**Probability and statistics, 2021, Computer Science Algorithmics, Undergraduate Course, Part II, lecturer: Mirosław Kutyłowski**

# I. Generating Random Numbers for a given probability distribution

**Chapter 5.2 in Byron**

**goal:**
**- simulations (e.g., pharma industry, weather forecast, system testing ...)**

## the case of invertible CDF

**Theorem 2** *Let $X$ be a continuous random variable with cdf $F_X(x)$. Define a random variable $U = F_X(X)$. The distribution of $U$ is Uniform(0,1).*

Proof: First, we notice that $0 \le F(x) \le 1$ for all $x$, therefore, values of $U$ lie in $[0,1]$. Second, for any $u \in [0,1]$, find the cdf of $U$,

$$
\begin{aligned}
F_U(u) &= \boldsymbol{P}\{U \le u\} \\
&= \boldsymbol{P}\{F_X(X) \le u\} \\
&= \boldsymbol{P}\left\{X \le F_X^{-1}(u)\right\} && \text{(solve the inequality for } X\text{)} \\
&= F_X(F_X^{-1}(u)) && \text{(by definition of cdf)} \\
&= u && (F_X \text{ and } F_X^{-1} \text{ cancel})
\end{aligned}
$$

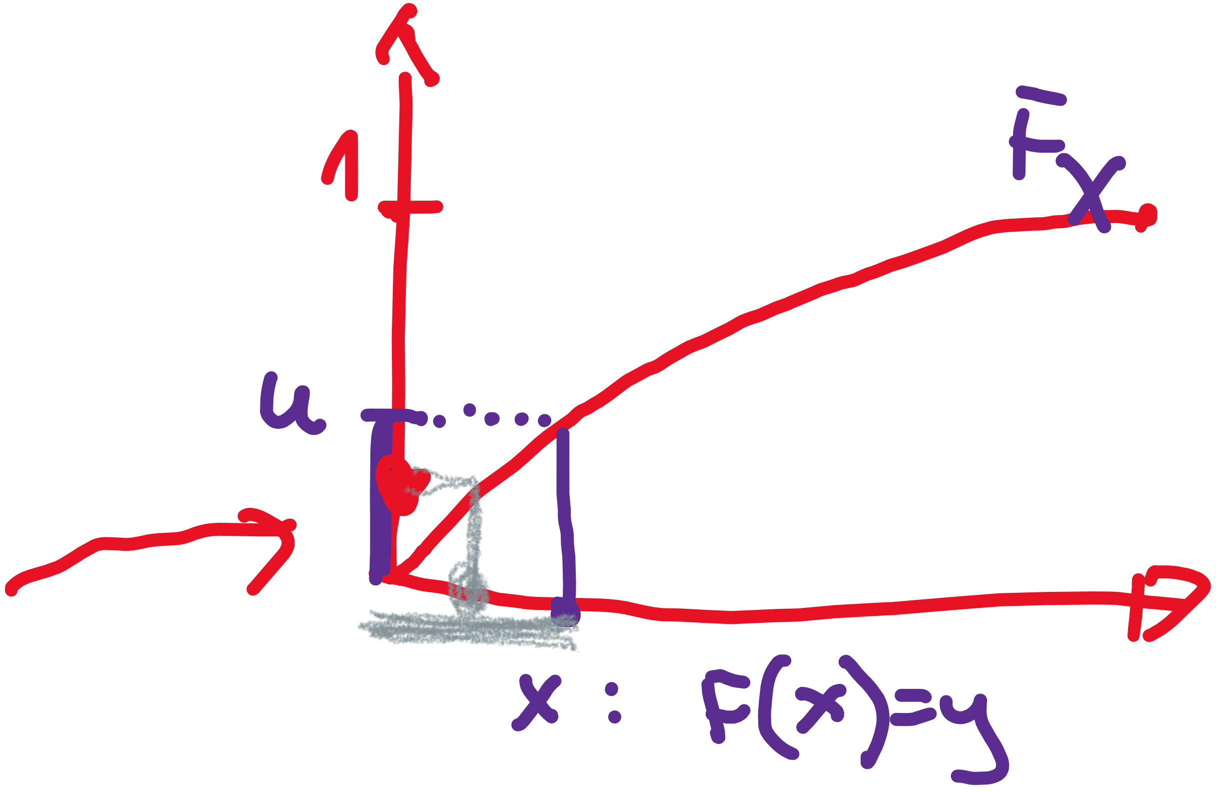
## the case of invertible CDF - continuous distribution X

**Procedure:**

1. choose u uniformly at random in [0,1]

2. take

$$x := F_X^{-1}(u)$$



## Example

**Exponential distribution** $F(x) = 1 - e^{\lambda x}$

**Procedure:**

1. choose $u$ uniformly at random in $[0,1]$

2. solve $u = 1 - e^{\lambda x}$

$u = F(x)$

**that is** $1 - u = e^{\lambda x}$

$\ln(1-u) = \lambda x$

$x = \ln(1-u)/\lambda$

Or simply $x = \ln(u)/\lambda$

# Example –warning

**Gamma distribution has complicated density function**

$$F(t) = \int_0^t f(x)dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^t x^{\alpha-1} e^{-\lambda x} dx.$$

**Inverting *F*?**

**Workaround:**

**a random variable with Gamma distribution α is is a sum of α independent random variables with exponential distribution**

**---**

# the case of invertible CDF - discrete distribution X

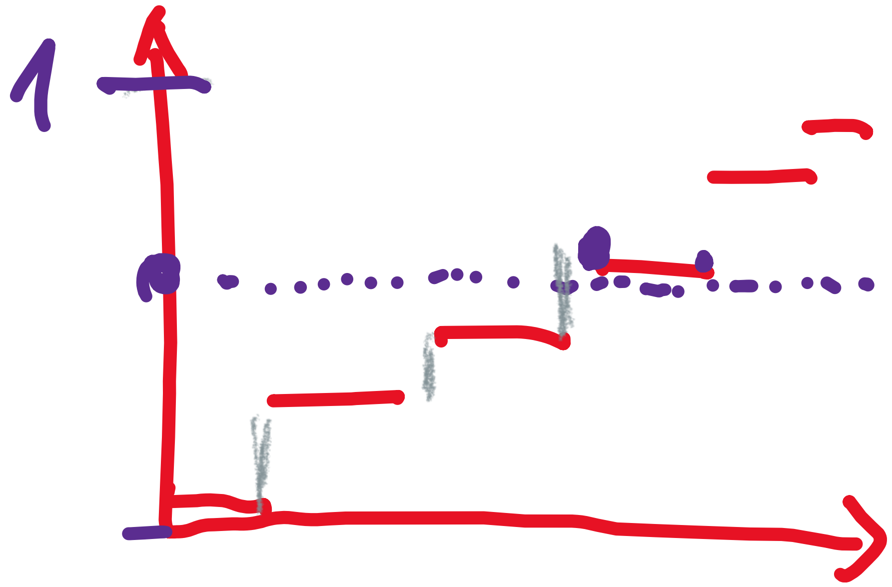
**Procedure:**

1. choose u uniformly at random in [0,1]

2. take

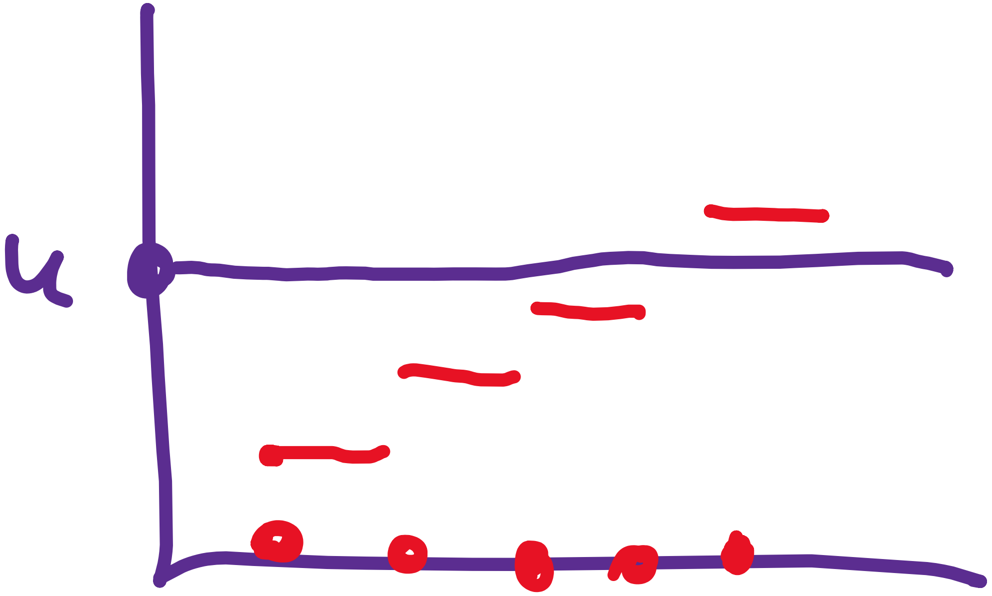
$$x := \min\{x : F(x) > u\}$$

so: $x = F^{-1}(u)$



# Example: geometric distribution

$$F(x) = 1 - (1-p)^x.$$




**Procedure:**
**Find the smallest *x* such that**

$$1 - (1-p)^x > U$$

$$1 - u > (1-p)^x$$

**Solution:**

$$X = \left\lceil \frac{\ln(1-U)}{\ln(1-p)} \right\rceil$$

$$\ln(1-u) > \ln(1-p) \cdot x$$

$$\frac{\ln(1-u)}{\ln(1-p)} > x$$

# Rejection method:

$X \in [a,b]$, $Y \in [0,c]$

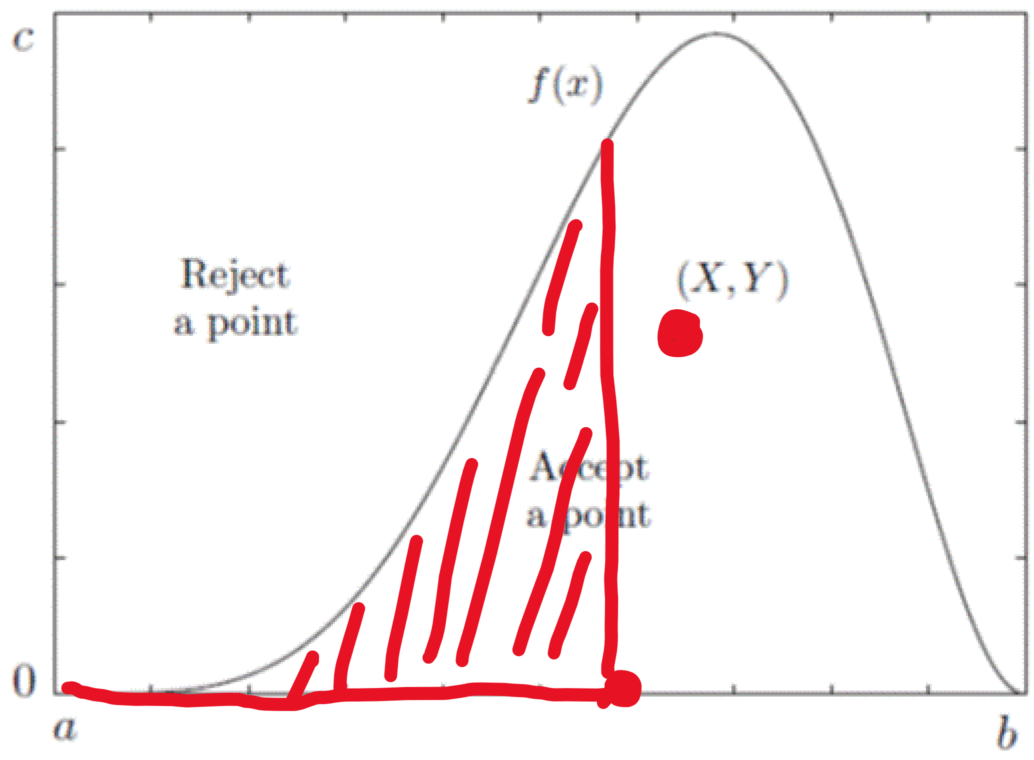
1. sample $X$ and $Y$ uniformly at random

2. if $Y>f(X)$, then goto 1

3. Output X

then density of $X = f(X)$ !

Ograniczenie dla funkcji gestosci




Pict. from Byron

Zmienna losowa $\in [a,b]$

# Application:

**distributions where density is computable (e.g. Beta distribution)**

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \text{ for } 0 \le x \le 1.$$

**but computing cdf is hard (numeric computations of the integral)**

```
alpha=5.5; beta=3.1; a=0; b=1; c=2.5;
X=0; Y=c;                    % Initial values
while Y > gamma(alpha+beta)/gamma(alpha)/gamma(beta)...
         * X.^(alpha-1) .* (1-X).^(beta-1);
   U=rand; V=rand; X=a+(b-a)*U; Y=c*V;
end; X
```

# Poisson distribution

**An example of a clever approach tailored to the particular case**

$$P(x) \quad = \quad e^{-\lambda}\frac{\lambda^x}{x!}, \; x = 0, 1, 2, \ldots$$

**but.. also can be understood as the number of rare events in an interval of time, where the time between events is exponential**

**Pragmatic computation:**

1. Obtain Uniform variables $U_1, U_2, \ldots$ from a random number generator.
2. Compute Exponential variables $T_i = -\frac{1}{\lambda}\ln(U_i)$.
3. Let $X = \max\{k : T_1 + \ldots + T_k \leq 1\}$.