Probability and statistics, 2022, Computer Science Algorithmics, Undergraduate Course, Part II, lecturer: Mirosław Kutyłowski

3- Stochastic Processes

Stochastic process

- Time dependent random variables : time+space
    - time: 1, 2, 3, 4, ...

      $t \in (0, +\infty)$
    - space: Ω
    - state: X(t,ω) where t∈Time, ω∈Ω

Examples:

- Trajectory of a particle

- Noise
- Rain
- Messages in a communication bus

Examples:

- **CPU usage**

- **microcontrollers power consumption**

1) **Discrete time process**

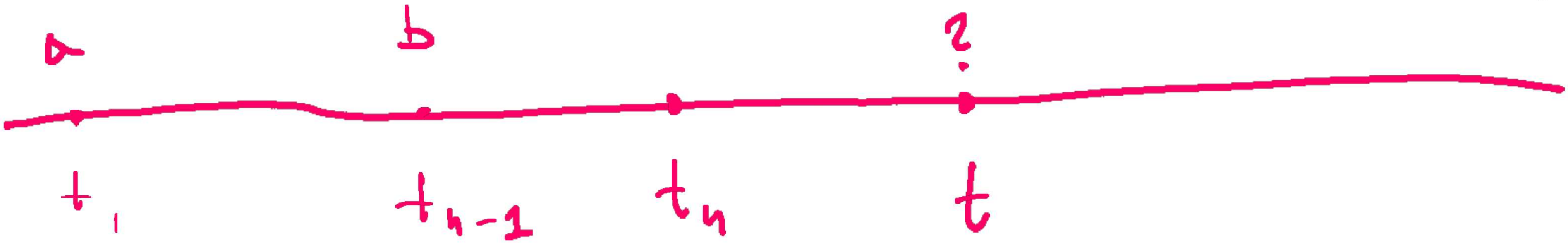
2) **Continuous time process**

Markov process

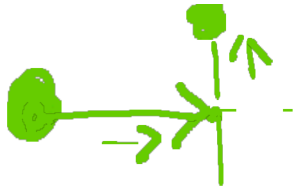
only the most recent state counts

Stochastic process $X(t)$ is Markov if for any $t_1 < ... < t_n < t$ and any sets $A; A_1, \ldots, A_n$

$$P\{X(t) \in A \mid X(t_1) \in A_1, \ldots, X(t_n) \in A_n\} = P\{X(t) \in A \mid X(t_n) \in A_n\}. \tag{6.1}$$



Markov chain



- **discrete Markov process**
- **the state at time t+1 depends only on the state at time t**

$$\begin{aligned} p_{ij}(t) &= P\{X(t+1) = j \mid X(t) = i\} \\ &= P\{X(t+1) = j \mid X(t) = i,\ X(t-1) = h,\ X(t-2) = g, \ldots\} \end{aligned}$$

Transition probability:

$$p_{ij}^{(\;)}(t) = P\{X(t+1) = j \mid X(t) = i\}$$

Homogenous Markov chain

- **Transition pbb does not depend on the time**

$p_{ij}(t)$ is constant, notation: p_{ij}

p_{ij} = pbb przejścia ze stanu i do j

- **Transition matrix**

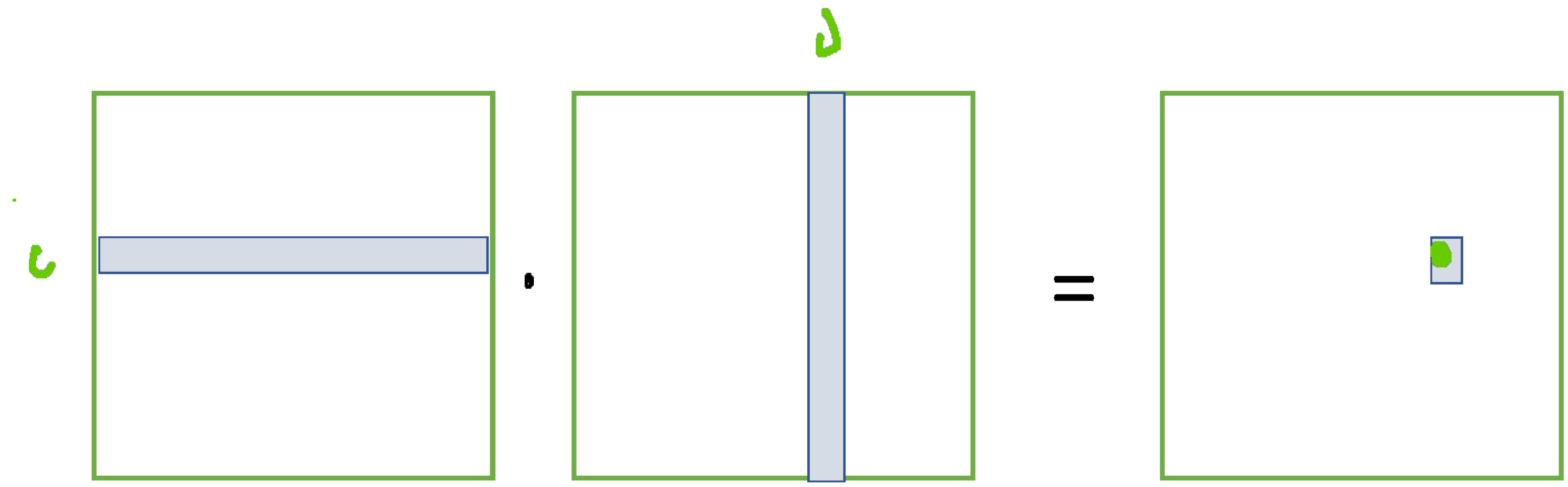
$$\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

Transition in 2 steps

$i \to k \to j$

$$
\begin{aligned}
p_{ij}^{(2)} &= P\{X(2)=j \mid X(0)=i\} \\
&= \sum_{k=1}^{n} P\{X(1)=k \mid X(0)=i\} \cdot P\{X(2)=j \mid X(1)=k\} \\
&= \sum_{k=1}^{n} p_{ik}p_{kj} = (p_{i1},\ldots,p_{in}) \begin{pmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{pmatrix}.
\end{aligned}
$$

Transition pbb in two steps



macierz przejścia T

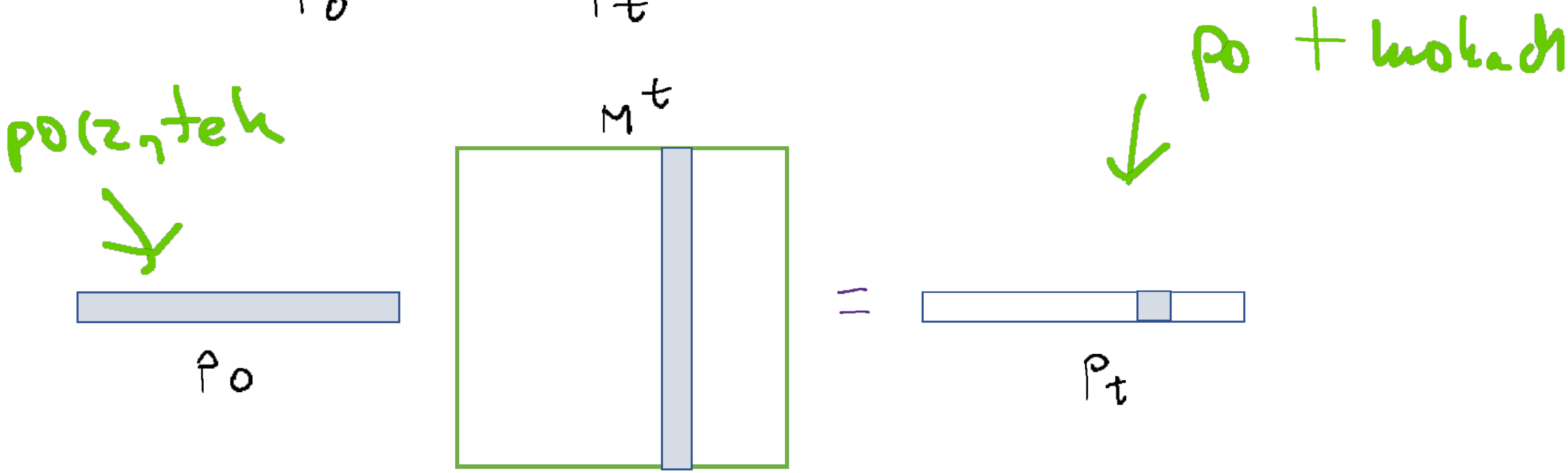
dla 2 kroków T^2

n kroków T^n

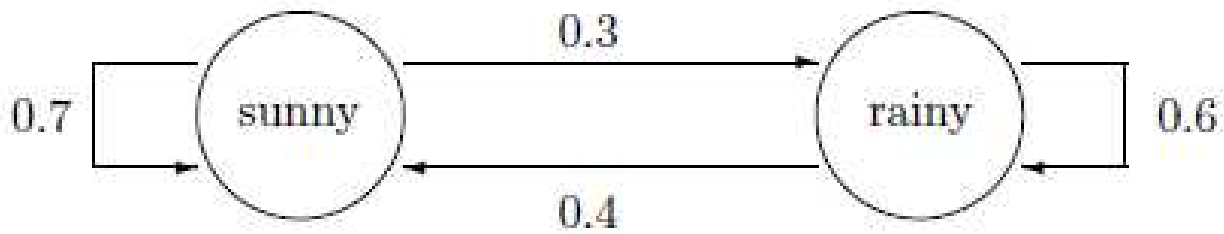
Probabilities at time t

- Transition matrix M of a homogenous chain

$$p_0 \cdot M^t = p_t$$



Description via a Transition diagram

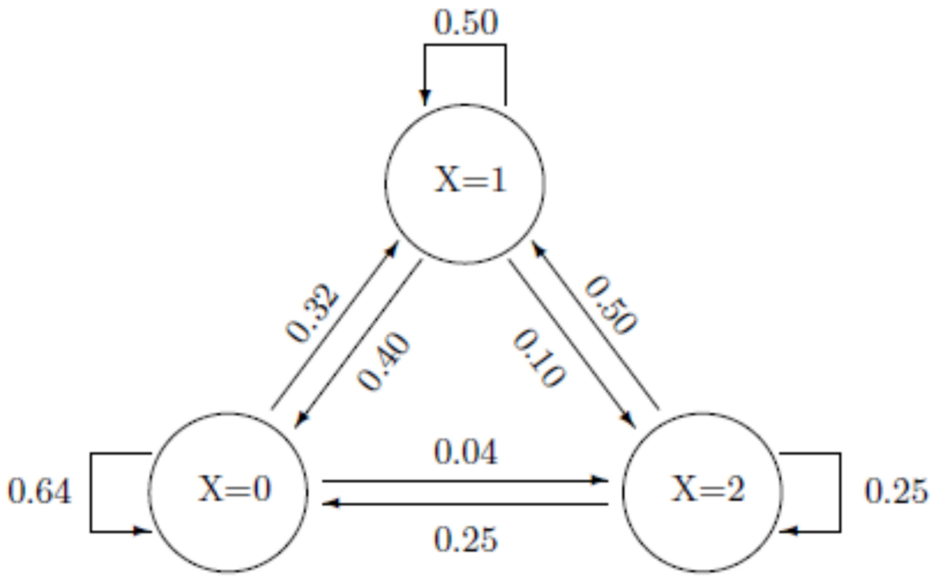


Transition diagram

2 users: active user disconnects with pbb 0.5

inactive user connects with ppb 0.2

X= number of active users



Steady state distribution

„eventually it does not depend on the initial state”

A collection of limiting probabilities

$$\pi_x = \lim_{h\to\infty} P_h(x)$$

is called a steady-state distribution of a Markov chain $X(t)$.

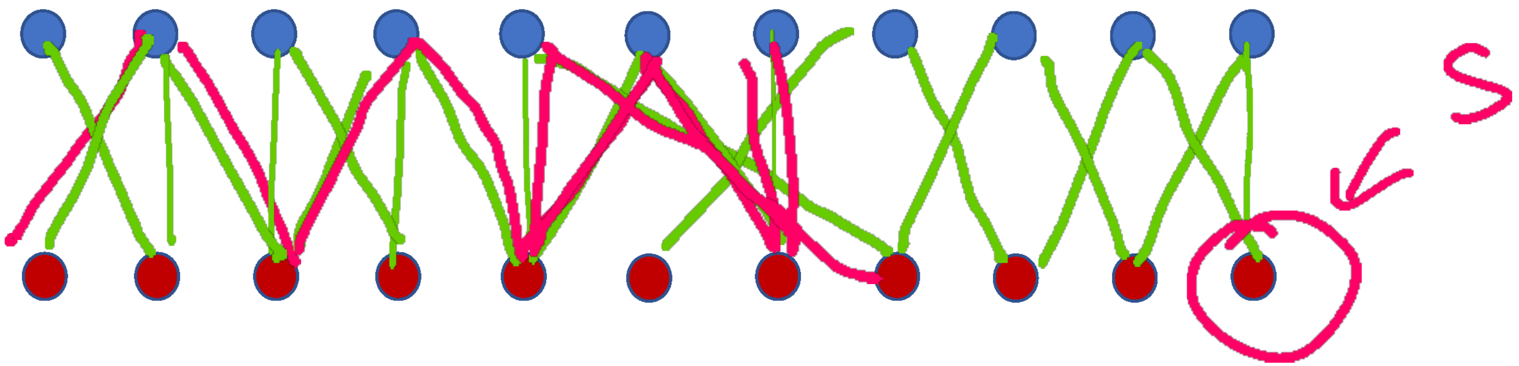
It is not clear in advance that a steady-state distribution exists

Another name used: *stationary distribution*

Example: no steady state distribution

proces periodyczny

random walk in a bipartite graph



parzyste kroki: na dole

nieparzyste na górze

$P_{17}(s)=0$

$P_{18}(s) \geq ?$

$P_{19}(s)=0$

Computing steady state distribution

$$P_h P = P_0 P^h P = P_0 P^{h+1} = P_{h+1}.$$

$$\pi P = \pi.$$

$$(\pi_0 \;\; \pi_1)\begin{pmatrix} a & \cdot \\ b & \cdot \end{pmatrix} = (\pi_0 \;\; \pi_1)$$

$$\begin{cases} \pi_0 \cdot a + \pi_1 b = \pi_0 \\ \end{cases}$$

this is a system of linear equations. Moreover:

$$\sum \pi_i = 1$$

(the probabilities must sum up to 1)

Weather example cnt

$$(\pi_1,\ \pi_2) = (\pi_1,\ \pi_2)\begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = (0.7\pi_1 + 0.4\pi_2,\ 0.3\pi_1 + 0.6\pi_2)\,.$$

$$\begin{cases} 0.7\pi_1 + 0.4\pi_2 &= \pi_1 \\ 0.3\pi_1 + 0.6\pi_2 &= \pi_2 \end{cases} \Leftrightarrow \begin{cases} 0.4\pi_2 &= 0.3\pi_1 \\ 0.3\pi_1 &= 0.4\pi_2 \end{cases} \Leftrightarrow \pi_2 = \frac{3}{4}\pi_1\,.$$

$$\pi_1 + \pi_2 = \pi_1 + \frac{3}{4}\pi_1 = \frac{7}{4}\pi_1 = 1,$$

$\underline{\pi_1 = 4/7}$ and $\underline{\pi_2 = 3/7}$.