 0 | 0 | 0 | 4 | 4 | 2 | 2 |
| 2 | 2 | 4 | 5 | 2 | 4 | 4 |
| 3 | 4 | 5 | 4 | 3 | 3 | 0 |
| 5 | 4 | 3 | 2 | 1 | 0 | 0 |
| 2 | 2 | 2 | 1 | 0 | 0 | 0 |

$P_S$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 0 | 0 | 0 |
| 3 | 4 | 4 | 2 | 0 | 0 | 0 |
| 2 | 2 | 4 | 4 | 0 | 0 | 0 |
| 4 | 4 | 2 | 5 | 4 | 2 | 2 |
| 0 | 3 | 3 | 4 | 5 | 4 | 3 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 1 | 2 | 2 | 2 |

# Example of Using the Lemma



$P_T$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 | 1 | 1 |
| 2 | 2 | 2 | 4 | 3 | 1 | 1 |
| 2 | 1 | 2 | 3 | 3 | 3 | 3 |
| 3 | 2 | 2 | 3 | 3 | 4 | 2 |
| 2 | 3 | 2 | 2 | 2 | 2 | 1 |
| 1 | 2 | 3 | 2 | 1 | 2 | 1 |
| 1 | 1 | 2 | 3 | 2 | 2 | 1 |

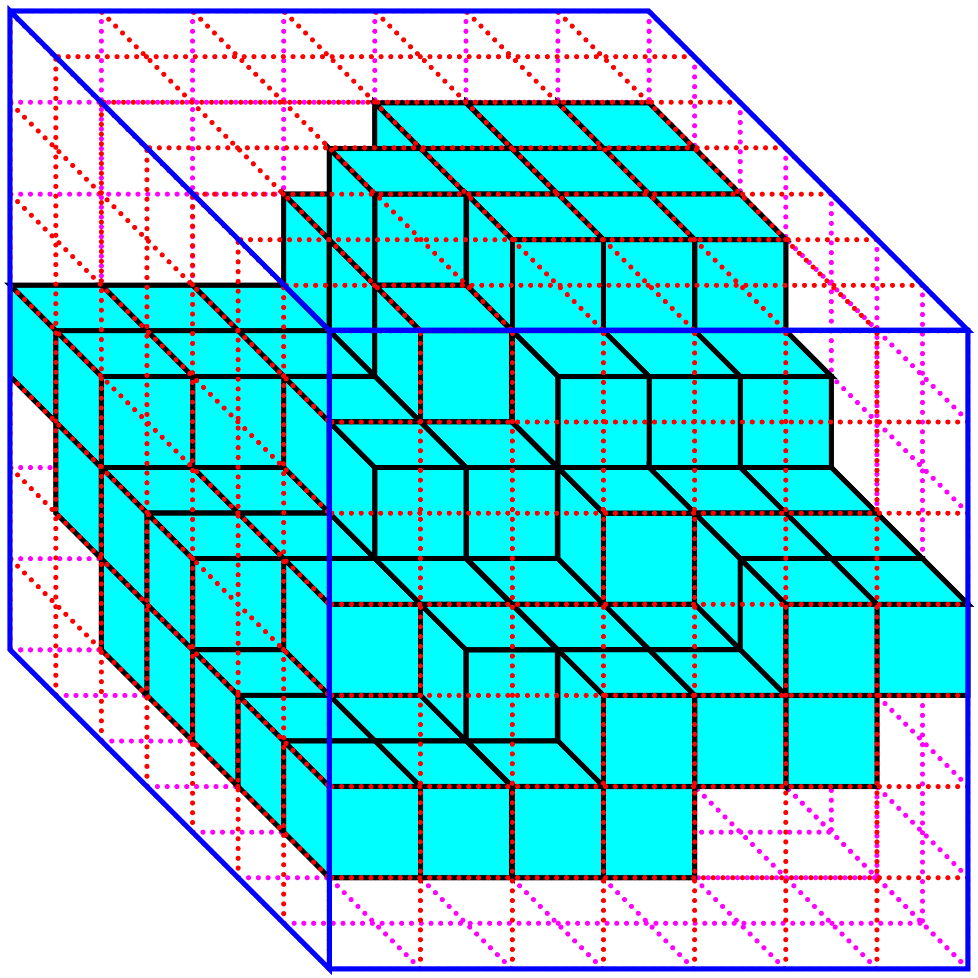
$P_F$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 3 | 2 | 1 |
| 0 | 0 | 0 | 2 | 4 | 4 | 3 |
| 0 | 0 | 0 | 4 | 4 | 2 | 2 |
| 2 | 2 | 4 | 5 | 2 | 4 | 4 |
| 3 | 4 | 5 | 4 | 3 | 3 | 0 |
| 5 | 4 | 3 | 2 | 1 | 0 | 0 |
| 2 | 2 | 2 | 1 | 0 | 0 | 0 |

$P_S$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 0 | 0 | 0 |
| 3 | 4 | 4 | 2 | 0 | 0 | 0 |
| 2 | 2 | 4 | 4 | 0 | 0 | 0 |
| 4 | 4 | 2 | 5 | 4 | 2 | 2 |
| 0 | 3 | 3 | 4 | 5 | 4 | 3 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 1 | 2 | 2 | 2 |

# Example of Start Positions



$P_T$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 | 1 | 1 |
| 2 | 2 | 2 | 4 | 3 | 1 | 1 |
| 2 | 1 | 2 | 3 | 3 | 3 | 3 |
| 3 | 2 | 2 | 3 | 3 | 4 | 2 |
| 2 | 3 | 2 | 2 | 2 | 2 | 1 |
| 1 | 2 | 3 | 2 | 1 | 2 | 1 |
| 1 | 1 | 2 | 3 | 2 | 2 | 1 |

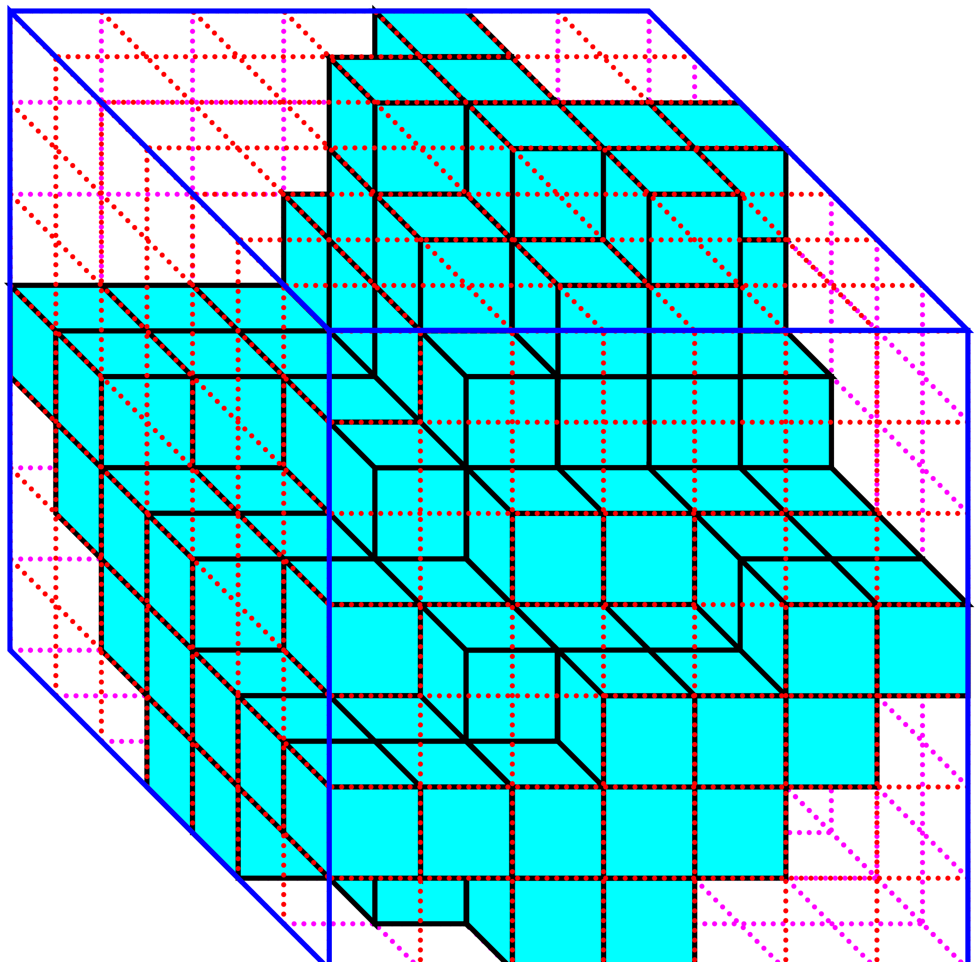
$P_F$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 3 | 2 | 1 |
| 0 | 0 | 0 | 2 | 4 | 4 | 3 |
| 0 | 0 | 0 | 4 | 4 | 2 | 2 |
| 2 | 2 | 4 | 5 | 2 | 4 | 4 |
| 3 | 4 | 5 | 4 | 3 | 3 | 0 |
| 5 | 4 | 3 | 2 | 1 | 0 | 0 |
| 2 | 2 | 2 | 1 | 0 | 0 | 0 |

$P_S$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 0 | 0 | 0 |
| 3 | 4 | 4 | 2 | 0 | 0 | 0 |
| 2 | 2 | 4 | 4 | 0 | 0 | 0 |
| 4 | 4 | 2 | 5 | 4 | 2 | 2 |
| 0 | 3 | 3 | 4 | 5 | 4 | 3 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 1 | 2 | 2 | 2 |

# Example of 3D Convex Polyomino



$P_T$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 | 1 | 1 |
| 2 | 2 | 2 | 4 | 3 | 1 | 1 |
| 2 | 1 | 2 | 3 | 3 | 3 | 3 |
| 3 | 2 | 2 | 3 | 3 | 4 | 2 |
| 2 | 3 | 2 | 2 | 2 | 2 | 1 |
| 1 | 2 | 3 | 2 | 1 | 2 | 1 |
| 1 | 1 | 2 | 3 | 2 | 2 | 1 |

$P_F$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 3 | 2 | 1 |
| 0 | 0 | 0 | 2 | 4 | 4 | 3 |
| 0 | 0 | 0 | 4 | 4 | 2 | 2 |
| 2 | 2 | 4 | 5 | 2 | 4 | 4 |
| 3 | 4 | 5 | 4 | 3 | 3 | 0 |
| 5 | 4 | 3 | 2 | 1 | 0 | 0 |
| 2 | 2 | 2 | 1 | 0 | 0 | 0 |

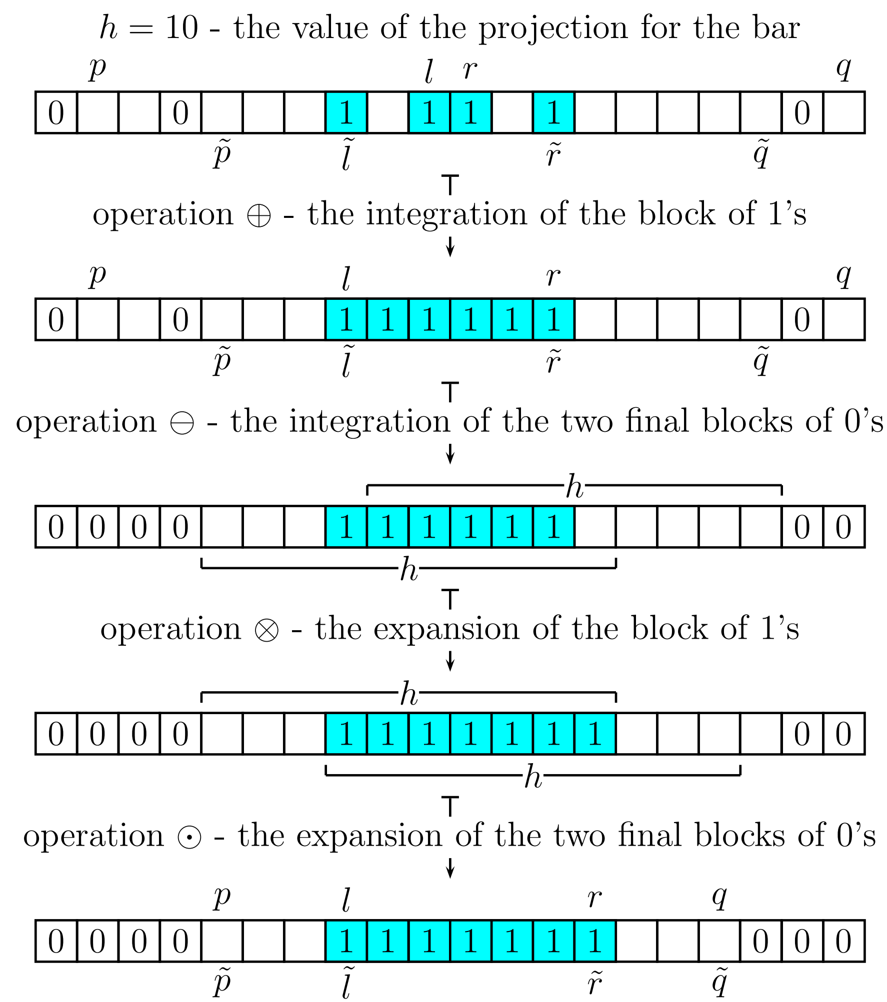
$P_S$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 0 | 0 | 0 |
| 3 | 4 | 4 | 2 | 0 | 0 | 0 |
| 2 | 2 | 4 | 4 | 0 | 0 | 0 |
| 4 | 4 | 2 | 5 | 4 | 2 | 2 |
| 0 | 3 | 3 | 4 | 5 | 4 | 3 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 1 | 2 | 2 | 2 |

# Filling Procedure - properties

- It returns *fail* if 3D convex polyomino with computing start positions and adequate matrices of projections does not exist.
- Otherwise it returns 3D convex polyomino.
- It works correctly if start positions contain at least one cell of polyomino in each vertical bar.

# Filling Procedure - the main idea



# Complexity

- $n^4$ – the number of different initial positions;
- $O(n^2)$ – the cost of computing of the start positions;
- $O(n^3 \log n)$ – the cost of the Filling Procedure.

Complexity of the Algorithm – $O(n^4 \cdot n^3 \log n)$.