Locality

Algo lecture 2021, M.Kutyłowski

Coloring problem

log*(n)+O(1) time algorithm already discussed for trees

PROBLEM: can we do any better?

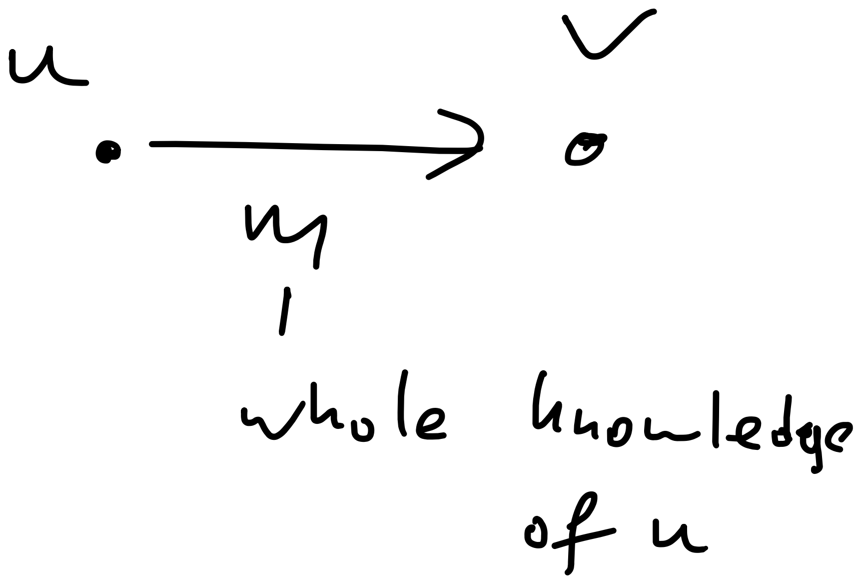
A LOWER BOUND: there is no algorithm that runs faster than …

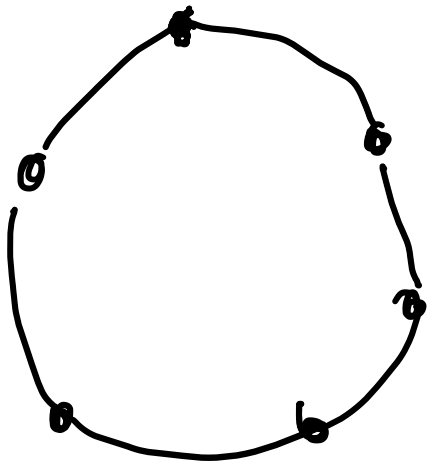
Model:

+ Unbounded communication

+ Unbounded local communication

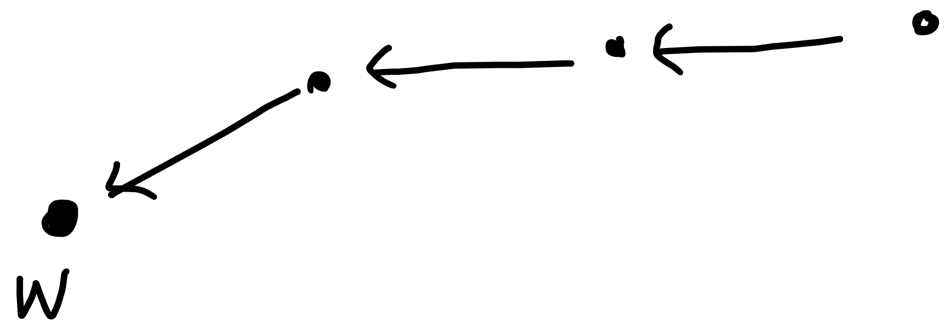
- Deterministic
- Ring architecture





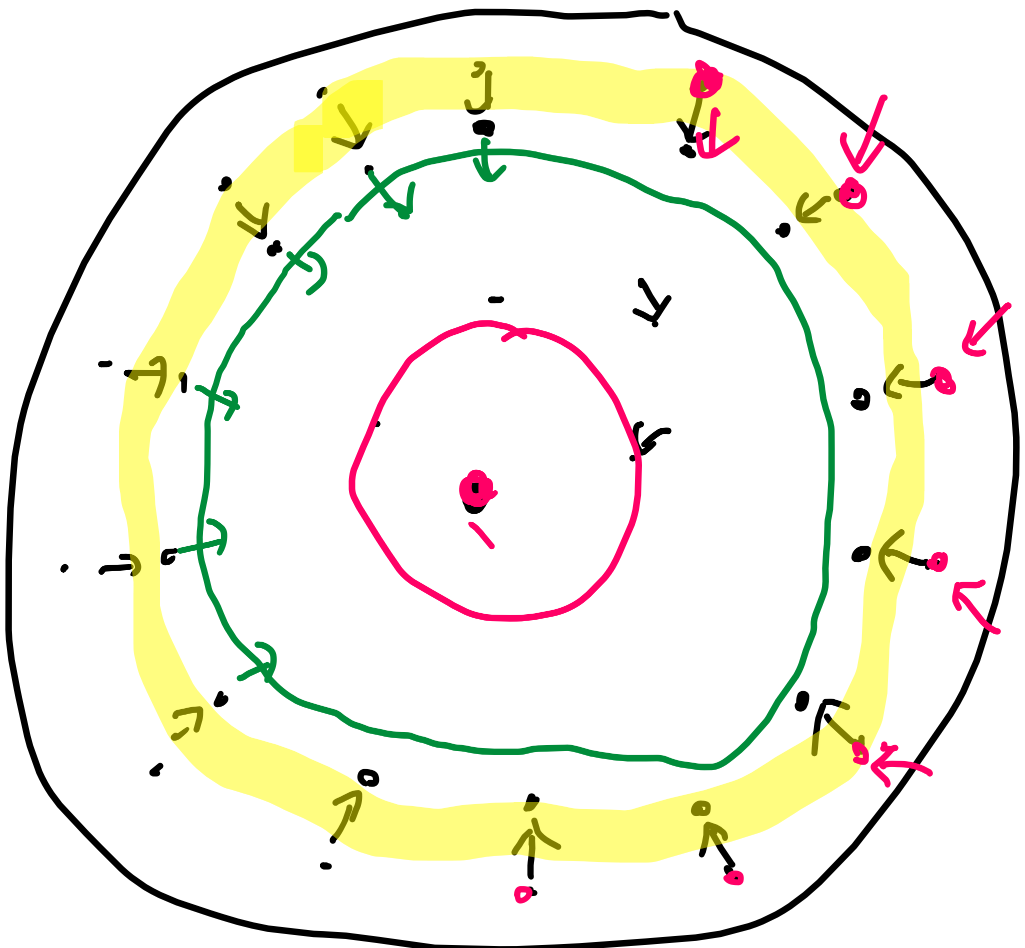
Algorithm 8.1 Synchronous Algorithm: Canonical Form

In r rounds: **send** complete initial state to nodes at distance at most r
// do all the communication first
Compute output based on complete information about r-neighborhood
// do all the computation in the end



time r : horizon is cube of radius r

Lemma 8.2. *If message size and local computations are not bounded, every deterministic, synchronous r-round algorithm can be transformed into an algorithm of the form given by Algorithm 8.1 (i.e., it is possible to first communicate for r rounds and then do all the computations in the end).*

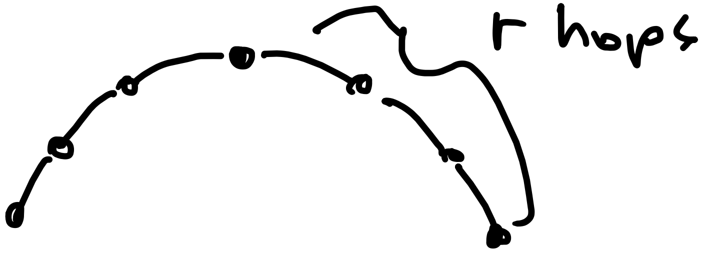


emul. original Algo.

Definition 8.3 (r-hop view). *We call the collection of the initial states of all nodes in the r-neighborhood of a node v, the r-hop view of v.*

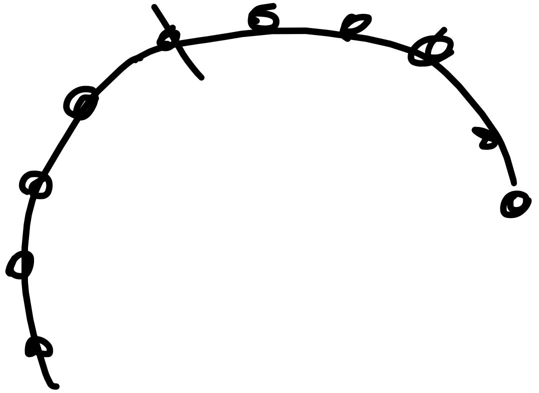
Corollary 8.4. *A deterministic r-round algorithm $\mathcal{A}$ is a function that maps every possible r-hop view to the set of possible outputs.*

r-hop views on a ring,
neighboring views

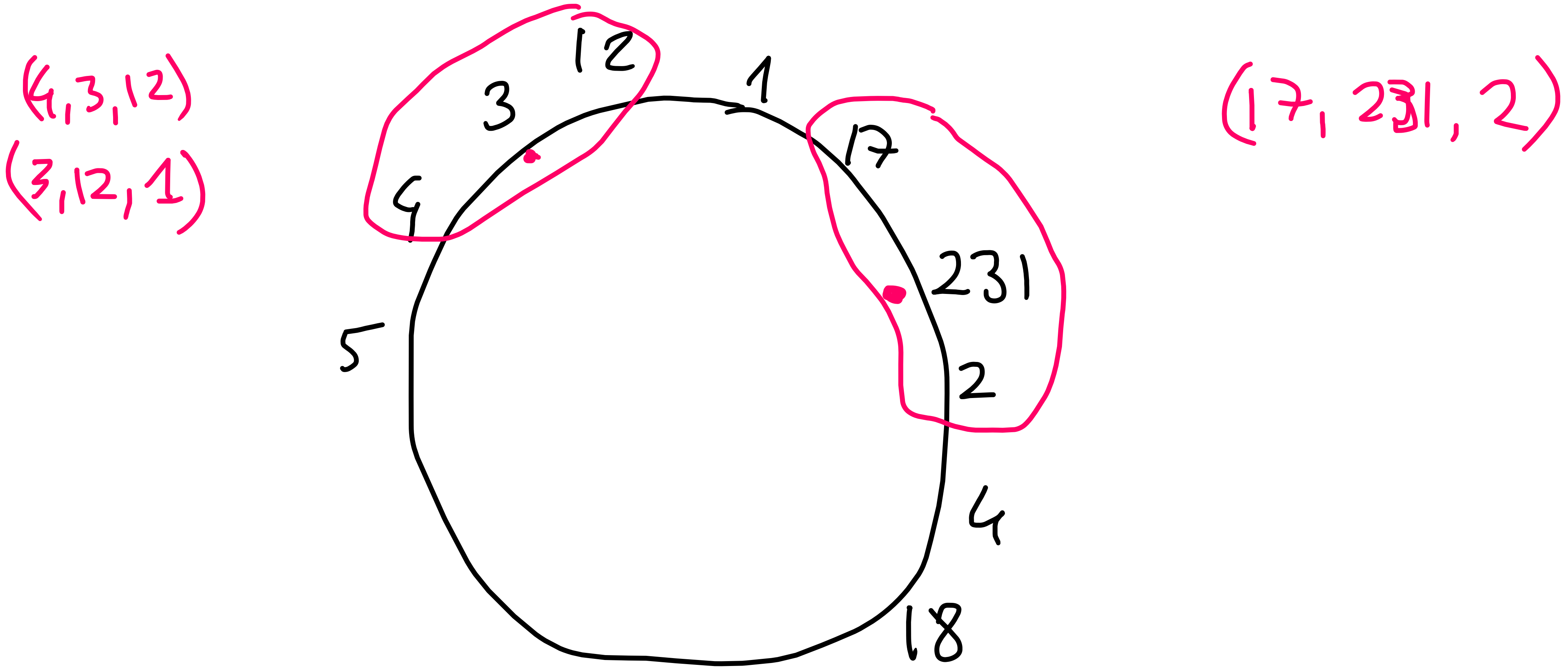


initial state of 2r+1 nodes

decision:



Definition 8.5 (Neighborhood Graph). *For a given family of network graphs $\mathcal{G}$, the r-neighborhood graph $\mathcal{N}_r(\mathcal{G})$ is defined as follows. The node set of $\mathcal{N}_r(\mathcal{G})$ is the set of all possible labeled r-neighborhoods (i.e., all possible r-hop views). There is an edge between two labeled r-neighborhoods $\mathcal{V}_r$ and $\mathcal{V}'_r$ if $\mathcal{V}_r$ and $\mathcal{V}'_r$ can be the r-hop views of two adjacent nodes.*



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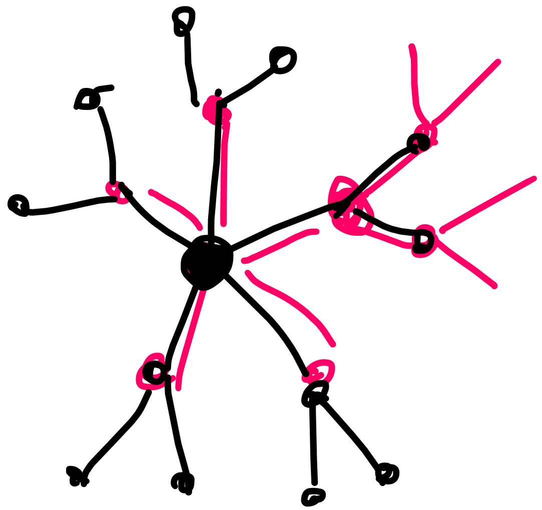
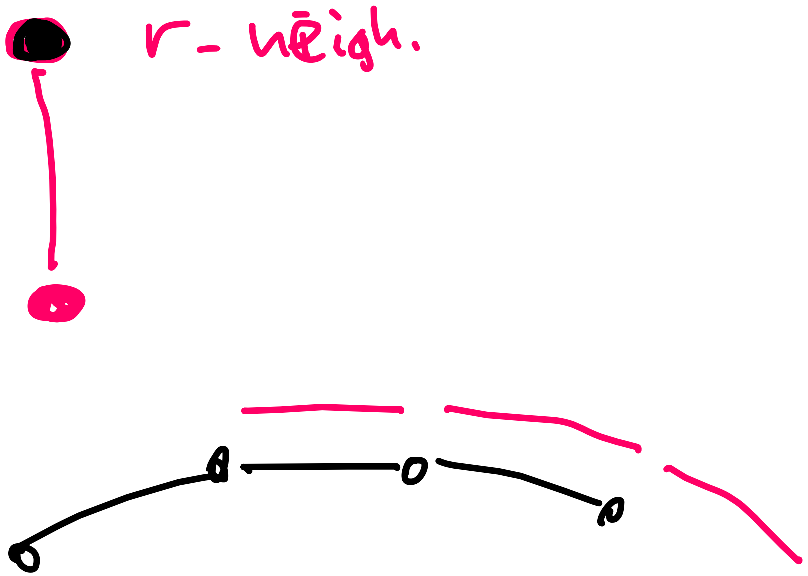




Lemma 8.6. *For a given family of network graphs $\mathcal{G}$, there is an r-round algorithm that colors graphs of $\mathcal{G}$ with c colors iff the chromatic number of the neighborhood graph is $\chi(\mathcal{N}_r(\mathcal{G})) \leq c$.*

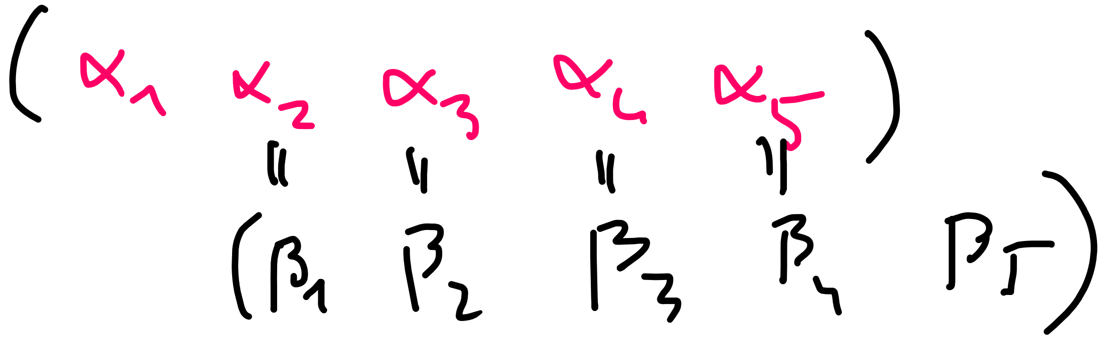
$N_r(G)$ - neighbors $\Rightarrow$ can be neighbors in G

$\Rightarrow$ A cannot assign same colour

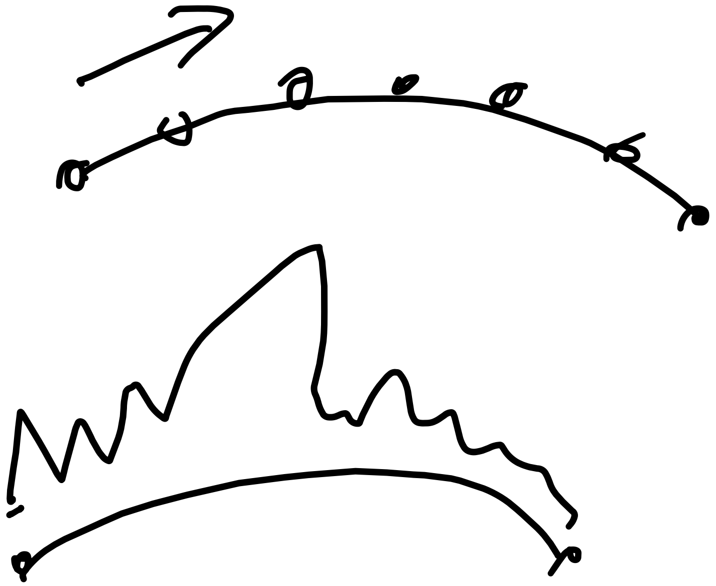
$$V[\mathcal{B}_k] \;=\; \{(\alpha_1, \ldots, \alpha_k) : \alpha_i \in [n], i < j \rightarrow \alpha_i < \alpha_j\} \tag{8.1}$$

For $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ and $\underline{\beta} = (\beta_1, \ldots, \beta_k)$ there is a directed edge from $\underline{\alpha}$ to $\underline{\beta}$ iff

$$\forall i \in \{1, \ldots, k-1\} : \beta_i = \alpha_{i+1}. \tag{8.2}$$



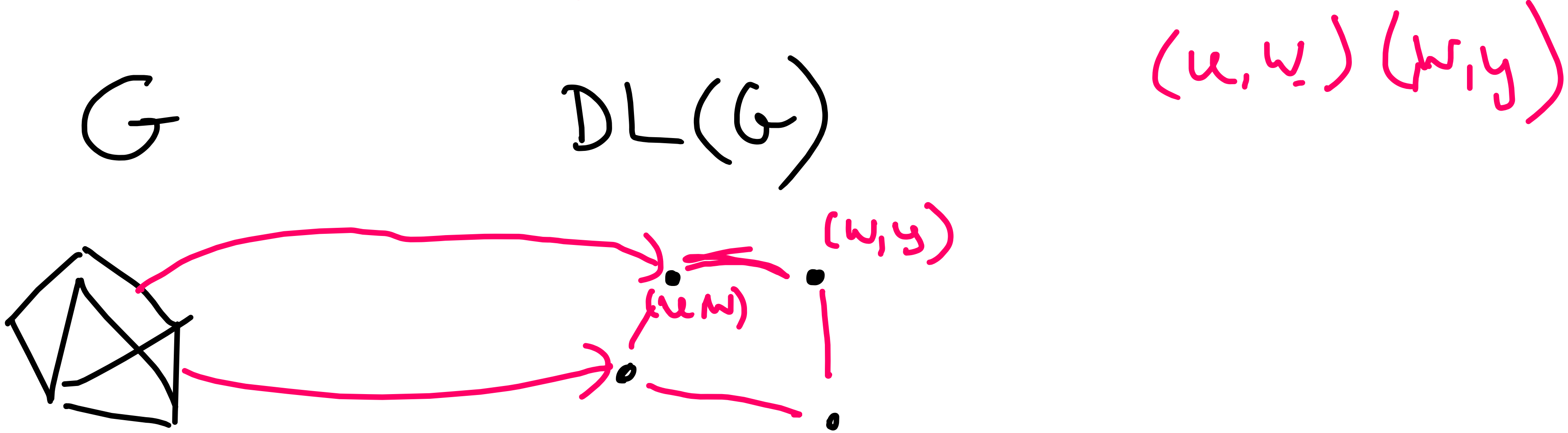
(**Lemma 8.7.** *Viewed as an undirected graph, the graph* $\mathcal{B}_{2r+1}$ *is a subgraph of the* r*-neighborhood graph of directed* n*-node rings with node labels from* $[n]$.



Definition 8.8 (Diline Graph). *The directed line graph (diline graph) $\mathcal{DL}(G)$ of a directed graph $G = (V, E)$ is defined as follows. The node set of $\mathcal{DL}(G)$ is $V[\mathcal{DL}(G)] = E$. There is a directed edge $((w, x), (y, z))$ between $(w, x) \in E$ and $(y, z) \in E$ iff $x = y$, i.e., if the first edge ends where the second one starts.*

Lemma 8.9. *If $n > k$, the graph $\mathcal{B}_{k+1}$ can be defined recursively as follows:*

$$\mathcal{B}_{k+1} = \mathcal{DL}(\mathcal{B}_k).$$



B_r $\quad (\alpha_1, \dots, \alpha_r)$

$\mathrm{DLIN}(B_r)$ $\quad (\alpha_1, \dots, \alpha_r), (\alpha_2, \alpha_3, \dots, \alpha_{r+1})$

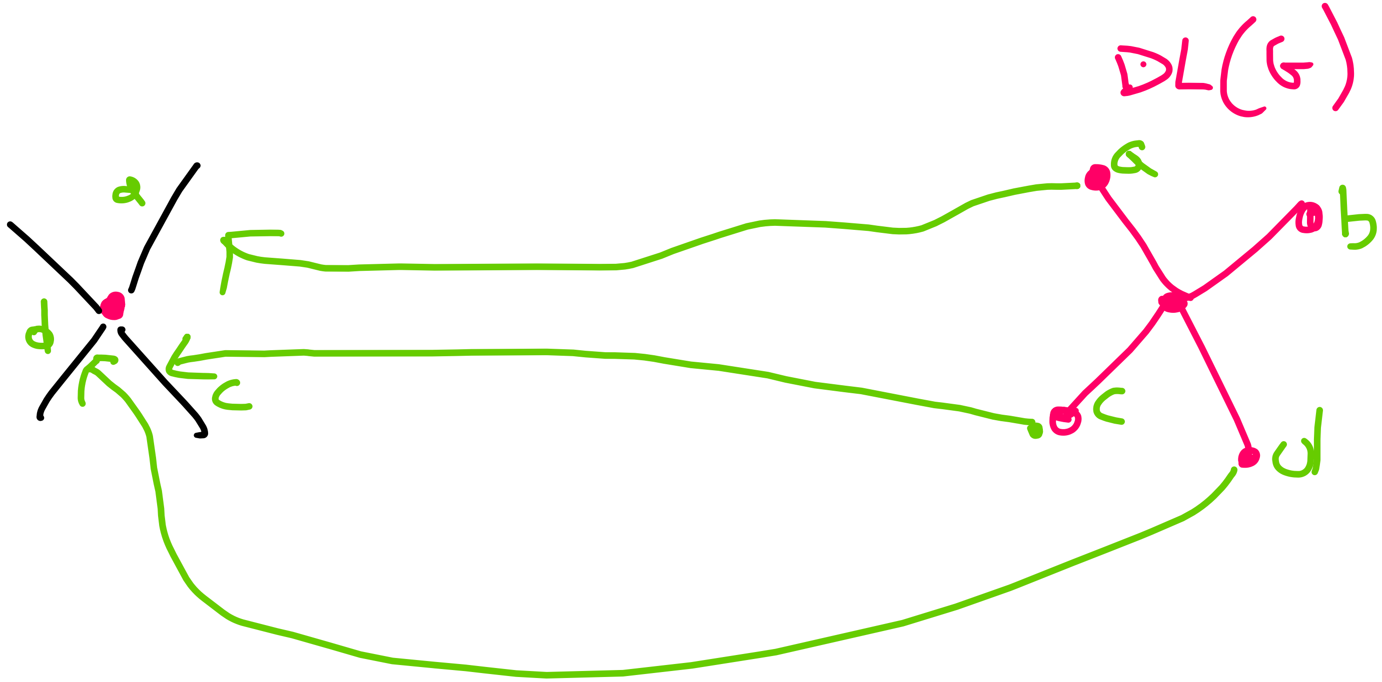
$\downarrow$

B_{r+1} $\quad (\alpha_1, \dots, \alpha_{r+1})$

Lemma 8.10. *For the chromatic numbers $\chi(G)$ and $\chi(\mathcal{DL}(G))$ of a directed graph G and its diline graph, it holds that*

$$\chi\big(\mathcal{DL}(G)\big) \geq \log_2\big(\chi(G)\big).$$

- c colors of DLIN(G) -- colors of the edges
- collection of colors of outgoing edges = node colors in G



Theorem 8.12. *Every deterministic, distributed algorithm to color a directed ring with 3 or less colors needs at least* $(\log^* n)/2 - 1$ *rounds.*

start: Chromatic number of B1 is n

$\chi(B2) \geq (\log n) = \log \chi(B1)$ $\quad n$

$\chi(B3) \geq \log(\chi(B_2)) = \log\log n$

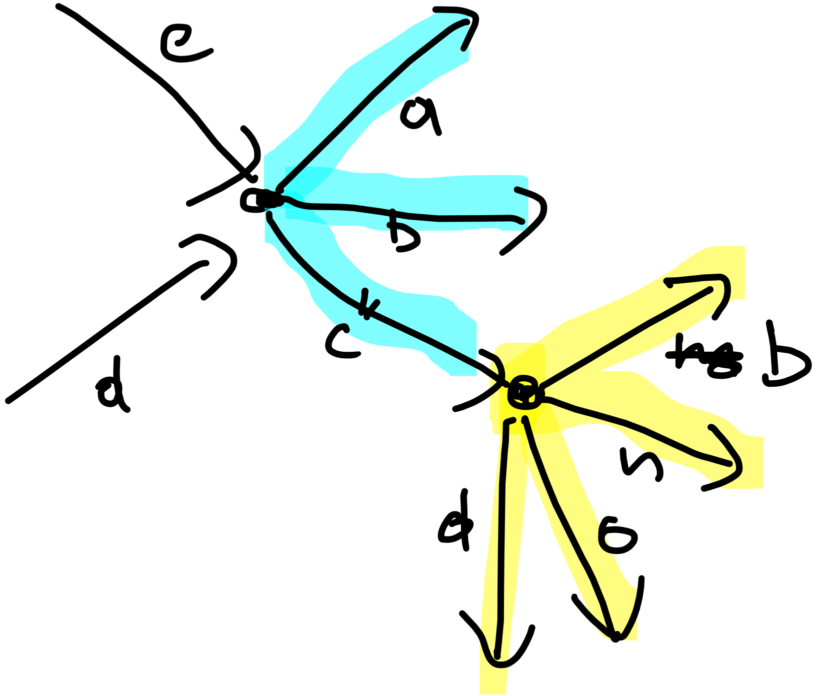
$\vdots$

$\chi(B(c \cdot \log^* n)) \geq 3$ $\quad 3$

colors of outgoing edges

$\{a, b, c\}$

$\{m, n, o, d\} \not\ni c$



$\{a, b, c\} \neq \{m, n, o, d\}$

$\ni c$ $\not\ni c$

Coloring with 2^c colors