Metody probabilistyczne I statystyka, 2021
informatyka algorytmiczna, WIiT PWr

3-Stochastic processes

Stochastic process

- **Time dependent random variables : time+space**
  - **time:**
  - **space:** Ω
  - **state:**

$$x(t, \omega) \quad \text{for } t \in \text{Time}, \ \omega \in \Omega$$

Examples:

- **Trajectory of an particle**

- **Electromagnetic noise**

- **Rain**

Examples:

- **CPU usage**
- **microcontrollers power consumption**

Discrete state process

Continuous state process

Discrete time process

Continuous time process

Markov process

only recent states count

Stochastic process $X(t)$ is Markov if for any $t_1 < ... < t_n < t$ and any sets $A; A_1, \ldots, A_n$

$$\begin{aligned} \boldsymbol{P}\{X(t) \in A \mid X(t_1) \in A_1, \ldots, X(t_n) \in A_n\} \\ = \boldsymbol{P}\{X(t) \in A \mid X(t_n) \in A_n\}. \end{aligned} \quad (6.1)$$

Markov chain

- **discrete Markov process**
- **the state at time t+1 depend only on state at time t**

$$
\begin{aligned}
p_{ij}(t) &= P\{X(t+1) = j \mid X(t) = i\} \\
&= P\{X(t+1) = j \mid X(t) = i,\ X(t-1) = h,\ X(t-2) = g, \ldots\}
\end{aligned}
$$

Transition probability:

$$p_{ij}^{(\cancel{h})}(t) = P\{X(t + \overset{1}{\cancel{h}}) = j \mid X(t) = i\}$$

Homogenous Markov chain

- **Transition pbb do not depend on time**

A Markov chain is **homogeneous** if all its transition probabilities are independent of t. Being homogeneous means that transition from i to j has the same probability at any time. Then $p_{ij}(t) = p_{ij}$ and $p_{ij}^{(h)}(t) = p_{ij}^{(h)}$.

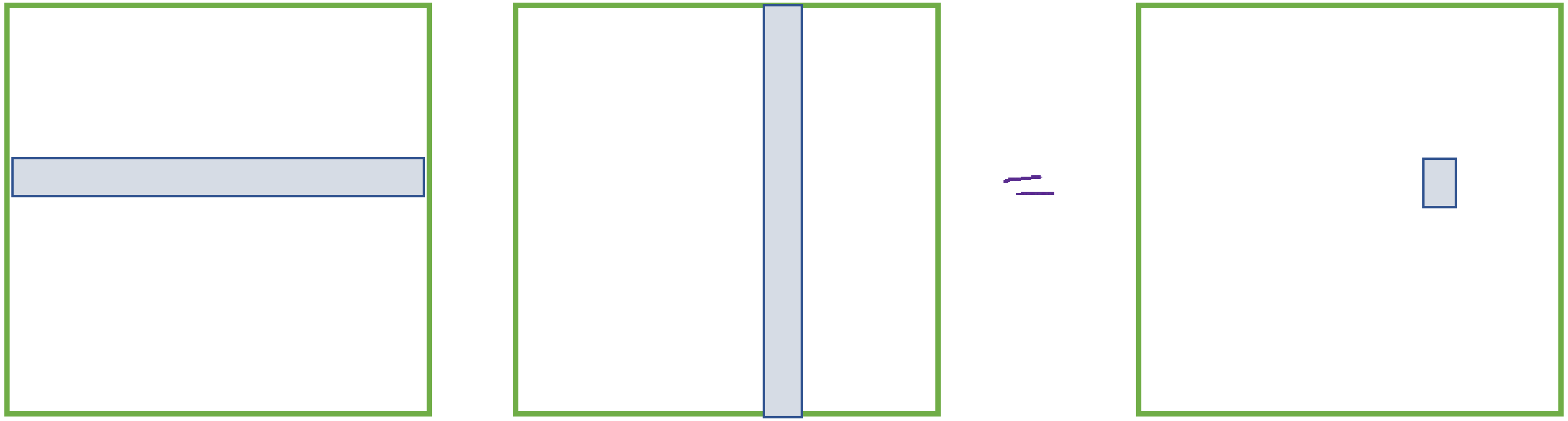
- **Transition matrix**

$$\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

Transition in 2 steps

$$
\begin{aligned}
p_{ij}^{(2)} &= P\{X(2)=j \mid X(0)=i\} \\
&= \sum_{k=1}^{n} P\{X(1)=k \mid X(0)=i\}\, P\{X(2)=j \mid X(1)=k\} \\
&= \sum_{k=1}^{n} p_{ik}p_{kj} = (p_{i1},\ldots,p_{in}) \begin{pmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{pmatrix}.
\end{aligned}
$$

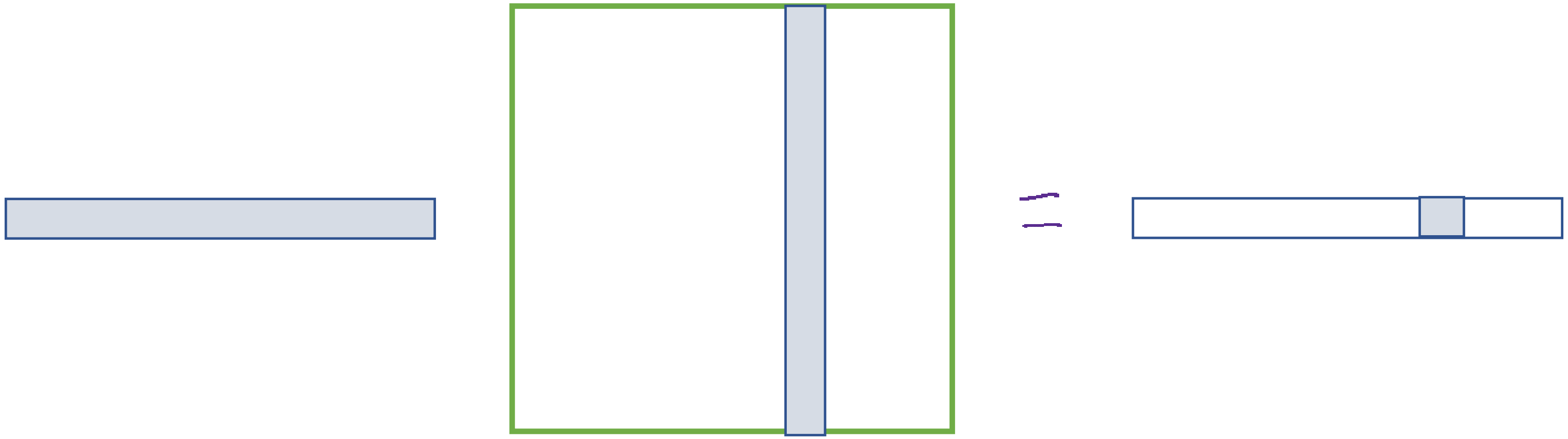
Transition pbb in two steps



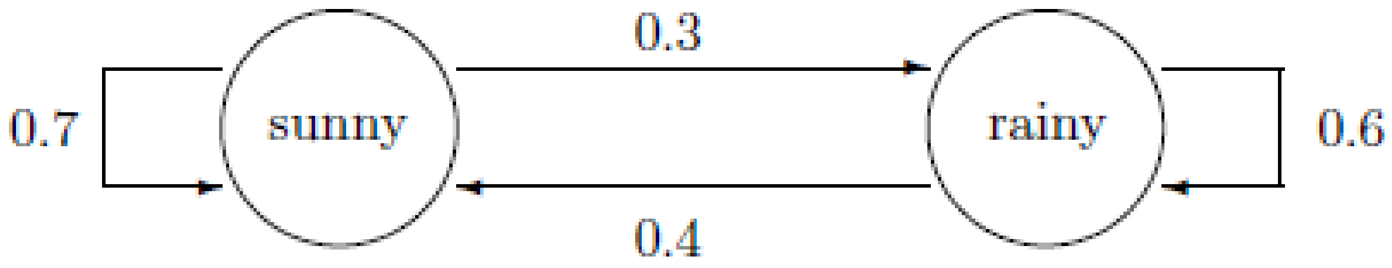
Probabilities at time t

- **Transition matrix M of a homogenous chain**

$$p_0 \cdot M^t = p_t$$



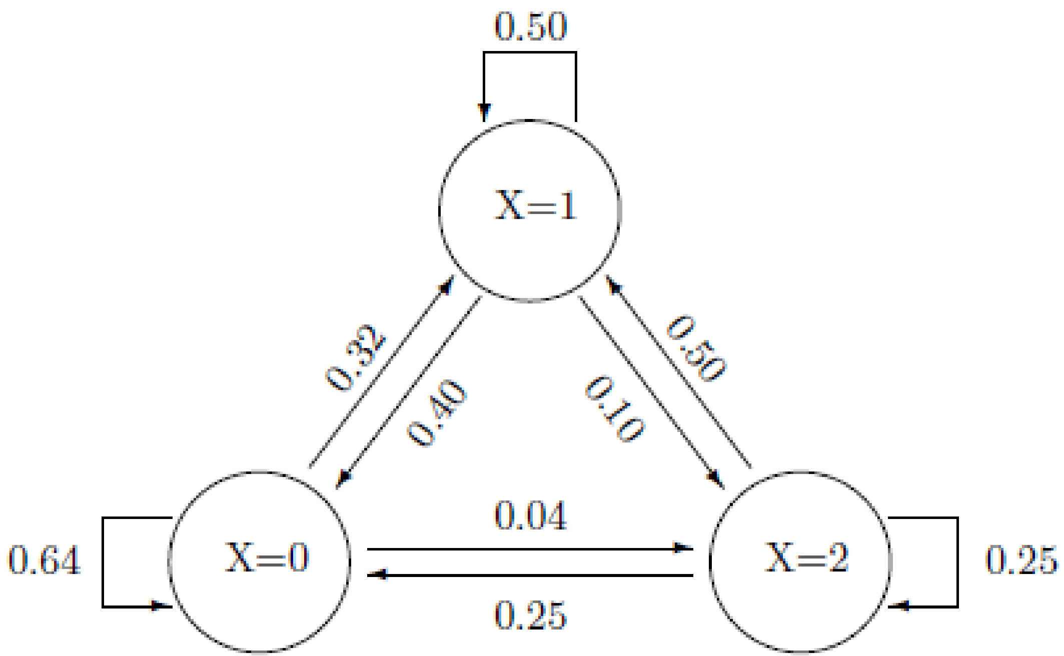
Transition diagram



Transition diagram

2 users: active user disconnects with pbb 0.5
inactive user connects with ppb 0.2

X= number of active users



Steady state distribution

A collection of limiting probabilities

$$\pi_x = \lim_{h \to \infty} P_h(x)$$

is called a **steady-state distribution** of a Markov chain $X(t)$.

Example: no steady state distribution

random walk in a bipartite graph

Computing steady state distribution

$$P_h P = P_0 P^h P = P_0 P^{h+1} = P_{h+1}.$$

$$\pi P = \pi.$$

linear system, recall that $\Sigma \pi_i = \underline{1}$

Weather example cnt

$$(\pi_1,\ \pi_2) = (\pi_1,\ \pi_2)\begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = (0.7\pi_1 + 0.4\pi_2,\ 0.3\pi_1 + 0.6\pi_2)\,.$$

$$\begin{cases} 0.7\pi_1 + 0.4\pi_2 & = & \pi_1 \\ 0.3\pi_1 + 0.6\pi_2 & = & \pi_2 \end{cases} \Leftrightarrow \begin{cases} 0.4\pi_2 & = & 0.3\pi_1 \\ 0.3\pi_1 & = & 0.4\pi_2 \end{cases} \Leftrightarrow \pi_2 = \frac{3}{4}\pi_1.$$

$$\pi_1 + \pi_2 = \pi_1 + \frac{3}{4}\pi_1 = \frac{7}{4}\pi_1 = 1,$$

$$\underline{\pi_1 = 4/7} \quad \text{and} \quad \underline{\pi_2 = 3/7}.$$

Existence of stationary distribution

A Markov chain is **regular** if

$$p_{ij}^{(h)} > 0$$

for some h and all i, j. That is, for some h, matrix $P^{(h)}$ has only non-zero entries, and h-step transitions from any state to any state are possible.

Any regular Markov chain has a steady-state distribution.

Algorithms based on Markov chains

Example: choose a maximal independent set in a graph at random (with uniform probability)

Difficult:

-

Approach:

Define a Markov chain:

- **states = independent sets**
- **transitions: simple modifications (removing or adding nodes)**
- **… so that the steady distribution is uniform**

Uniform steady distribution

Case of double stochastic matrix:

sum of each row is 1 (must be)

sum of each column is 1 (not all transition matrices)

THM: stationary distribution is uniform for a double stochastic transition matrix

$$\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) \cdot \boxed{\phantom{\rule{3em}{3em}}} \stackrel{?}{=} \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$$

$$\frac{1}{n} p_{1j} + \frac{1}{n} p_{2j} + \ldots + \frac{1}{n} p_{nj} = \frac{1}{n}\left(p_{1j} + p_{2j} + \ldots + p_{nj}\right) = \frac{1}{n}$$

Uniform steady distribution

symmetric matrix:

$$P_{ij} = P_{ji}$$

then: transition matrix is double stochastic!

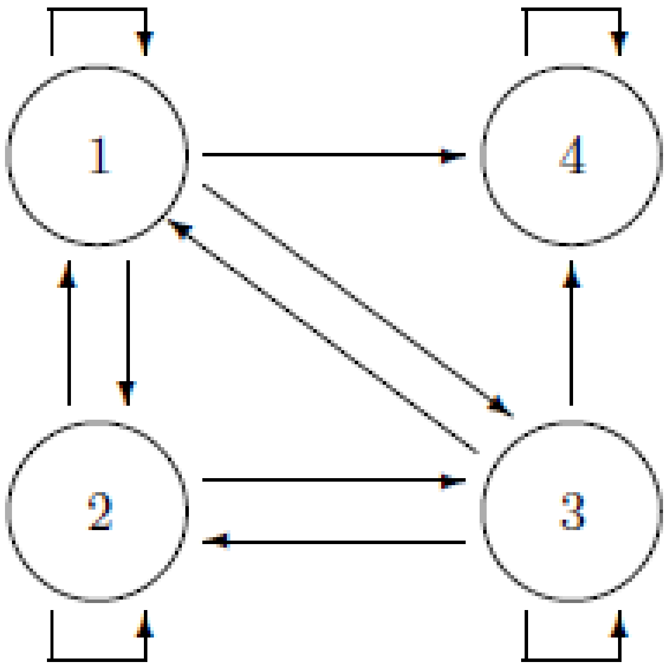
$\Rightarrow$ uniform steady distribution

Absorbing states

No exit from an absorbing state

Absorbing state example

a)



Algorithms based on absorbing state

Random walk through states based on Markov chain

Counting processes

e.g. Bernoulli trials
State in time t = number of succeses in steps 1 through t

Expected number: p*t

Counting processes

time frame Δ: one Bernoulli trial per Δ seconds

Expected number of successes:

$$\mathbf{E}\left\{X\left(\frac{t}{\Delta}\right)\right\} = \frac{t}{\Delta}p$$

Expected number of successes per second (arrival rate):

$$\lambda = \frac{p}{\Delta}$$

Bernoulli counting process - interarrival time

$$T = Y\Delta.$$

gdzie Y ma rozkład g

$$\begin{aligned}
\mathrm{E}(T) &= \mathrm{E}(Y)\Delta = \frac{1}{p}\Delta = \frac{1}{\lambda};\\
\mathrm{Var}(T) &= \mathrm{Var}(Y)\Delta^2 = (1-p)\left(\frac{\Delta}{p}\right)^2 \text{ or } \frac{1-p}{\lambda^2}.
\end{aligned}$$

Continuous counting process

a limit of Bernoulli counting process with time frame Δ → 0

The number of frames during time t increases to infinity,

$$n = \frac{t}{\Delta} \uparrow \infty \text{ as } \Delta \downarrow 0.$$

The probability of an arrival during each frame is proportional to Δ, so it also decreases to 0,

$$p = \lambda\Delta \downarrow 0 \text{ as } \Delta \downarrow 0.$$

Continuous counting process

Then, *the number of arrivals* during time t is a Binomial(n, p) variable with expectation

$$\mathbf{E}\,X(t) = np = \frac{tp}{\Delta} = \lambda t.$$

$$X(t) = \text{ Binomial}(n, p) \to \text{ Poisson}(\lambda)$$

$$\lim_{\substack{n \to \infty \\ p \to 0 \\ np \to \lambda}} \binom{n}{x} p^x (1-p)^{n-x} = e^{-\lambda} \frac{\lambda^x}{x!}$$

Continuous counting process

The *interarrival time* T becomes a random variable with the c.d.f.

$$\begin{aligned} F_T(t) &= P\{T \le t\} = P\{Y \le n\} && \text{because } T = Y\Delta \text{ and } t = n\Delta \\ &= 1 - (1-p)^n && \text{Geometric distribution of } Y \\ &= 1 - \left(1 - \frac{\lambda t}{n}\right)^n && \text{because } p = \lambda\Delta = \lambda t/n \\ &\to 1 - e^{\lambda t}. \end{aligned}$$

This is the "Euler limit":
$(1 + x/n)^n \to e^x$ as $n \to \infty$

Continuous counting process

$$P\{T_k \le t\} = P\{\ k\text{-th arrival before time } t\ \} = P\{X(t) \ge k\}$$

where T_k is Gamma(k, λ) and $X(t)$ is Poisson(λt).

Applications

What is the probability that in time T more than k requests arrive for a webpage P?

We assume that λ is known (λ requests per minute)

Solution for λ=7 hits per minute , assumed Poisson process

Pbb for 10000 hits within 24 hours?

Solution. The time of the 10,000-th hit T_k has Gamma distribution with parameters $k = 10,000$ and $\lambda = 7\ \text{min}^{-1}$. Then, the expected time of the k-th hit is

$$\mu = \mathrm{E}(T_k) = \frac{k}{\lambda} = \underline{1{,}428.6 \text{ min or } 23.81 \text{ hrs.}}$$

$$\sigma = \mathrm{Std}(T_k) = \frac{\sqrt{k}}{\lambda} = 14.3 \text{ min.}$$

Pbb of more than 10000 hits?

A shortcut: CLT (we do not have to care about Gamma distribution!)

$$P\{T_k < 1440\} = P\left\{\frac{T_k - \mu}{\sigma} < \frac{1440 - 1428.6}{14.3}\right\} = P\{Z < 0.80\} = \underline{0.7881}.$$

Conclusion

Easy way to solve many problems regarding required capacity …