Metody probabilistyczne i statystyka, 2021
informatyka algorytmiczna, WIiT PWr

6-Statistical Inference

Goal: parameter estimation

- **population given**
- **distribution may be known (because of the nature of the model)**
- **parameters of the model are to be determined**

Example:
λ of the Poisson distribution
Easy: λ=E(X), so estimate the mean

Generally: expressions for mean, variance ,… may contain parameters to be estimated

Strategic question:

which function(s) apply to the sample to get a reliable information?

Methods of moments

The k-th population moment is defined as

$$\mu_k = \mathrm{E}(X^k).$$

The k-th sample moment

$$m_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

Central moments

$$\mu'_k = \mathbf{E}(X - \mu_1)^k$$

$$m'_k = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^k$$

Method of moments

$$\begin{cases} \mu_1 & = & m_1 \\ \dots & \dots & \dots \\ \mu_k & = & m_k \end{cases}$$

In this system:

- concrete values on the right side
- expressions with parameters on the left side

Method of moments – example

Gamma distribution with parameters α, λ:

$$\begin{cases} \mu_1 &= \mathrm{E}(X) &= \alpha/\lambda &= m_1 \\ \mu_2' &= \mathrm{Var}(X) &= \alpha/\lambda^2 &= m_2'. \end{cases}$$

Pareto distribution

Well describes the distribution of file sizes sent in the internet

$$F(x) = 1 - \left(\frac{x}{\sigma}\right)^{-\theta} \quad \text{for } x > \sigma.$$

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$$f(x) = F'(x) = \frac{\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-\theta-1} = \theta\sigma^{\theta}x^{-\theta-1}$$

Pareto distribution

$$\begin{aligned}\mu_1 &= \mathrm{E}(X) = \int_\sigma^\infty x\, f(x)\, dx = \theta\sigma^\theta \int_\sigma^\infty x^{-\theta} dx \\ &= \theta\sigma^\theta \left. \frac{x^{-\theta+1}}{-\theta+1} \right|_{x=\sigma}^{x=\infty} = \frac{\theta\sigma}{\theta-1}, \quad \text{for } \theta > 1,\end{aligned}$$

$$\mu_2 = \mathrm{E}(X^2) = \int_\sigma^\infty x^2\, f(x)\, dx = \theta\sigma^\theta \int_\sigma^\infty x^{-\theta+1} dx = \frac{\theta\sigma^2}{\theta-2}, \quad \text{for } \theta > 2.$$

Pareto distribution

$$\begin{cases} \mu_1 & = & \dfrac{\theta\sigma}{\theta - 1} & = & m_1 \\ \mu_2 & = & \dfrac{\theta\sigma^2}{\theta - 2} & = & m_2 \end{cases}$$

$$\hat{\theta} = \sqrt{\frac{m_2}{m_2 - m_1^2}} + 1 \quad \text{and} \quad \hat{\sigma} = \frac{m_1(\hat{\theta} - 1)}{\hat{\theta}}.$$

Method of Maximum Likelihood

Find parameters for which the obtained sample has the highest probability

Method of Maximum Likelihood –Discrete Case

Maximizing the probability of the observed sample:

$$P\{X = (X_1, \ldots, X_n)\} = P(X) = P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i),$$

A trick: it is easier to maximize a sum than a product, so take logarithms:

$$\ln \prod_{i=1}^{n} P(X_i) = \sum_{i=1}^{n} \ln P(X_i)$$

Method of Maximum Likelihood –example Poison distribution

Probability:

$$P(x) = e^{-\lambda} \frac{\lambda^x}{x!},$$

Taking logarithms:

$$\ln P(x) = -\lambda + x \ln \lambda - \ln(x!).$$

Maximize:

$$\ln P(X) = \sum_{i=1}^{n} (-\lambda + X_i \ln \lambda) + C = -n\lambda + \ln \lambda \sum_{i=1}^{n} X_i + C,$$

Finding local maximum:

$$\frac{\partial}{\partial \lambda} \ln P(X) = -n + \frac{1}{\lambda} \sum_{i=1}^{n} X_i = 0.$$

Solution:

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}.$$

Method of Maximum Likelihood – continuous case

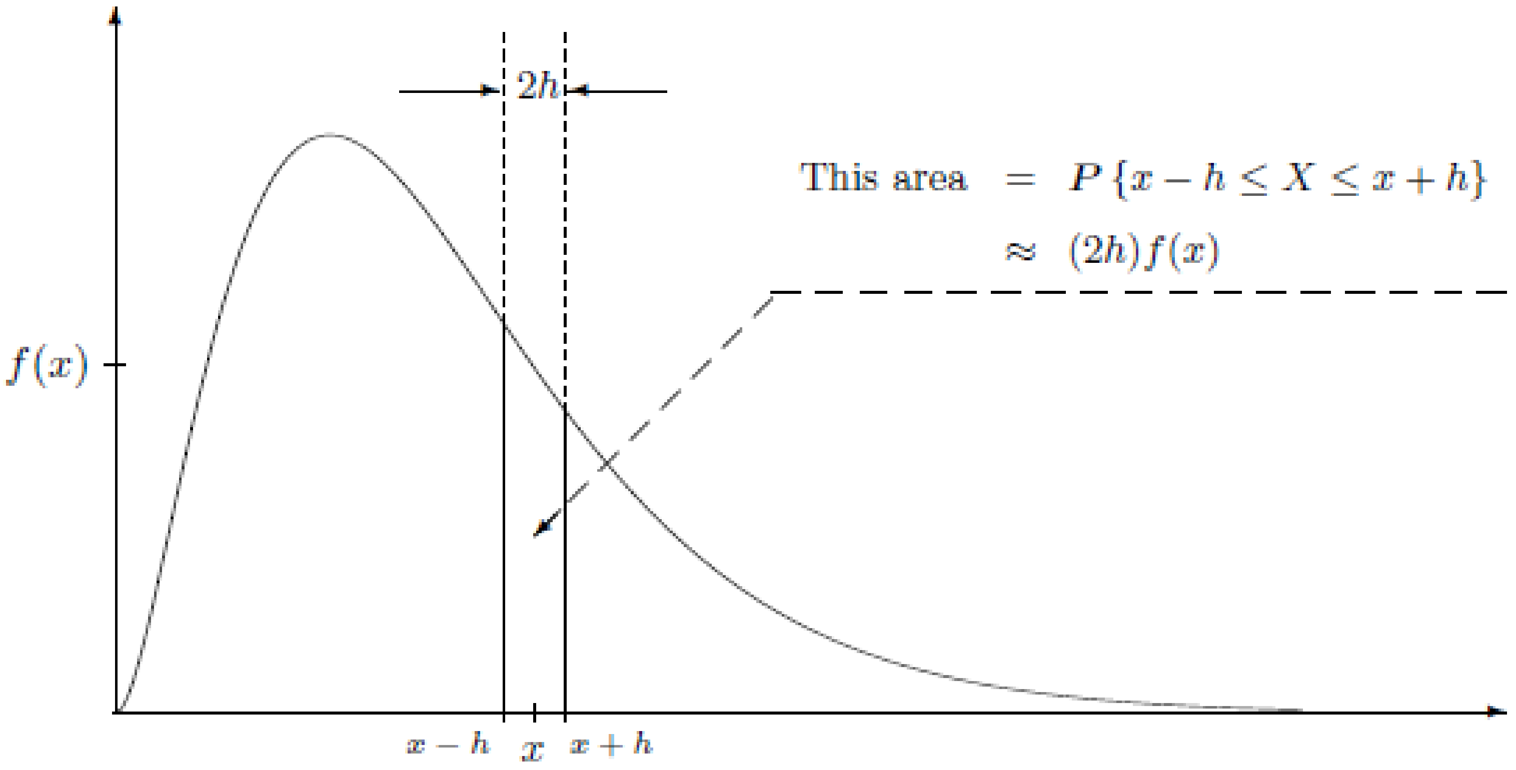


FIGURE 9.1: *Probability of observing "almost" $X = x$.*

Conclusion: take parameters such that f(X) is maximal

Method of Maximum Likelihood – example: exponential density

density: $$f(x) = \lambda e^{-\lambda x},$$

sample density ln: $$\ln f(X) = \sum_{i=1}^{n} \ln\left(\lambda e^{-\lambda X_i}\right) = \sum_{i=1}^{n} (\ln \lambda - \lambda X_i) = n \ln \lambda - \lambda \sum_{i=1}^{n} X_i.$$

derivative: $$\frac{\partial}{\partial \lambda} \ln f(X) = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0,$$

solution: $$\hat{\lambda} = \frac{n}{\sum X_i} = \frac{1}{\bar{X}}.$$

Estimating the error of an estimate

estimator is a random variable and we wish to know how concentrated is this estimator value around the true value

Estimating the error of an estimate- example Poisson distribution

already we have obtained an estimator for λ: $\hat{\lambda} = \bar{X}$

Approach 1: $\sigma = \sqrt{\lambda}$ for the Poisson(λ)

so: $\sigma(\hat{\lambda}) = \sigma(\bar{X}) = \sigma/\sqrt{n} = \sqrt{\lambda/n},$

Thus: $$s_1(\hat{\lambda}) = \sqrt{\frac{\bar{X}}{n}} = \frac{\sqrt{\sum X_i}}{n}.$$

Estimating the error of an estimate- example Poisson distribution

Approach 2: $\sigma(\bar{X}) = \sigma/\sqrt{n},$ **so estimated by** $s(\bar{X}) = s/\sqrt{n}$

so:

$$s_2(\hat{\lambda}) = \frac{s}{\sqrt{n}} = \sqrt{\frac{\sum(X_i - \bar{X})^2}{n(n-1)}}.$$

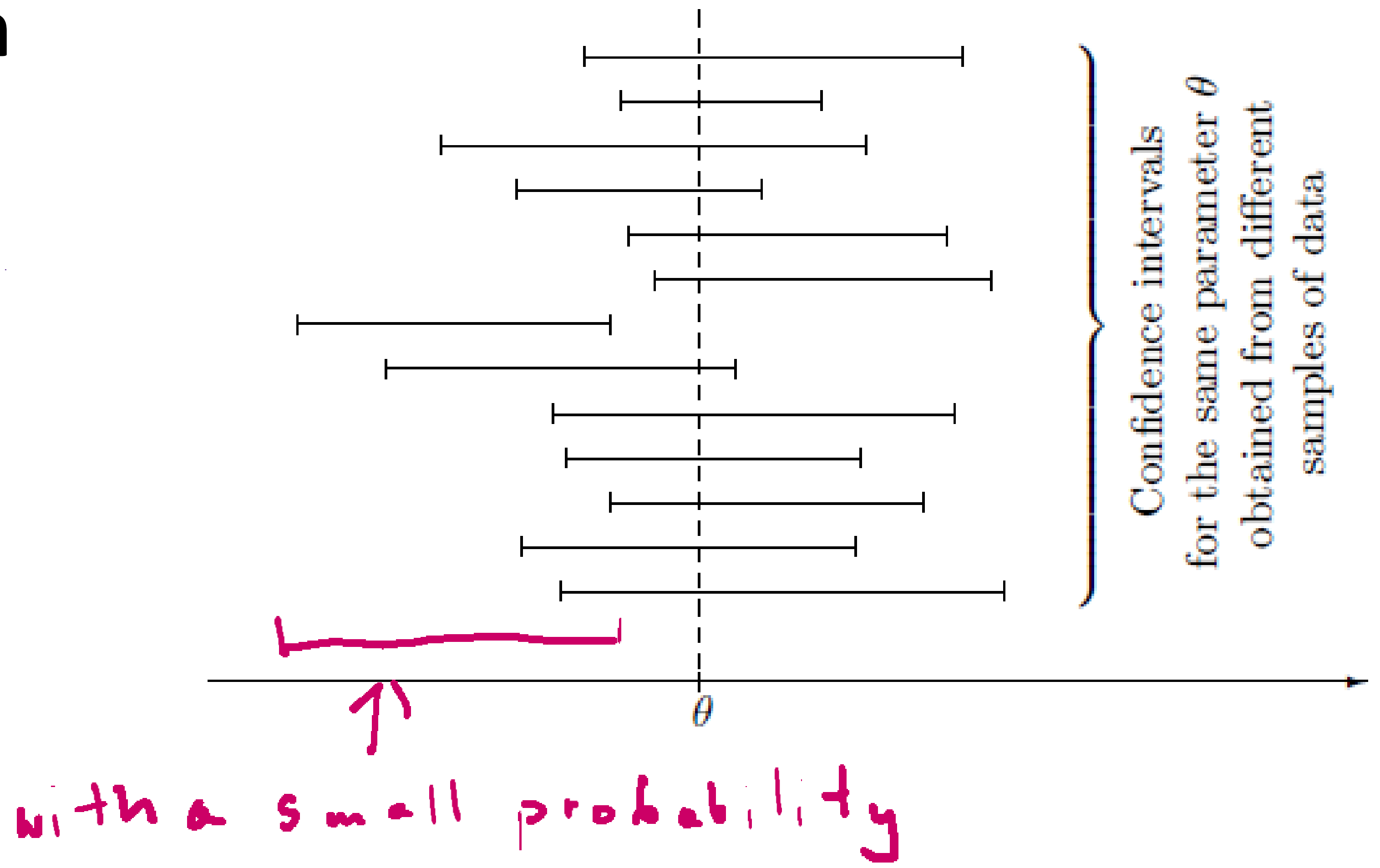
Confidence interval

An interval $[a, b]$ is a $(1-\alpha)100\%$ **confidence interval** for the parameter θ if it contains the parameter with probability $(1-\alpha)$,

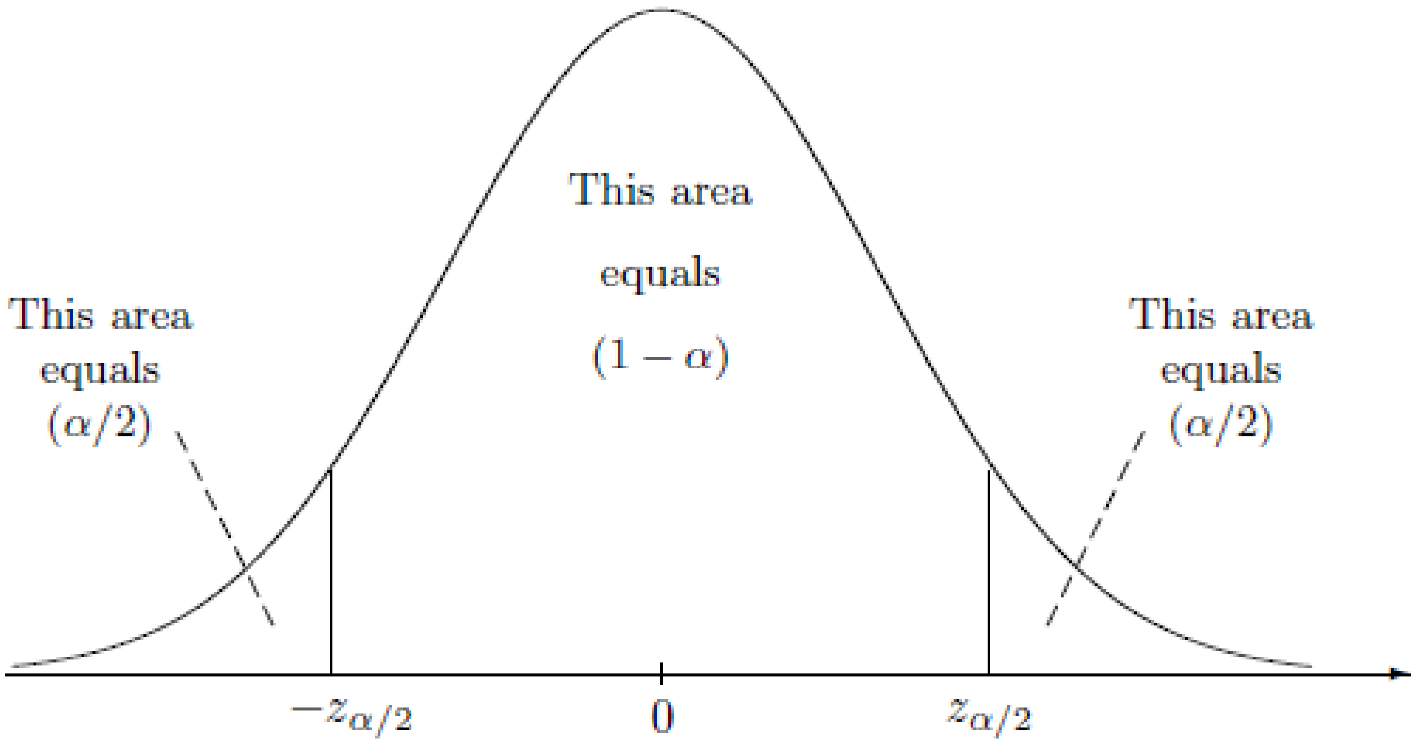
$$\boldsymbol{P}\{a \le \theta \le b\} = 1 - \alpha.$$

The **coverage probability** $(1-\alpha)$ is also called a **confidence level.**

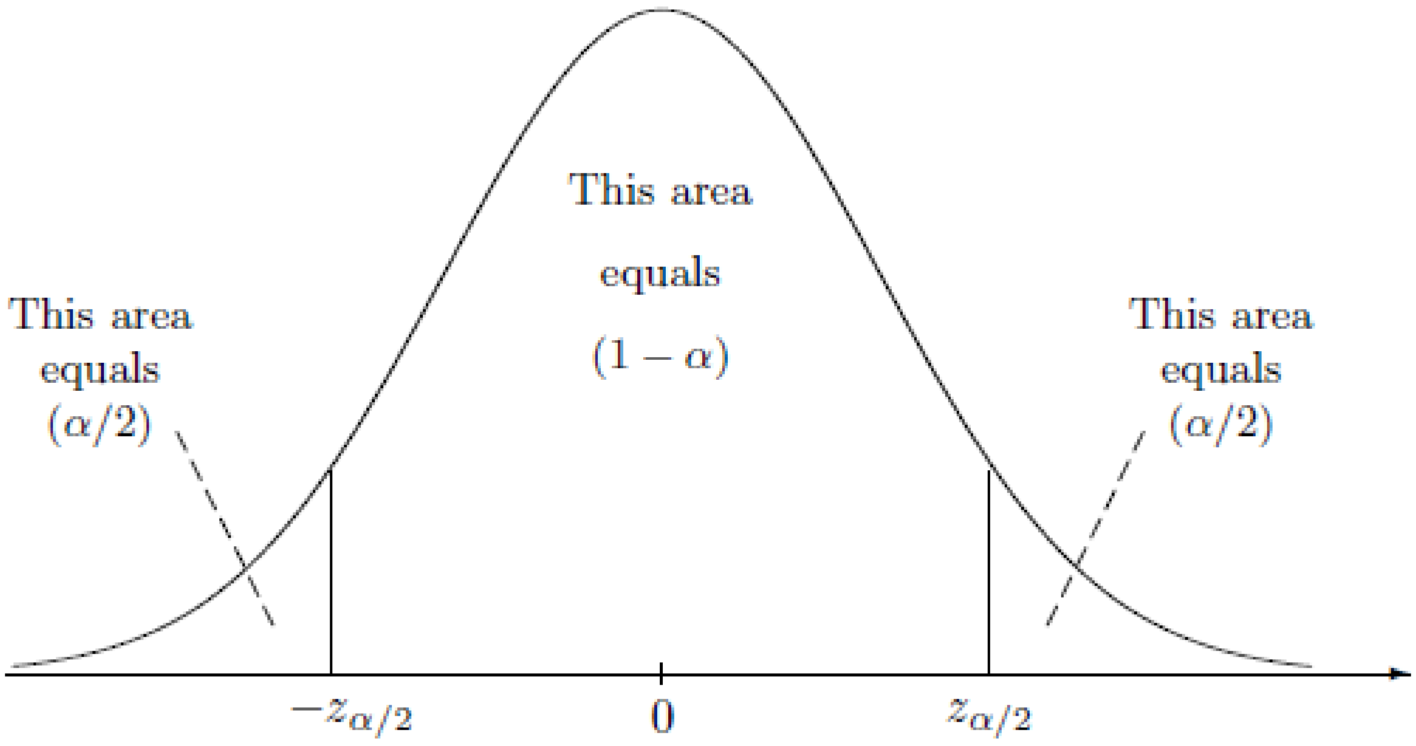
Situation


Confidence intervals
for the same parameter θ
obtained from different
samples of data
θ
with a small probability

Confidence interval for normal distribution



Confidence interval for normal distribution



Confidence interval – for unbiased estimator with normal distribution

after normalizing to Standard Normal distribution:

$$P\left\{-z_{\alpha/2} \leq \frac{\hat{\theta}-\theta}{\sigma(\hat{\theta})} \leq z_{\alpha/2}\right\} = 1-\alpha.$$

$$P\left\{\hat{\theta} - z_{\alpha/2}\cdot\sigma(\hat{\theta}) \leq \theta \leq \hat{\theta} - z_{\alpha/2}\cdot\sigma(\hat{\theta})\right\} = 1-\alpha.$$

Confidence interval [a,b] where:

$$\begin{aligned} a &= \hat{\theta} - z_{\alpha/2}\cdot\sigma(\hat{\theta}) \\ b &= \hat{\theta} + z_{\alpha/2}\cdot\sigma(\hat{\theta}) \end{aligned}$$

Application: confidence level for a sample mean

when it applies:

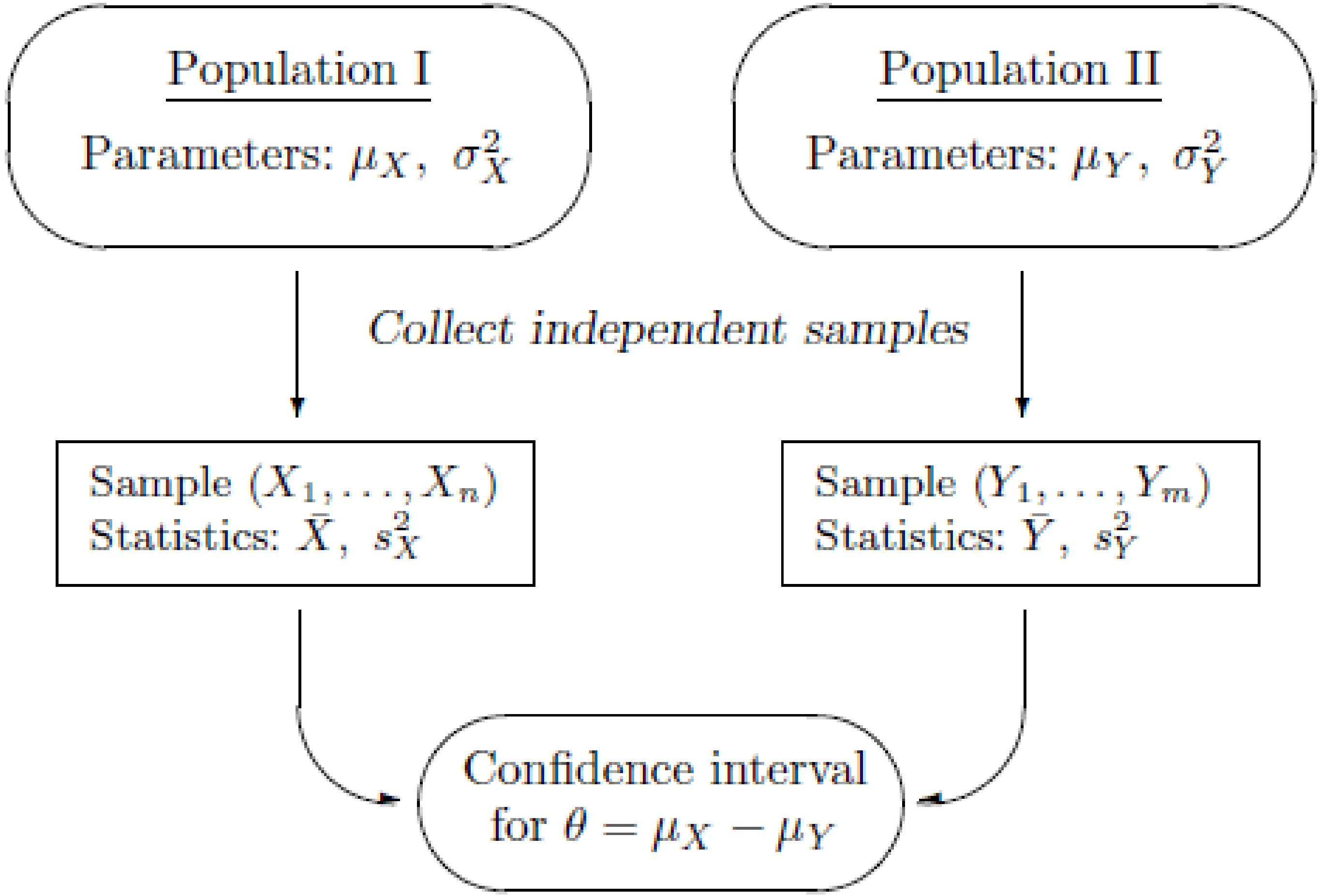
- **sum of random variables with normal distribution**
- **a large number of samples for any random variable due to CLT**

Recall that:

$$\begin{aligned} \mathbf{E}(\bar{X}) &= \mu \\ \sigma(\bar{X}) &= \sigma/\sqrt{n}. \end{aligned}$$

So the confidence interval with endpoints: $\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

Confidence interval for difference between two means:



Steps

1. **estimator of mean value:** $\hat{\theta} = \bar{X} - \bar{Y}$ **(it is unbiased)**

2. **if the sample is large, then approximately normal distribution**
3. **estimate variance:**

$$\sigma(\hat{\theta}) = \sqrt{\mathrm{Var}\left(\bar{X} - \bar{Y}\right)} = \sqrt{\mathrm{Var}\left(\bar{X}\right) + \mathrm{Var}\left(\bar{Y}\right)} = \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}.$$

4. **Confidence interval with endpoints:**

$$\bar{X} - \bar{Y} \pm z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}$$

How big should be the sample size?

How big should be the sample size?

Confidence interval depends on sample size n:

$$\text{margin} = z_{\alpha/2} \cdot \sigma / \sqrt{n}.$$

So a simple rule:

In order to attain a margin of error Δ for estimating a population mean with a confidence level $(1 - \alpha)$,

a sample of size $\quad n \geq \left(\frac{z_{\alpha/2} \cdot \sigma}{\Delta} \right)^2 \quad$ is required.

Confidence interval for unknown variance

Example: population with fraction p of objects with property A

Sample proportion: $$\hat{p} = \frac{\text{number of sampled items from } A}{n}$$

So:
$$X_i = \begin{cases} 1 & \text{if} \quad i \in A \\ 0 & \text{if} \quad i \notin A \end{cases}$$

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\mathrm{Var}\,(\hat{p}) = \frac{p(1-p)}{n},$$

Finally:
$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Problem of small sample size

CLT does not apply anymore .

Recall normalization (for normal distribution):

$$Z = \frac{\hat{\theta} - \mathbf{E}(\hat{\theta})}{\sigma(\hat{\theta})} = \frac{\hat{\theta} - \theta}{\sigma(\hat{\theta})},$$

For small sample we consider so called T-ratio:

$$t = \frac{\hat{\theta} - \theta}{s(\hat{\theta})}$$

Student’s distribution

Introduced by W. Gosset (pseudonym Student):

for T-ratio:

$$t = \frac{\hat{\theta} - \theta}{s(\hat{\theta})}$$

computed for a sample of size *n* for random variable with normal distribution

Subtle issue: T-ratio is not normal (observe that denominator is also an estimator)

True distribution: Student’s distribution with “*n-1* degrees of freedom”

Using Students distribution:

For each *n* there are precomputed values for any confidence interval – except that follow the same steps as for normal distribution

Example: difference between two variables with the same variance:

assumption: $\sigma_X^2 = \sigma_Y^2 = \sigma^2.$

sample variance:

$$s_p^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{m}(Y_i - \bar{Y})^2}{n+m-2} = \frac{(n-1)s_X^2 + (m-1)s_Y^2}{n+m-2}$$

confidence interval from Student's distribution:

$$\bar{X} - \bar{Y} \pm t_{\alpha/2}\, s_p \sqrt{\frac{1}{n} + \frac{1}{m}}$$

easy..

Example: difference between two variables with the different variance:

problem: not the Student distribution anymore!
no compact and clean solution

Approximation (only to see):

1. **computing "degree of freedom"**

$$\nu = \frac{\left(\frac{s_X^2}{n} + \frac{s_Y^2}{m}\right)^2}{\frac{s_X^4}{n^2(n-1)} + \frac{s_Y^4}{m^2(m-1)}}.$$

2. **Proceed with formulas for Student's distribution with this degree**

$$\bar{X} - \bar{Y} \pm t_{\alpha/2}\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}$$

Hypothesis testing

Population -- claimed property H0

-- alternative property H1

so that both cannot hold at the same time

Case 1:

Data from the whole population available:

one can say which of them is false

Case 2:

Only a sample available -- this is the most frequent case

which of H0 and H1 is true, which is false???

medicine,

Example

H0= the proportion of defect chips is 3%
H1 = the proportion of defect chips is >3%

Test outcomes

	Result of the test	
	Reject H_0	Accept H_0
H_0 is true	*Type I error*	correct
H_0 is false	correct	*Type II error*

Example: biometric recognition, AI

Test outcomes

Example: biometric recognition

Significance level of a test

For type 1 error:

$$\alpha = P\{\text{reject } H_0 \mid H_0 \text{ is true}\}$$

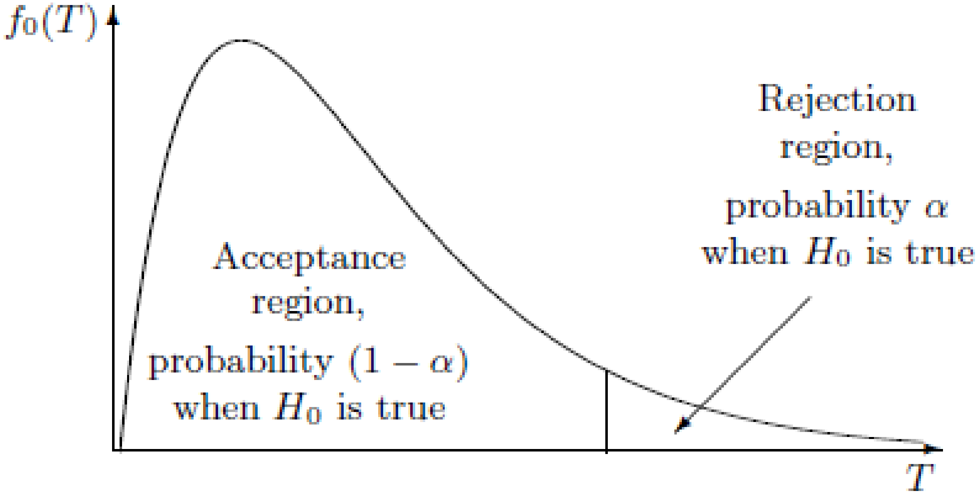
Power of the test

Alternative test H_A with parameters θ

$$p(\theta) = P\{\text{reject } H_0 \mid \theta;\ H_A \text{ is true}\}.$$

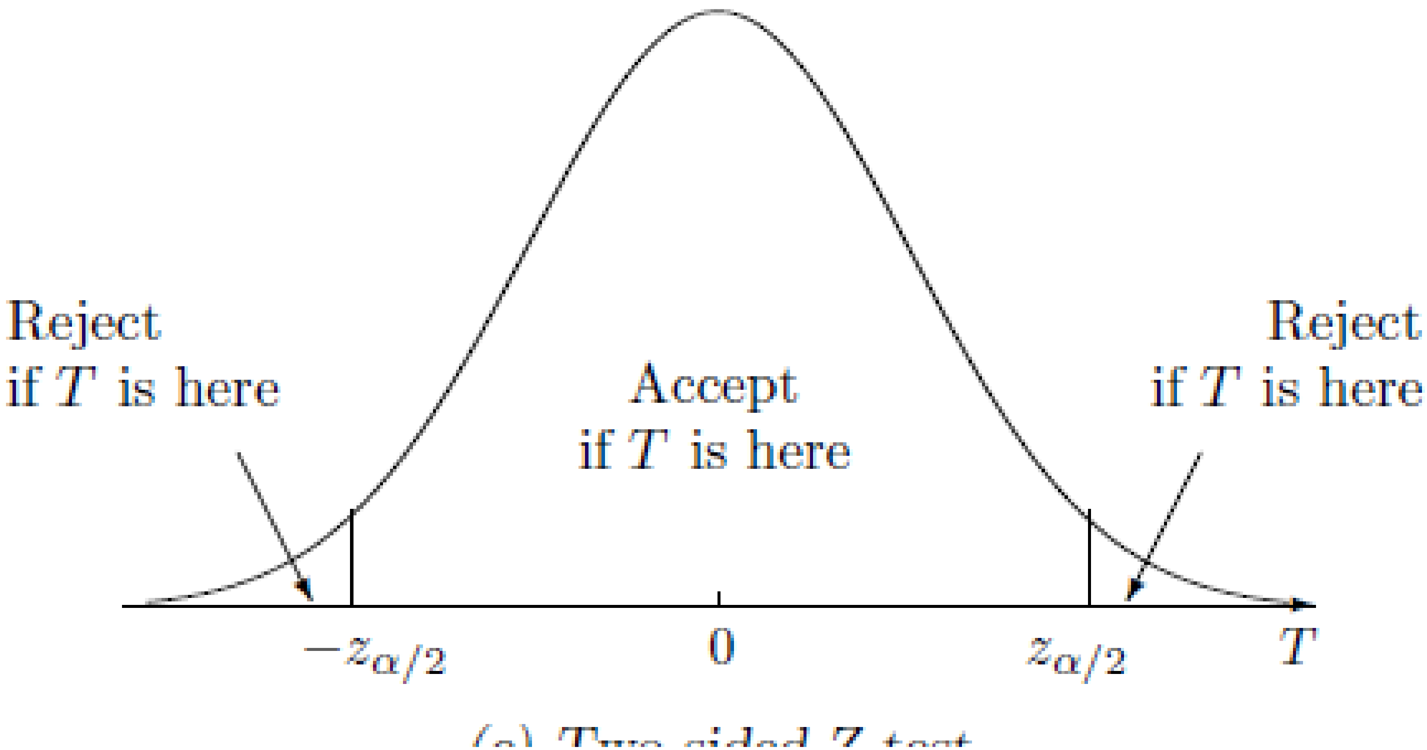
General approach

- H_0 corresponds to some distribution F_0
- define statistic T
- define acceptance and rejection regions so that probability of values from rejection regions is at most α



$$\begin{aligned} \text{Significance level} &= P\{\text{ Type I error }\} \\ &= P\{\text{ Reject } \mid H_0\} \\ &= P\{T \in \mathcal{R} \mid H_0\} \\ &= \alpha. \end{aligned}$$

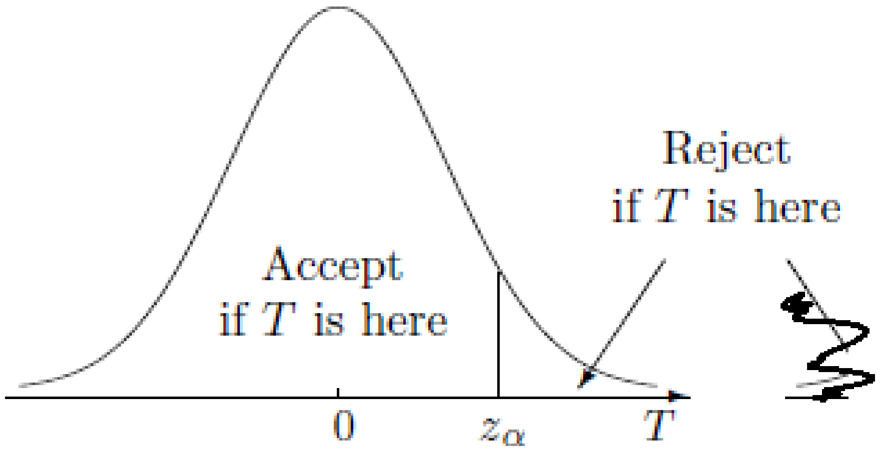
For normal distribution mean 0 – two sided Z test



Right tail alternative

(a) A level α test with a **right-tail alternative** should

$$\begin{cases} \text{reject } H_0 & \text{if} \quad Z \geq z_\alpha \\ \text{accept } H_0 & \text{if} \quad Z < z_\alpha \end{cases}$$

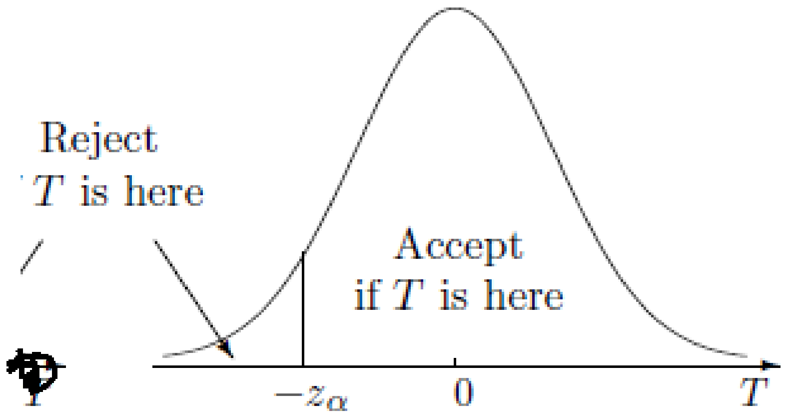


(a) *Right-tail Z-test*

Left tail alternative

With a **left-tail alternative**, we should

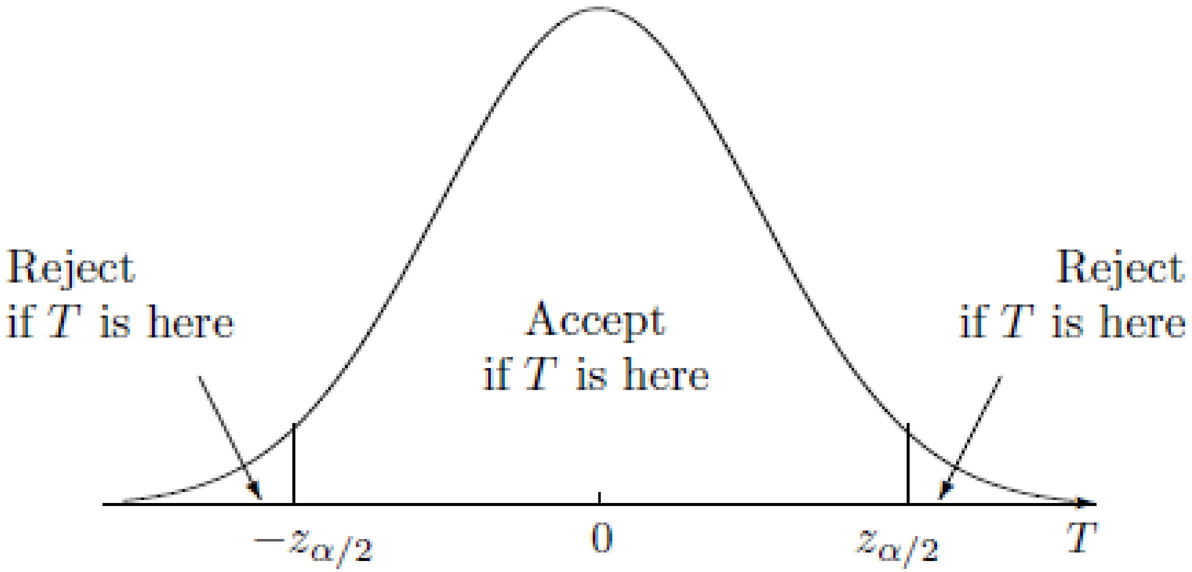
$$\begin{cases} \text{reject } H_0 & \text{if} \quad Z \le -z_\alpha \\ \text{accept } H_0 & \text{if} \quad Z > -z_\alpha \end{cases}$$



(b) *Left-tail Z-test*

Choosing α

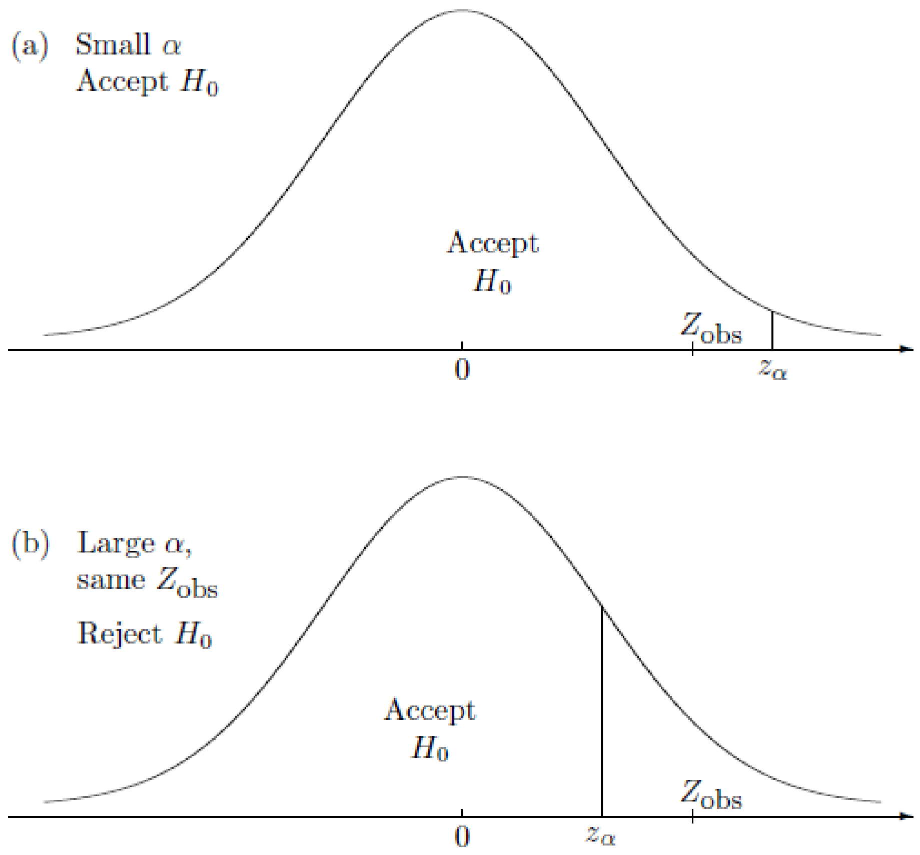
Delicate issue, a tradeoff between errors of type 1 and 2



P-value

For a given observation which values of α force rejection of H_0 and which force acceptance of H_0?

P-value is the boundary between these regions of α



P-value

Testing H_0 with a P-value

For	$\alpha < P$,	accept H_0
For	$\alpha > P$,	reject H_0

Practically,

If	$P < 0.01$,	reject H_0
If	$P > 0.1$,	accept H_0

Confidence intervals and testing for the variance

Important for making decisions based on a sample:

-- system reliability

-- quality testing

.. no room in cyber-physical systems

Variance unbiased estimator

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

the values $(X_i - \bar{X})^2$ are not independent:

- each X_i occurs in the sample mean
- CLT can be applied only for large *n*
- distribution of s^2 is not even symmetric

Distribution of variance?

Assumption: $X_1, \ldots, X_n$ -- independent, normally distributed with variance σ

$$\frac{(n-1)s^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma} \right)^2$$

is *Chi-square with* $(n-1)$ *degrees of freedom*

Density:

$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, \qquad x > 0,$$

Chi-square distribution

A case of Gamma distribution:

$$\text{Chi-square}(\nu) = \text{Gamma}(\nu/2, 1/2),$$

Deriving from general formulas for Gamma distribution:

$$\mathrm{E}(X) = \nu \quad \text{and} \quad \text{Var}(X) = 2\nu.$$

Chi-square distribution

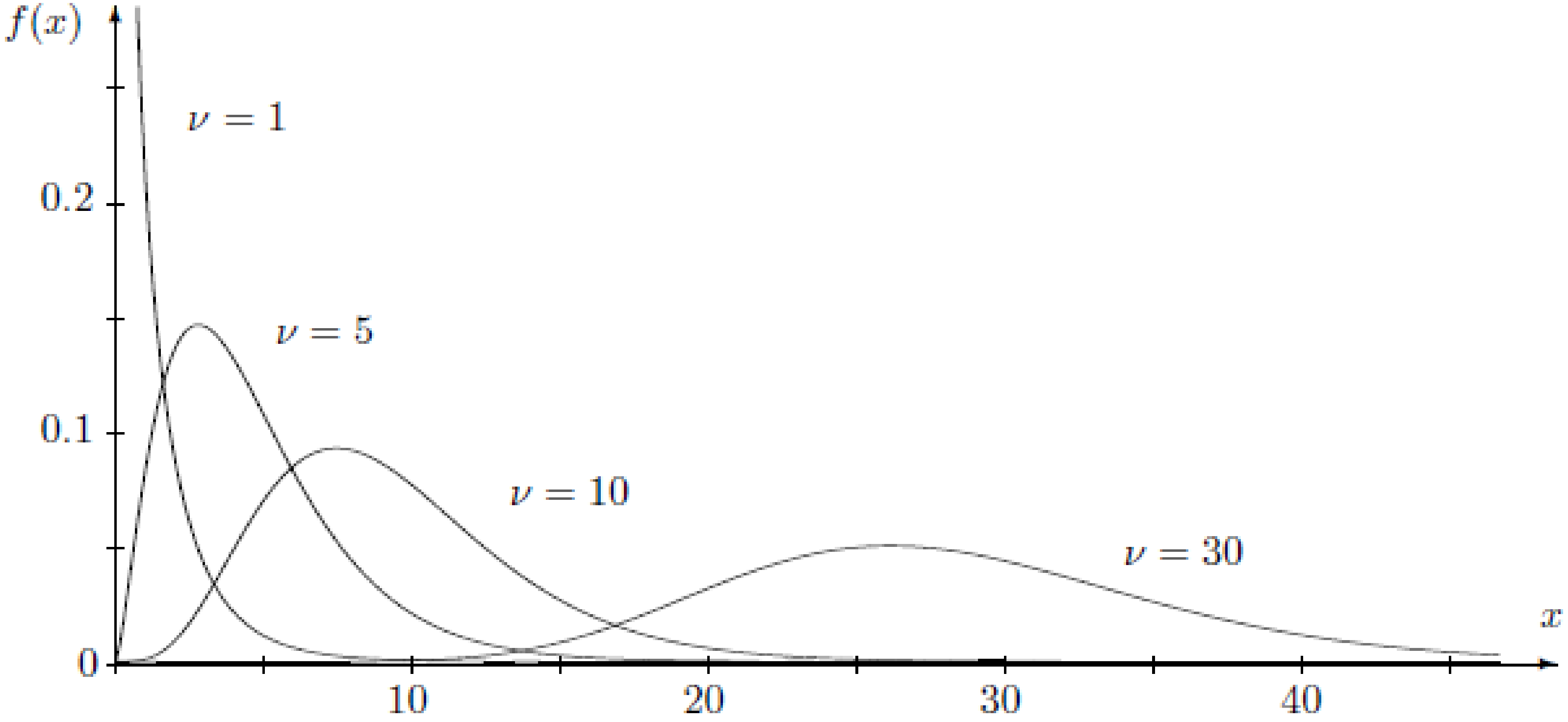
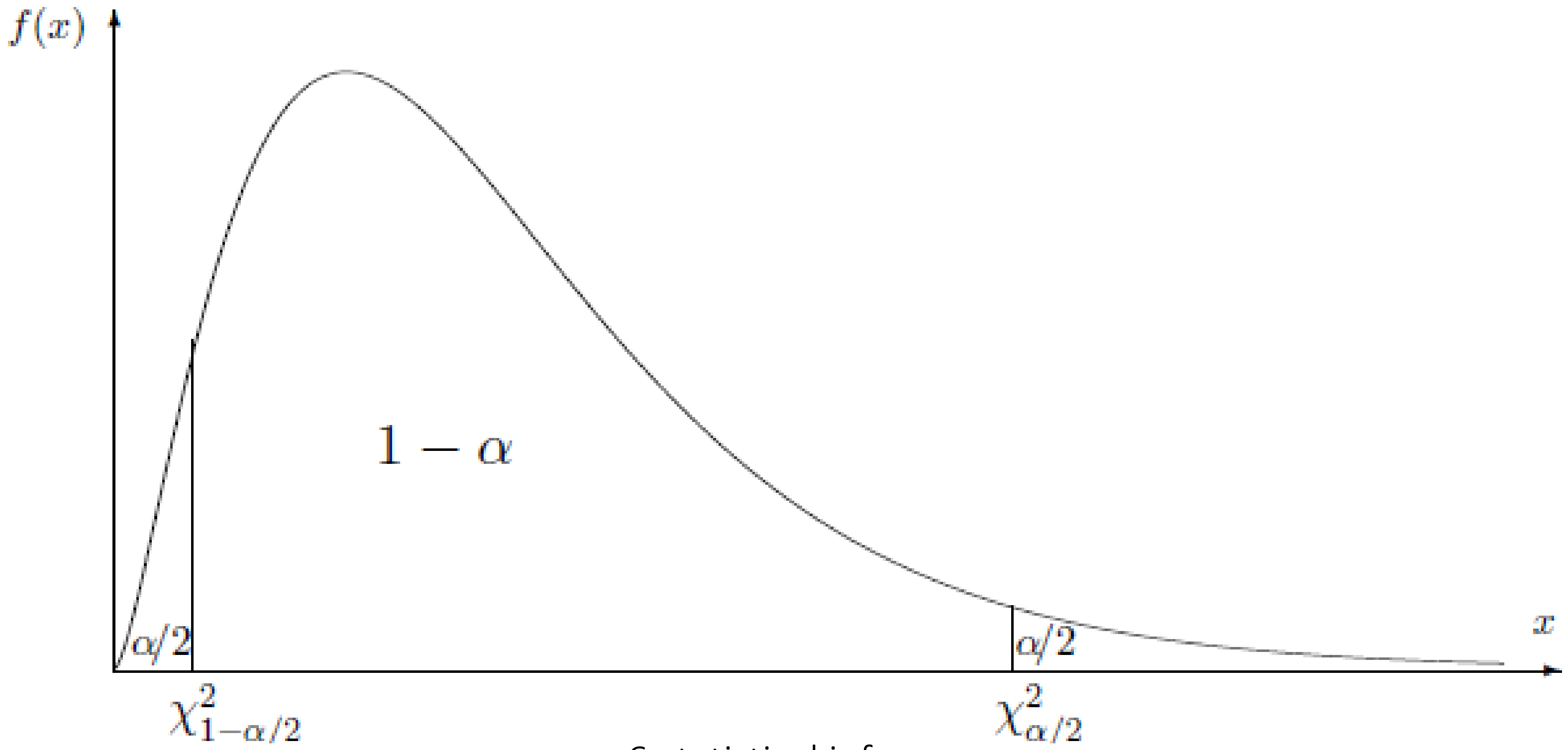


FIGURE 9.12: *Chi-square densities with ν = 1, 5, 10, and 30 degrees of freedom. Each distribution is right-skewed. For large ν, it is approximately Normal.*

Confidence interval

distribution not symmetrical, so the confidence interval is not of the form $s \mp \Delta$

- two values must be read from precomputed lookup tables



Confidence interval

Confidence interval for the variance

$$\left[\frac{(n-1)s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}\right]$$