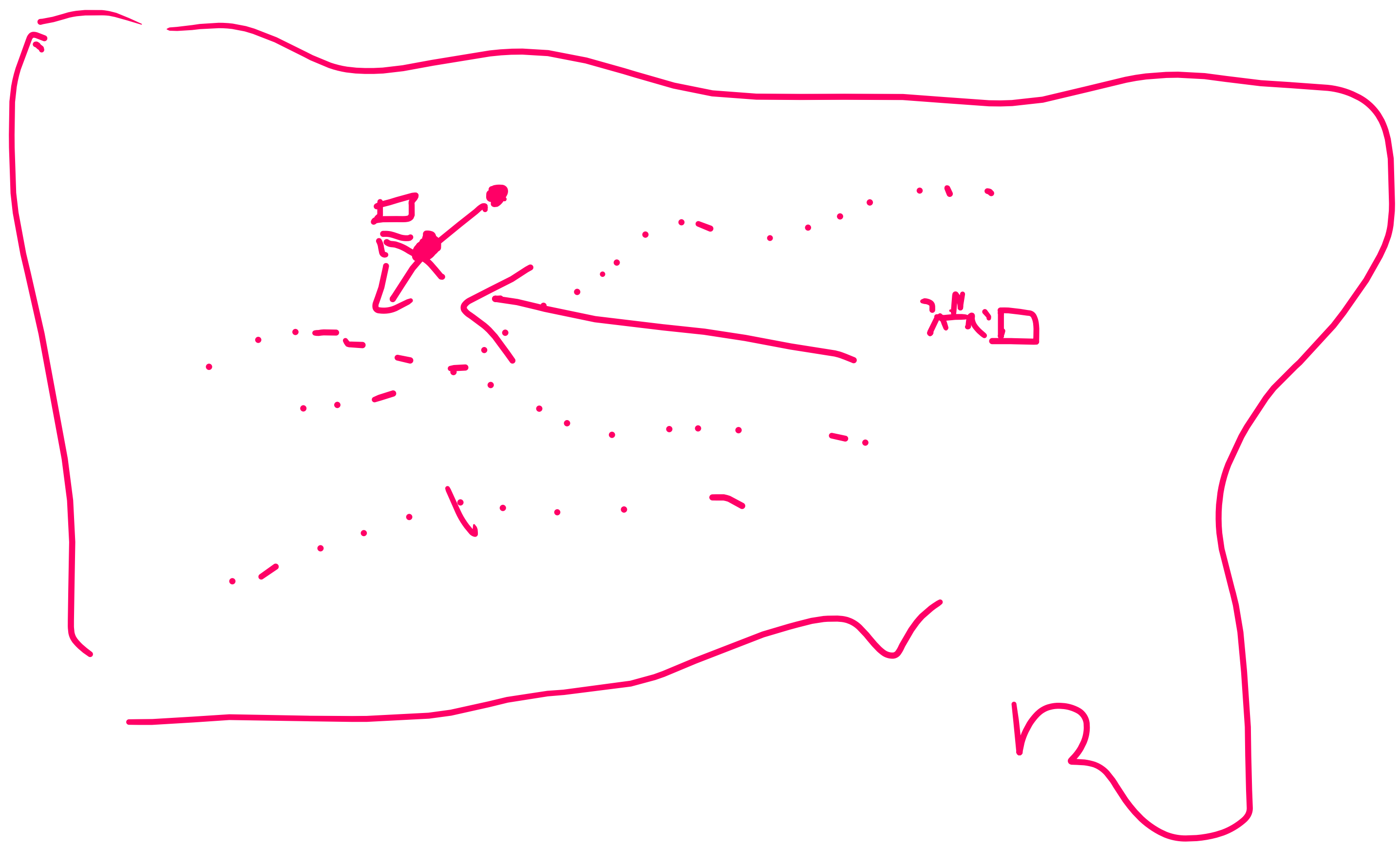
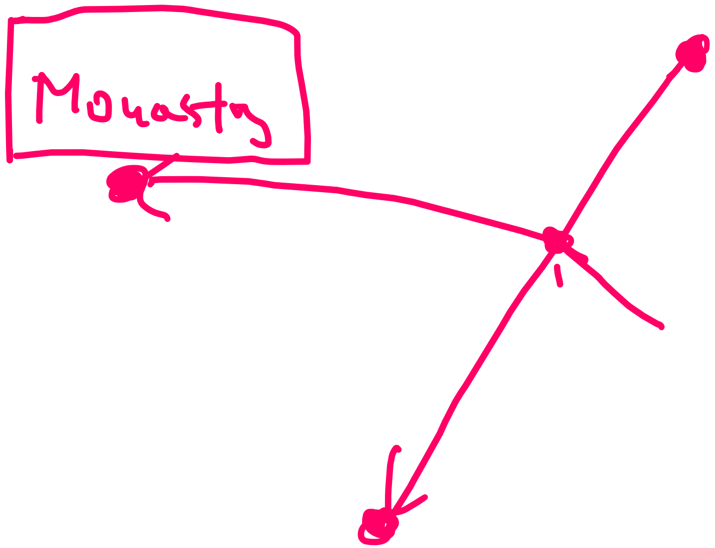
Small world

Algo lecture 21

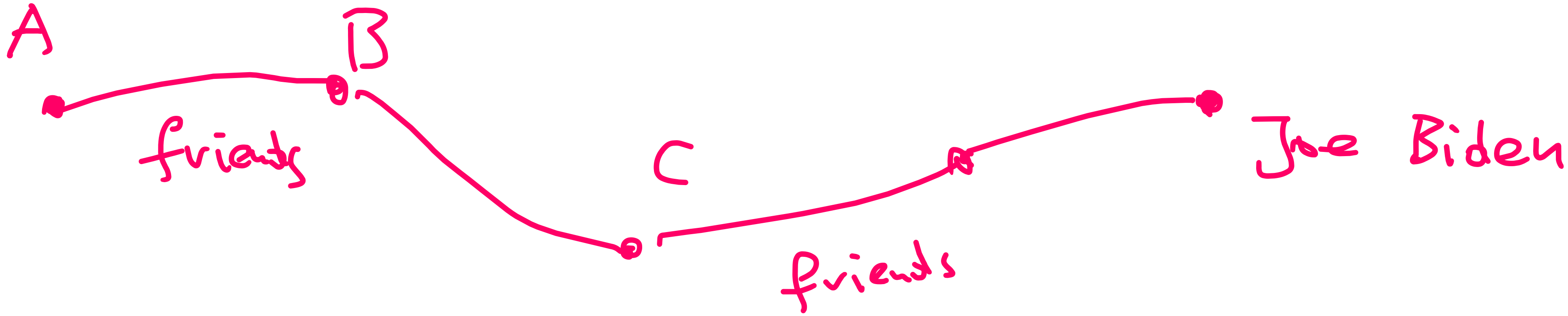
Milgram's experiment

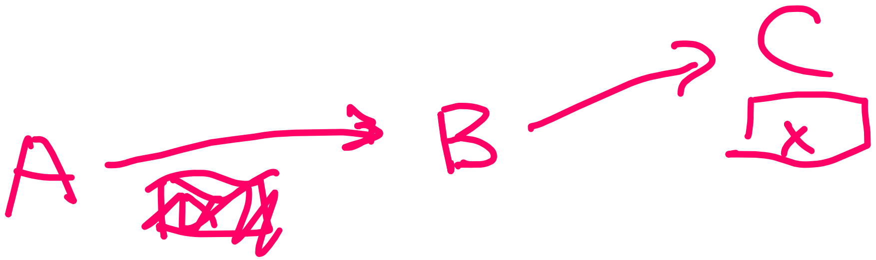



destination


Monastry

"SMALL WORLD"





Social networks

- experiments
- understand the graphs
  $\rightarrow$ model

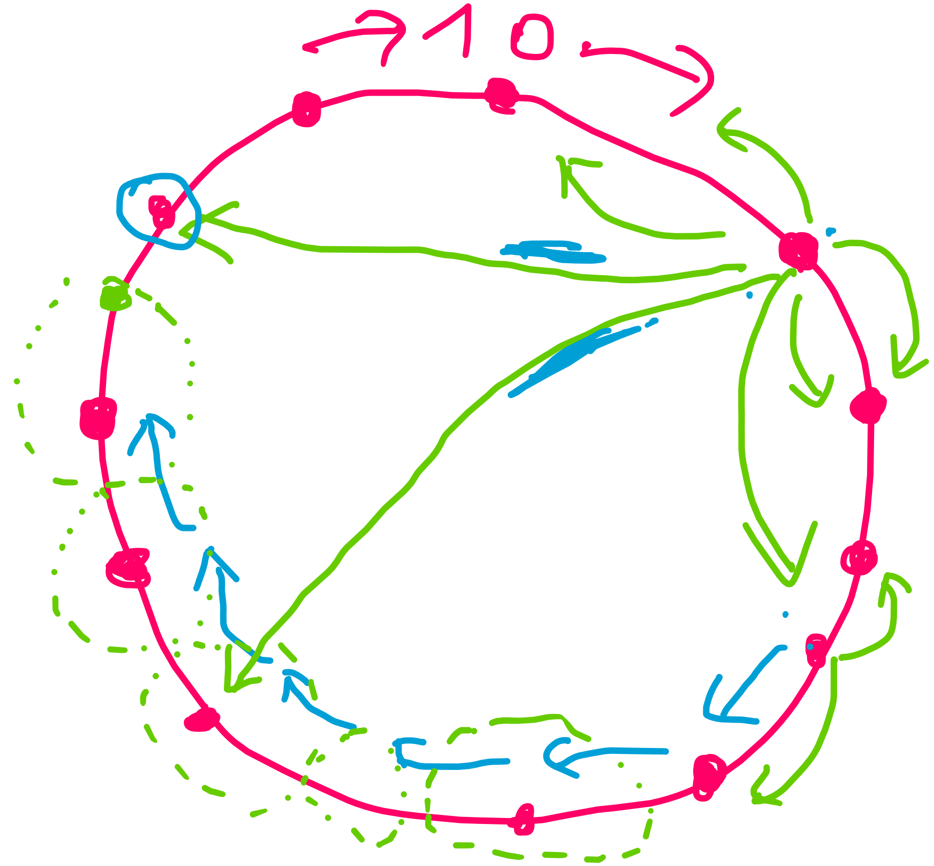
many models:

power law model

degree d - pbb $\sim \frac{1}{d^{\alpha}}$

$\alpha = \dots$

P2P network



mod 1

Augmented mesh

is modeled using a parameter $\alpha \geq 0$. *Each node* u *has a directed edge to every lattice neighbor. These are the* local contacts *of a node. In addition, each node also has an additional random link (the* long-range contact*). For all* u *and* v, *the long-range contact of* u *points to node* v *with probability proportional to* $d(u,v)^{-\alpha}$, *i.e., with probability* $d(u,v)^{-\alpha} / \sum_{w \in V \setminus \{u\}} d(u,w)^{-\alpha}$. *Figure 9.3 illustrates the model.*

$$\Pr(u \to v) = f(d(u,v))$$

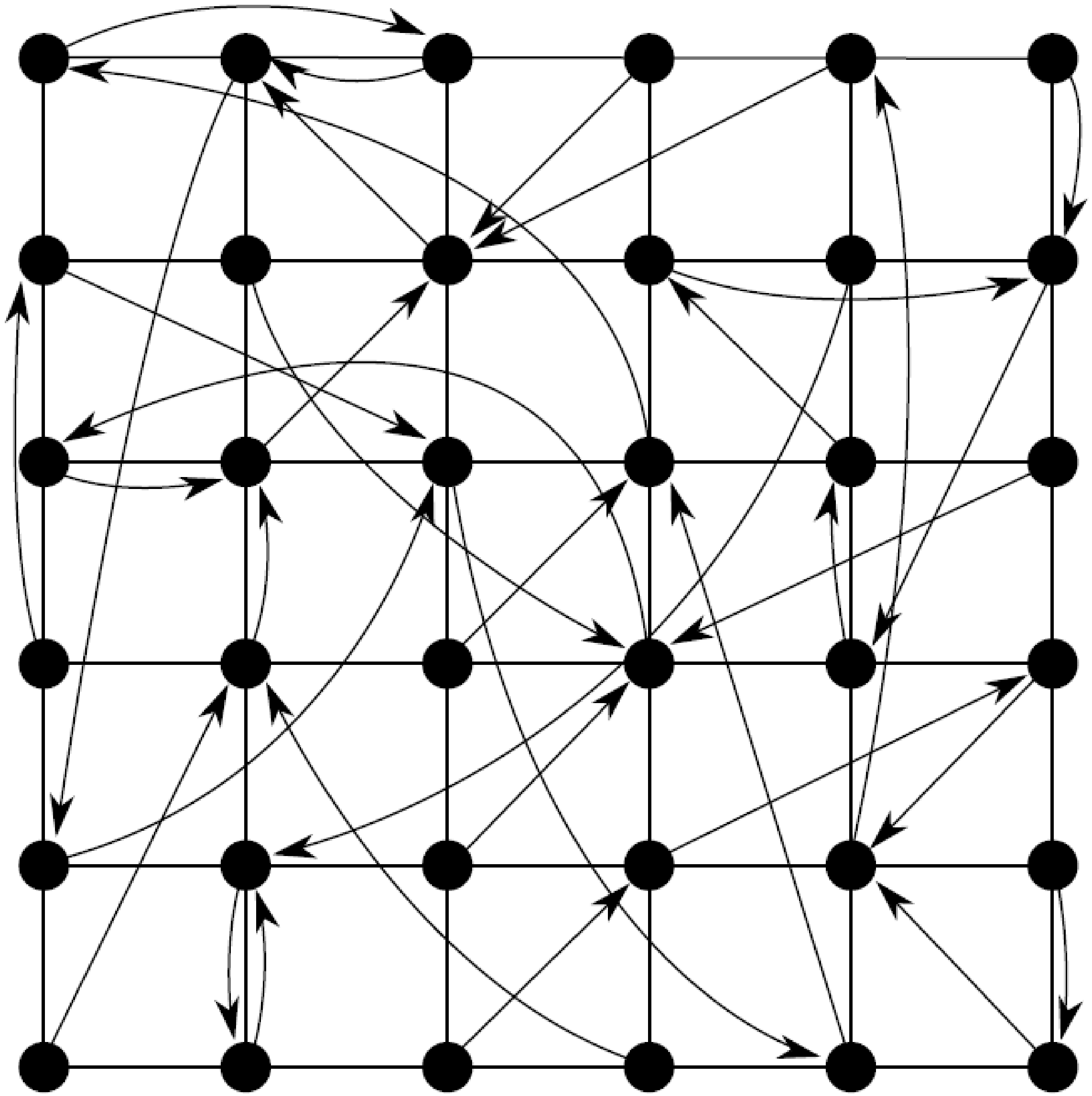
Augmented mesh

is modeled using a parameter $\alpha \geq 0$. Each node u has a directed edge to every lattice neighbor. These are the local contacts *of a node. In addition, each node also has an additional random link (the* long-range contact*). For all u and v, the long-range contact of u points to node v with probability proportional to $d(u,v)^{-\alpha}$, i.e., with probability $d(u,v)^{-\alpha} / \sum_{w \in V \setminus \{u\}} d(u,w)^{-\alpha}$. Figure 9.3 illustrates the model.*

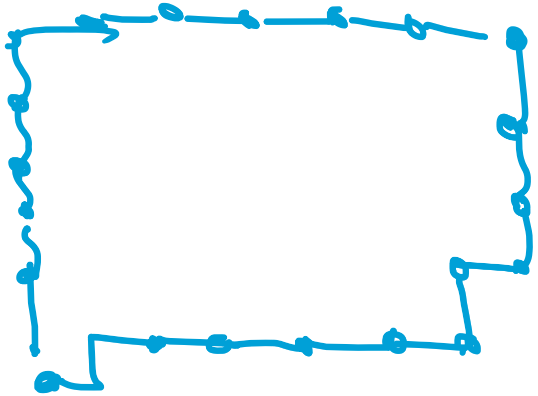
$$Z_u = \sum_{w \neq u} \frac{1}{d(u,w)^\alpha}$$

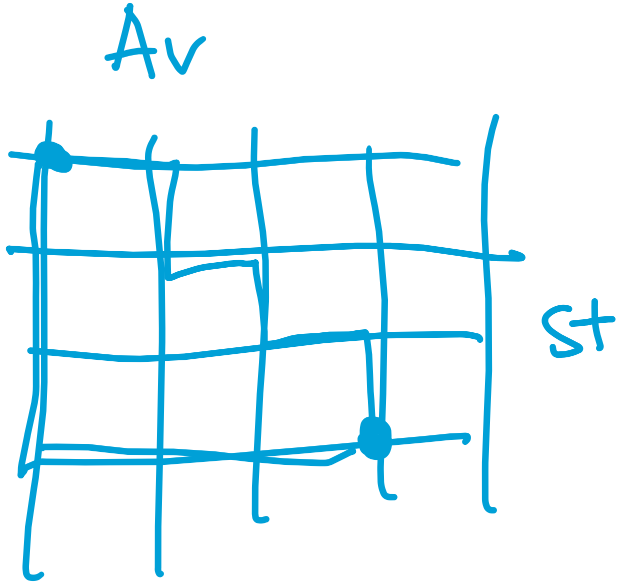
$\alpha = \text{const}$

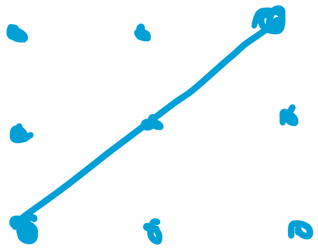
$$\Pr(u \to v) = \frac{\frac{1}{d(u,v)^\alpha}}{Z_u}$$



Manhattan distance







$\sqrt{\ldots}$

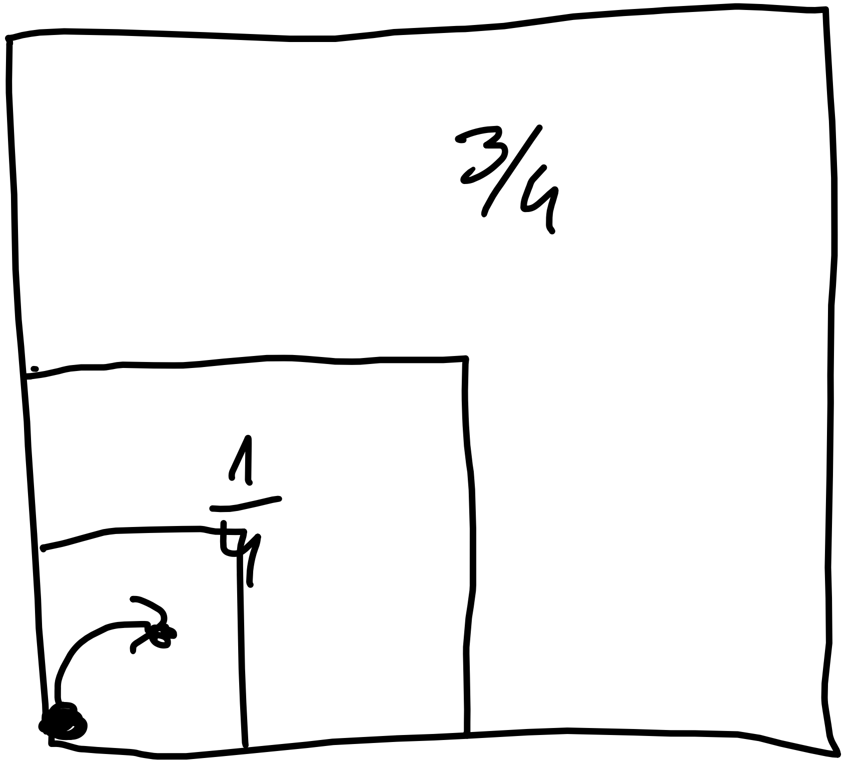
Freedom with α:

$\alpha = 0$

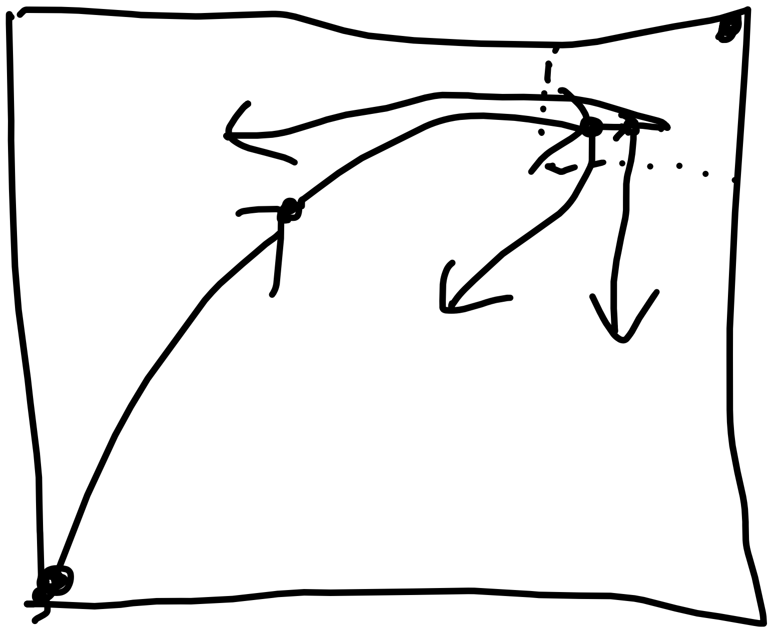
$$\sum \frac{1}{d(u,w)^0} = \sum \frac{1}{1} = \#\text{nodes} - 1$$

$$Pr(u \to v) = \frac{1}{\#\text{nodes} - 1}$$

uniform!

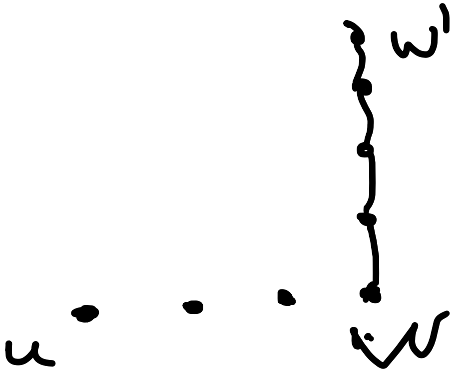


overshooting



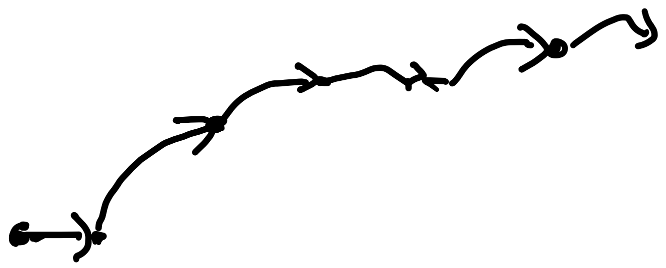
$\alpha > 0$, even α large e.g. $\alpha = 4$

$$\frac{1}{d(u,w)^4}$$

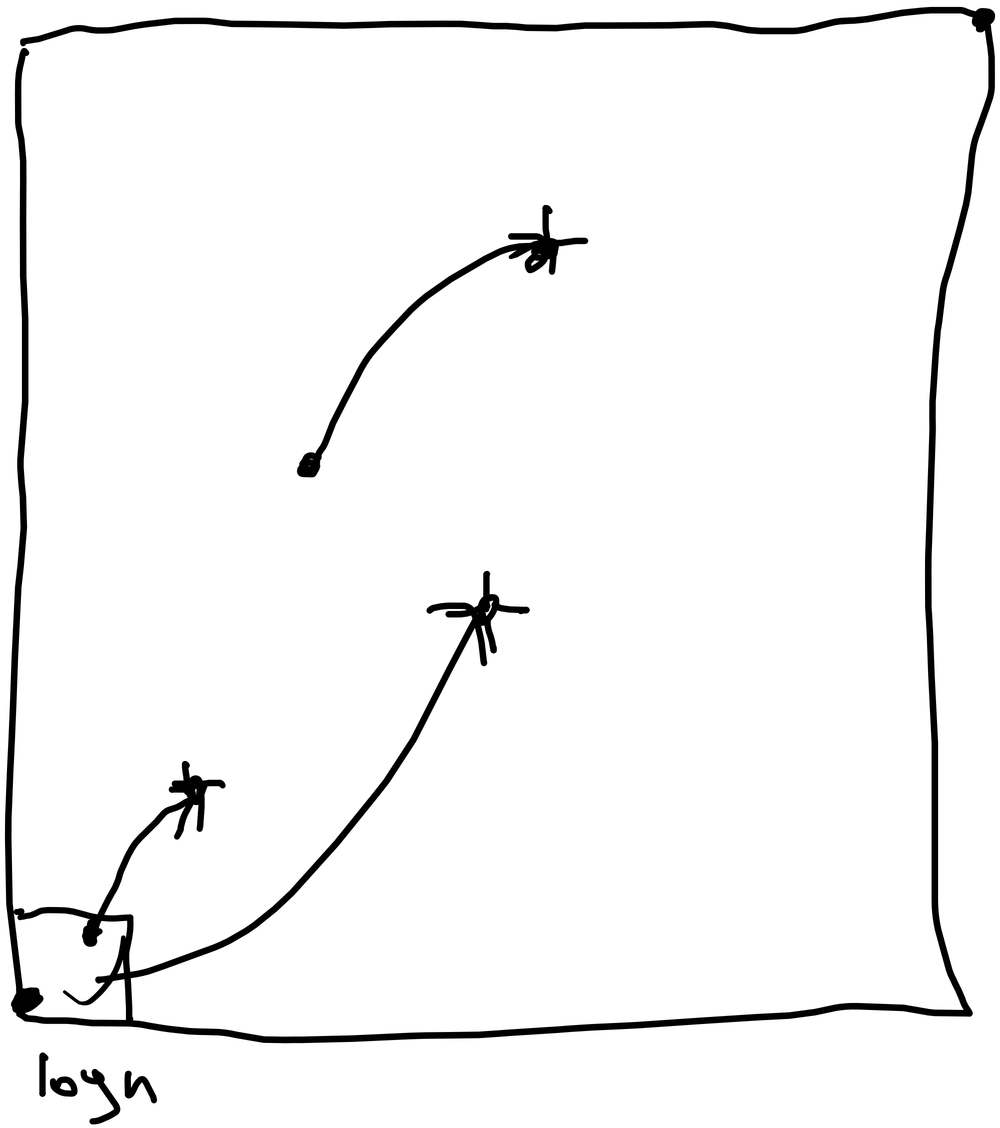


$$d = \frac{1}{4^4} \qquad d = \frac{1}{8^4} = \frac{1}{4^4} \cdot \frac{1}{2^4}$$

$$d = \frac{1}{32^4} = \frac{1}{4^4} \cdot \frac{1}{8^4}$$



Theorem 9.6. *The diameter of the augmented grid with $\alpha = 0$ is $\mathcal{O}(\log n)$ with high probability.*

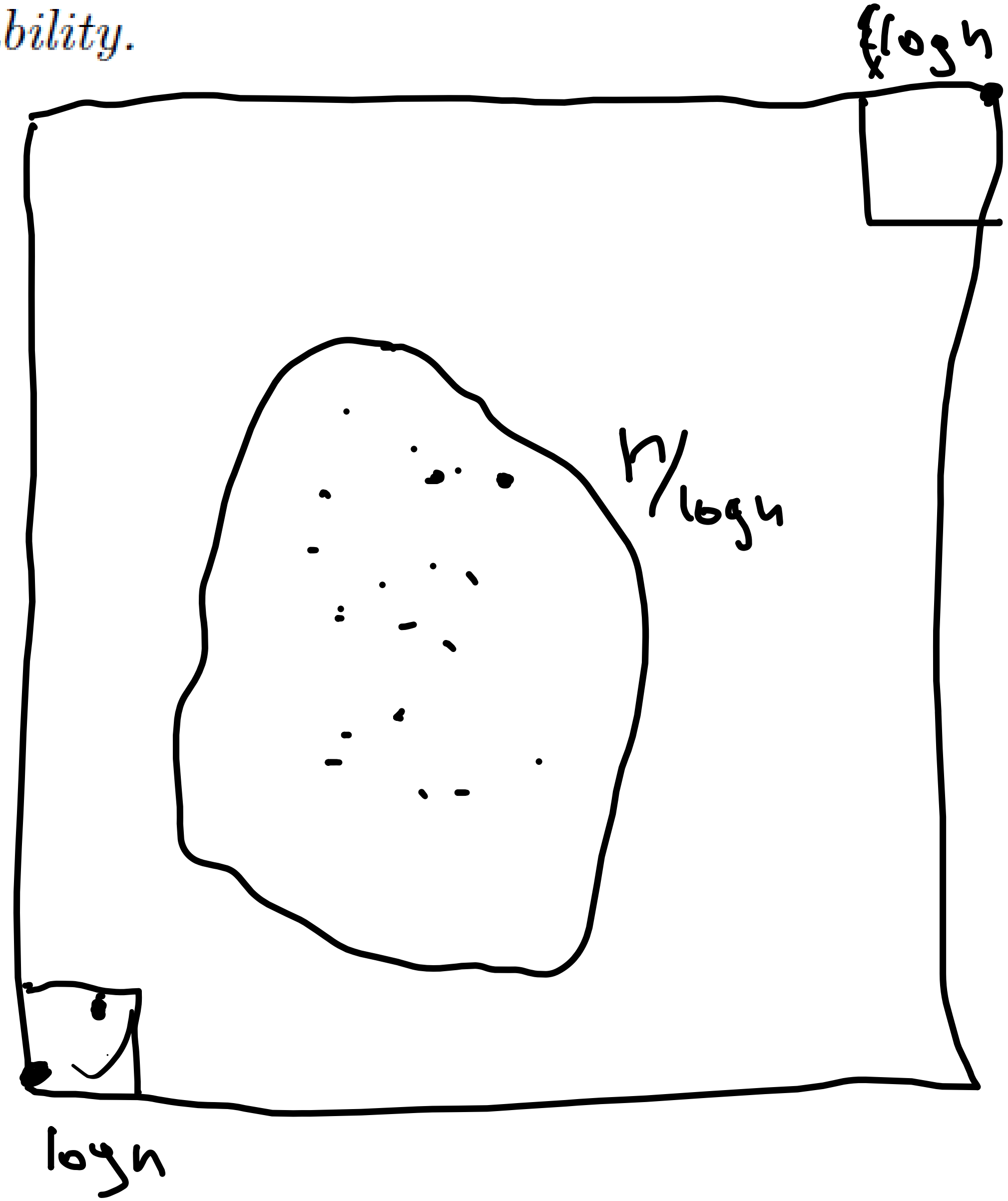


$\approx (\log n)^2$ points

$\approx 6 \cdot (\log n)^2$ points

$\approx 6 \cdot 6 \cdot (\log n)^2$ points

Theorem 9.6. *The diameter of the augmented grid with $\alpha = 0$ is $\mathcal{O}(\log n)$ with high probability.*

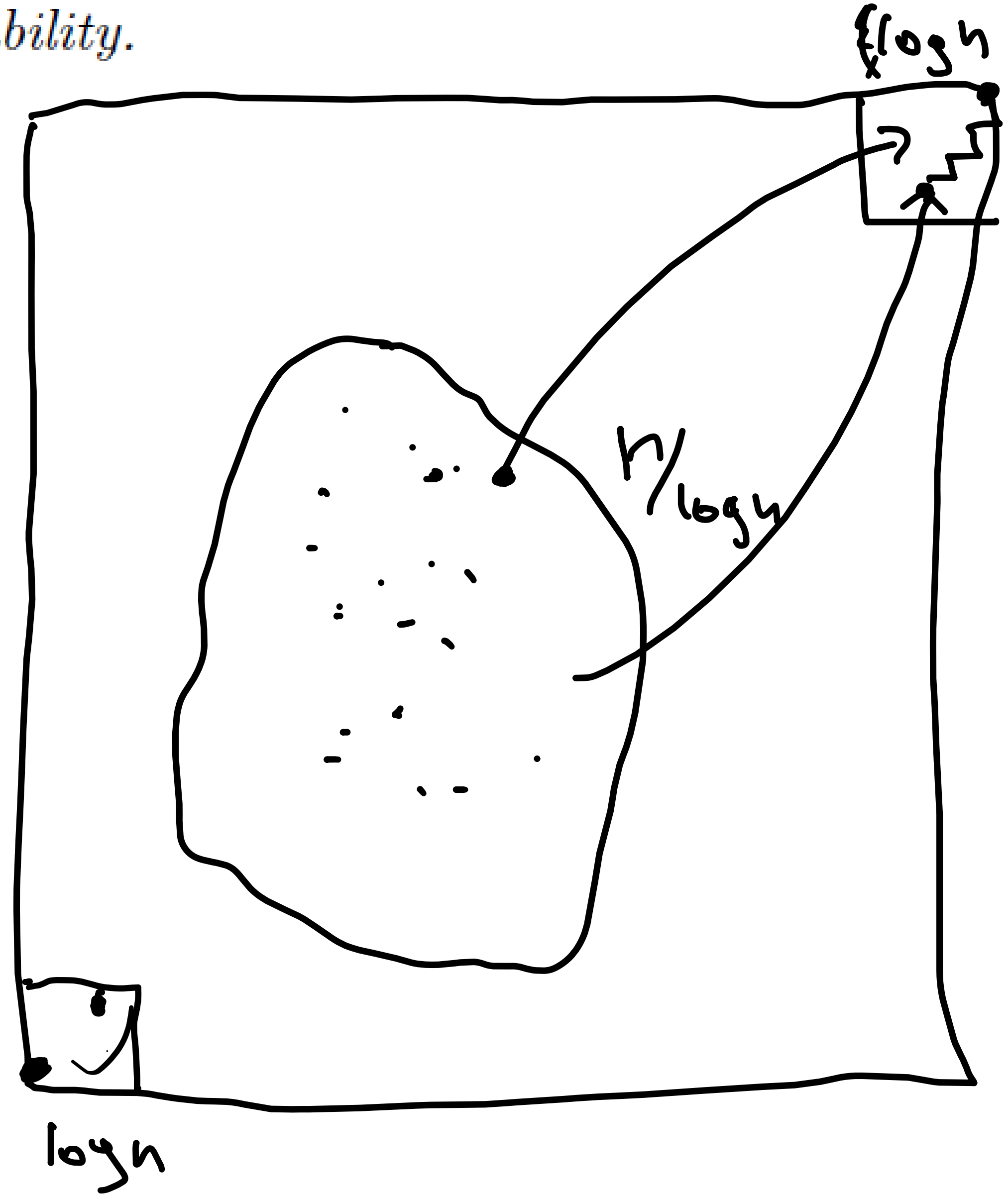


$\approx (\log n)^2$ points

$\gtrsim 5 \cdot (\log n)^2$ points

$\gtrsim 6 \cdot 6 \cdot (\log n)^2$ points

Theorem 9.6. *The diameter of the augmented grid with $\alpha = 0$ is $\mathcal{O}(\log n)$ with high probability.*



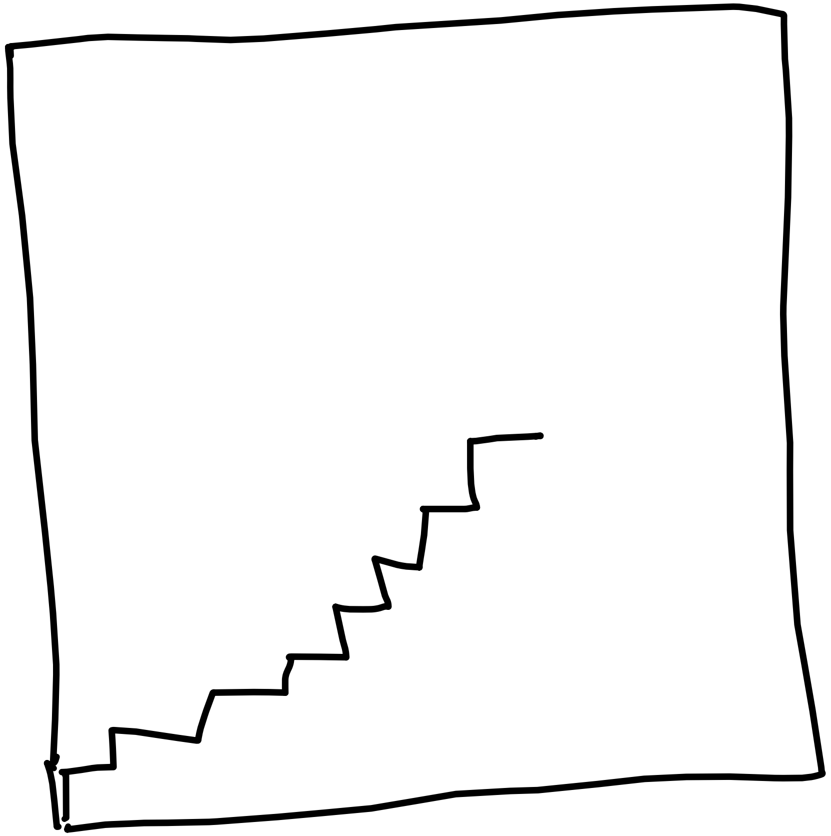
$$\frac{(\log n)^2}{n} \cdot \frac{n}{\log n} =$$

$$\log n$$

Algorithm 9.7 Greedy Routing

```
while not at destination do
    go to a neighbor which is closest to destination (considering grid distance only)
end while
```

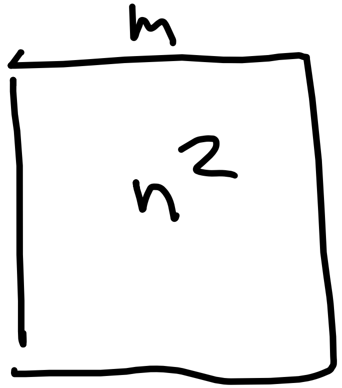
Lemma 9.8. *In the augmented grid, Algorithm 9.7 finds a routing path of length at most* $2(m-1) \in O(\sqrt{n})$.



$2\sqrt{n}$

Lemma 9.9. *Node u's random link points to a node v with probability*

- $\Theta(1/(d(u,v)^\alpha m^{2-\alpha}))$ *if* $\alpha < 2$. $1/m^{2-\alpha}$
- $\Theta(1/(d(u,v)^2 \log n))$ *if* $\alpha = 2$,
- $\Theta(1/d(u,v)^\alpha)$ *if* $\alpha > 2$.

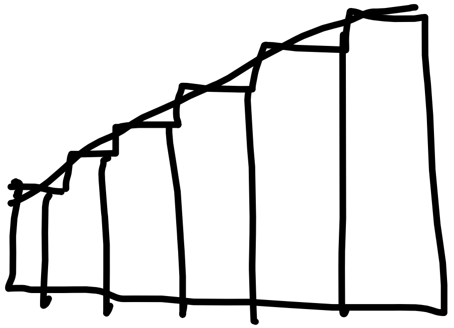


Moreover, if $\alpha > 2$*, the probability to see a link of length at least* d *is in* $\Theta(1/d^{\alpha-2})$.

$$c_1 \cdot \frac{1}{d(u,v)^\alpha} < Pr < c_2 \cdot \frac{1}{d(u,v)^\alpha}$$

Proof. For a constant $\alpha \neq 2$, we have that

$$\sum_{w\in V\setminus\{u\}} \frac{1}{d(u,w)^\alpha} \in \sum_{r=1}^{m} \frac{\Theta(r)}{r^\alpha} = \Theta\left(\int_{r=1}^{m} \frac{1}{r^{\alpha-1}}\,dr\right) = \Theta\left(\left[\frac{r^{2-\alpha}}{2-\alpha}\right]_1^m\right).$$



If $\alpha < 2$, this gives $\Theta(m^{2-\alpha})$, if $\alpha > 2$, it is in $\Theta(1)$.

If $\alpha = 2$,

$$\sum_{w \in V \setminus \{u\}} \frac{1}{d(u,w)^\alpha} \in \sum_{r=1}^{m} \frac{\Theta(r)}{r^2} = \Theta(1) \cdot \sum_{r=1}^{m} \frac{1}{r} = \Theta(\log m) = \Theta(\log n).$$

if $\alpha > 2$,

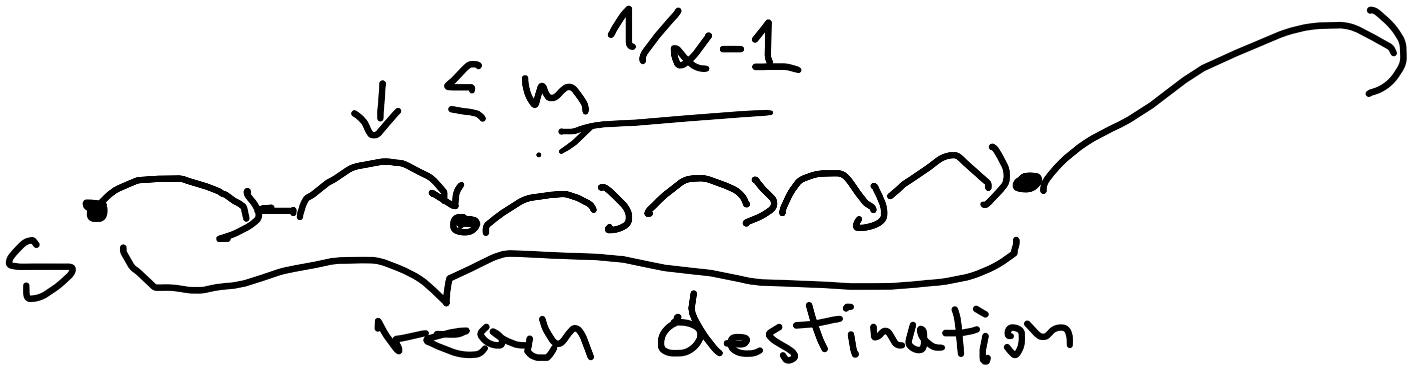
$$\sum_{\substack{v \in V \\ d(u,v) \geq d}} \Theta(1/d(u,v)^{\alpha}) = \Theta\left(\int_{r=d}^{m} \frac{r}{r^{\alpha}}\, dr\right) = \Theta\left(\left[\frac{r^{2-\alpha}}{2-\alpha}\right]_{d}^{m}\right) = \Theta(1/d^{\alpha-2}).$$

If $\alpha > 2$, according to the lemma, the probability to see a random link of length at least $d = m^{1/(\alpha-1)}$ is $\Theta(1/d^{\alpha-2}) = \Theta(1/m^{(\alpha-2)/(\alpha-1)})$.

In expectation we have to take $\Theta(m^{(\alpha-2)/(\alpha-1)})$ hops until we see a random link of length at least d.

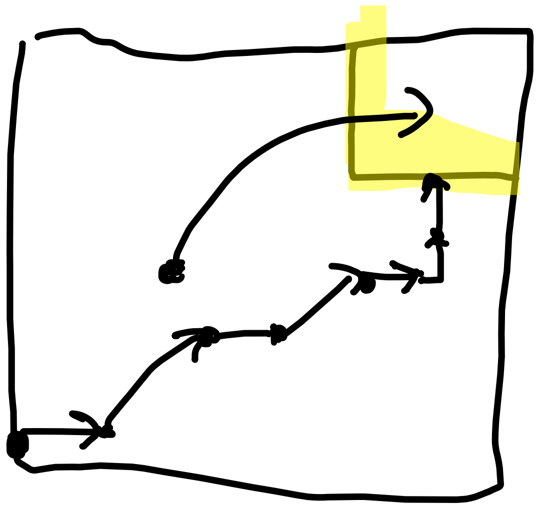
When just following links of length

less than d, it takes more than $m/d = m/m^{1/(\alpha-1)} = m^{(\alpha-2)/(\alpha-1)}$



$\alpha < 2$,

draw a border around the nodes in distance $m^{(2-\alpha)/3}$ to the target.

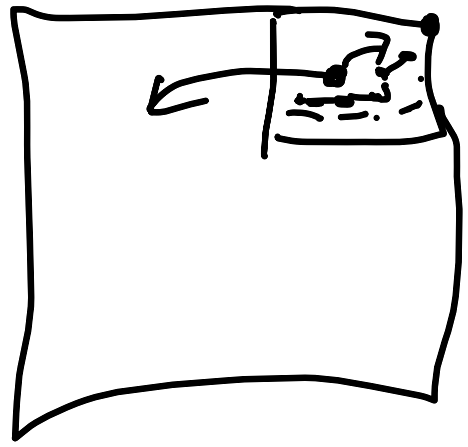


about $m^{2(2-\alpha)/3}$ many nodes

probability to find a random link that leads directly inside the target area is according to the lemma at most $m^{2(2-\alpha)/3} \cdot \Theta(1/m^{2-\alpha}) = \Theta(1/m^{(2-\alpha)/3})$,

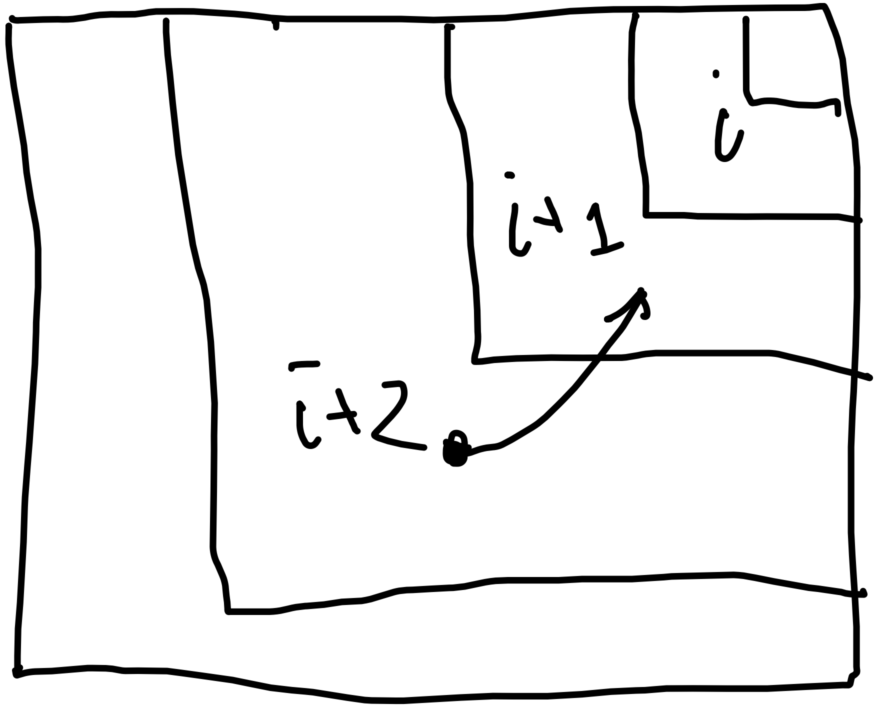
$\frac{1}{m^{-}}$

In other words, until we find a random link that leads into the target area, in expectation, we have to do $\Theta(m^{(2-\alpha)/3})$ hops. This is too slow, and our greedy strategy is probably faster, Once inside the target area, again the probability of short-cutting our trip by a random long-range link is $\Theta(1/m^{(2-\alpha)/3})$, so we probably just follow grid links, $m^{(2-\alpha)/3} = m^{\Omega(1)}$ many of them.



$\alpha = 2$

Definition 9.10 (Phase). *Consider routing from source s to target t and assume that we are at some intermediate node w. We say that we are in phase j at node w if the lattice distance $d(w,t)$ to the target node t is between $2^j < d(w,t) \leq 2^{j+1}$.*



Lemma 9.11. *Assume that we are in phase j at node w when routing from s to t. The probability for getting (at least) to phase $j-1$ in one step is at least $\Omega(1/\log n)$.*

For all nodes $x \in B_j$, we have $d(w,x) \le d(w,t) + d(x,t) \le 2^{j+1} + 2^j < 2^{j+2}$. Hence, for each node $x \in B_j$, the probability that the long-range contact of w points to x is $\Omega(1/2^{2j+4}\log n)$.

the number of nodes in B_j is at least

$(2^j)^2/2 = 2^{2j-1}$.

$$\Omega\left(|B_j| \cdot \frac{1}{2^{2j+4}\log n}\right) = \Omega\left(\frac{2^{2j-1}}{2^{2j+4}\log n}\right) = \Omega\left(\frac{1}{\log n}\right).$$

Theorem 9.12. *Consider the greedy routing path from a node s to a node t on an augmented grid with parameter $\alpha = 2$. The expected length of the path is $\mathcal{O}(\log^2 n)$.*

phase $\approx \log n / 2$

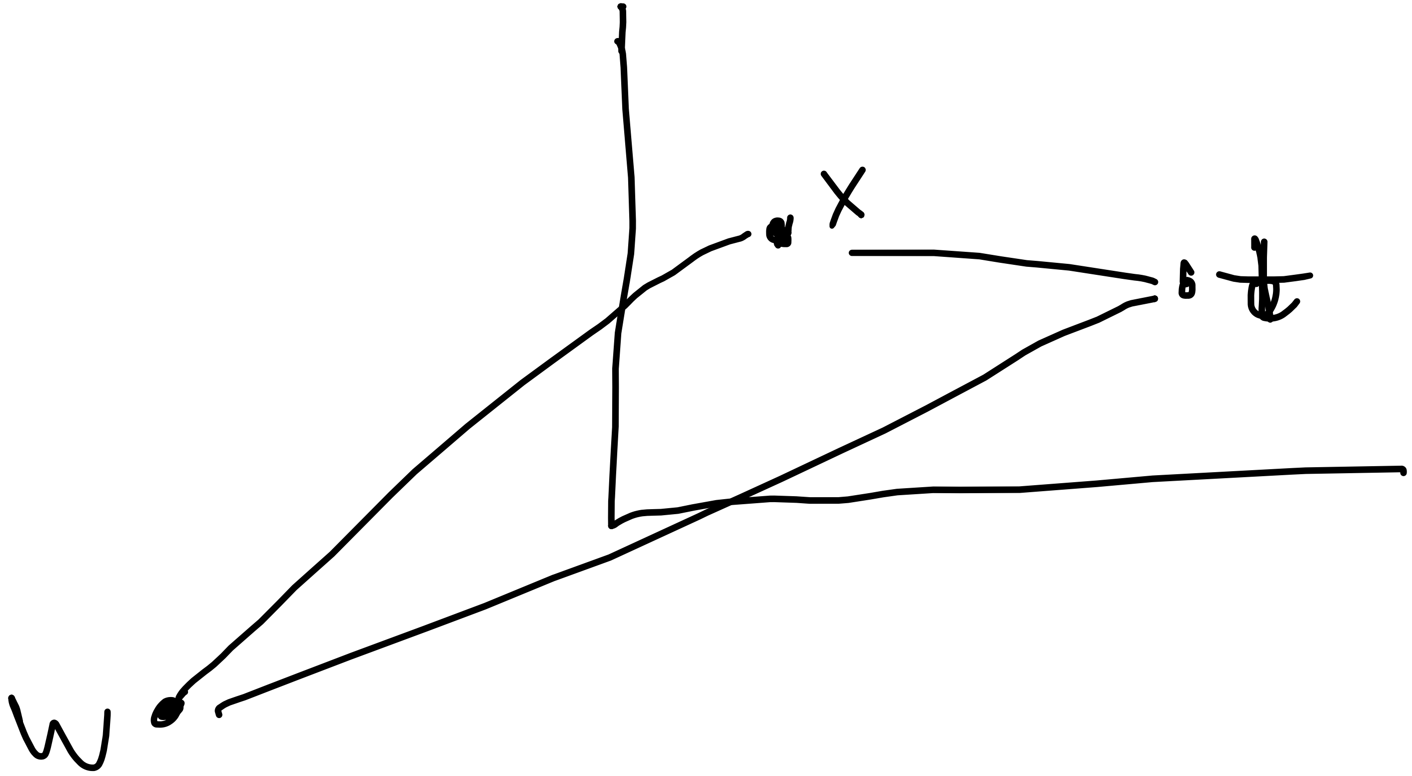
$2^{\frac{\log n}{2}} = \sqrt{n}$

$E(\text{steps}) = \log n)$

⋮

⋮

phase $\underline{\log n}$


X
W

$\alpha \neq 2 \Rightarrow$ time $\Theta(n^{\cdots})$

$\alpha = 2 \Rightarrow$ " $\Theta(\log^2 n)$