ADVANCED TOPICS IN ALGEBRA

LECTURE 1

(lecture and problems to solve)

2020/21

EIGENVALUES AND EIGENVECTORS

Let V be a linear space (over a field $\mathcal{K}$) and let $f : V \to V$ be a linear mapping. A nonzero vector $\boldsymbol{v} \in V$ is called an *eigenvector* of f if

$$f(\boldsymbol{v}) = \alpha\boldsymbol{v},$$

for some nonzero scalar $\alpha \in \mathcal{K}$.

Theorem 1. Let $f : V \to V$ be a linear mapping. The set of eigenvectors E_α having the same eigenvalue $\alpha \in \mathcal{K}$ (for the mapping f) with zero vector added to this set is a linear subspace of V. Moreover, it is f-invariant, i.e. $f[E_\alpha] = E_\alpha$.

Proof. Let $\boldsymbol{v}, \boldsymbol{w} \in E_\alpha$. Then

$$f(\beta\boldsymbol{v} + \gamma\boldsymbol{w}) = f(\beta\boldsymbol{v} + \gamma\boldsymbol{w}) = f(\beta\boldsymbol{v}) + f(\gamma\boldsymbol{w}) = \beta f(\boldsymbol{v}) + \gamma f(\boldsymbol{w}) =$$

$$\beta\alpha\boldsymbol{v} + \gamma\alpha\boldsymbol{v} = \alpha(\beta\boldsymbol{v} + \gamma\boldsymbol{w}).$$

Thus every linear combination of vectors from E_α belongs to E_α.

Let us check now that the image via f of a vector $\boldsymbol{v}$ from E_α again belongs to E_α. We have

$$f(\boldsymbol{v}) = \alpha\boldsymbol{v}.$$

Thus

$$f(f(\boldsymbol{v})) = f(\alpha\boldsymbol{v}) = \alpha f(\boldsymbol{v}).$$

$\square$

The dimension of E_α, $\dim(E_\alpha)$ is called the *multiplicity* of α.

If x is an eigenvalue of f and $\boldsymbol{v} \in E_x$, then $(f - x \cdot \mathrm{id})(\boldsymbol{v}) = \boldsymbol{0}$ and if F is a matrix of f in some basis $\langle \boldsymbol{a_1}, \ldots, \boldsymbol{a_n} \rangle$, then

$$\det(F - xI) = 0$$

because $(F - xI)\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} \neq \boldsymbol{0}$. The matrix $(F - xI)$ is called the *characteristic matrix* of f. The determinant

$$w_f(x) := \det(F - xI)$$

is called the *characteristic polynomial* of f.

Theorem 2. Let V be a linear space of finite dimension. Let $f : V \to V$ be a linear mapping. Then the characteristic polynomial $w_f(x)$ does not depend on a choice of a basis for the matrix F representing f.

Proof. Let us consider two bases $B = \langle \boldsymbol{b_1}, \ldots, \boldsymbol{b_n} \rangle$ and $B' = \langle \boldsymbol{b'_1}, \ldots, \boldsymbol{b'_n} \rangle$. Let D be the matrix transforming B to B'. If a matrix F represents f in the basis B then the matrix DFD^{-1} represents f in the basis B'. We have

$$\det(DFD^{-1} - xI) = \det(DFD^{-1} - xDID^{-1}) = \det(D(F - xI)D^{-1}) =$$

$$\det(D)\det(F - xI)\det(D^{-1}) = \det(D)\det(D^{-1})\det(F - xI) = \det(F - xI).$$

□

Problem 1. Let us consider two bases of the linear space $\mathbb{R}^2$:

$$B = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$$

and

$$B' = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\rangle.$$

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping such that

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Find the matrices F and F' od the mapping f in the bases B and B' and directly from the form of these matrices calculate the characteristic polynimial(s). Verify if the result is the same.

Theorem 3. A scalar x is an eigenvalue of a linear mapping $f : V \to V$ if and only if it is a root of the characteristic polynomial of f.

Proof. We have already shown that if $x \in \mathcal{K}$ is an eigenvalue then $w_f(x) = 0$. Assume now that $w_f(x) = 0$. This means that

$$\det(F - xI) = 0$$

which means that the matrix $\det(F - xI)$ has order smaller that $\dim(V)$. As the matrix $(F - xI)$ is the matrix of the linear mapping $f - x \cdot \mathrm{id}$ there must be a nonzero vector $\boldsymbol{v}$ such that

$$\mathbf{0} = (f - x \cdot \mathrm{id})(\boldsymbol{v}) = f(\boldsymbol{v}) - x \cdot \mathrm{id}(\boldsymbol{v}) = f(\boldsymbol{v}) - x \cdot \boldsymbol{v},$$

and hence

$$f(\boldsymbol{v}) = x \cdot \boldsymbol{v},$$

which means that the vector $\boldsymbol{v}$ is an eigenvector with the eigenvalue x.

□

Problem 2. Let us consider the linear space $\mathbb{R}^3$ and the linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ that is represented in the standard basis by the matrix

$$F = \begin{bmatrix} 4 & -1 & -1 \\ -3 & 2 & 3 \\ 5 & -1 & -2 \end{bmatrix}.$$

Calculate the characteristic polynomial $w_f(x)$, find its roots, find the eigenvectors of f, represent f in the basis consisting of the eigenvectors (if such a basis exists).

Theorem 4 Let V be a linear space of finite dimension n. Let $f : V \to V$ be a linear mapping and let α be an eigenvalue of the mapping f. The multiplicity of the eigenvalue α is equal to $n - \mathrm{rank}(F - \alpha I)$.

Proof. Let m be the multiplicity of α. Let $\langle \boldsymbol{e_1}, \ldots, \boldsymbol{e_m} \rangle$ be the basis of E_α. Let $\langle \boldsymbol{e_{m+1}}, \ldots, \boldsymbol{e_n} \rangle$ be a complement of the vector system $\langle \boldsymbol{e_1}, \ldots, \boldsymbol{e_m} \rangle$ to the basis $\langle \boldsymbol{e_1}, \ldots, \boldsymbol{e_n} \rangle$ of the whole space V. We have (a theorem in a basic course in linear algebra)

$$\mathrm{rank}(F - \alpha I) = \dim((f - \alpha \cdot \mathrm{id})[V]).$$

We shall show that the right-hand side of the above equality is equal to $n-m$. This will, of course, imply the conclusion of the theorem.

First let us notice that $(f - \alpha \cdot \mathrm{id})(\boldsymbol{e_i}) = \mathbf{0}$ for each $i \leqslant m$. Thus as the image space $(f - \alpha \cdot \mathrm{id})[V]$ is spanned over the vectors $f(\boldsymbol{e_i})$, $i \leqslant n$, the space $(f - \alpha \cdot \mathrm{id})[V]$ is the the space spanned by the vectors $\langle (f - \alpha \cdot \mathrm{id})(\boldsymbol{e_{m+1}}), \ldots, (f - \alpha \cdot \mathrm{id})(\boldsymbol{e_n}) \rangle$. We will show that these vectors are linearly independent. Let us assume that

$$\beta_1 (f - \alpha \cdot \mathrm{id})(\boldsymbol{e_{m+1}}) + \ldots + \beta_{n-m} (f - \alpha \cdot \mathrm{id})(\boldsymbol{e_n}) = \mathbf{0}.$$

Transforming this equality we obtain

$$f(\beta_1 \boldsymbol{e_{m+1}} + \ldots + \beta_1 \boldsymbol{e_n}) = \alpha(\beta_1 \boldsymbol{e_{m+1}} + \ldots + \beta_1 \boldsymbol{e_n})$$

which means that the vector $\beta_1 \boldsymbol{e_{m+1}} + \ldots + \beta_1 \boldsymbol{e_n}$ is an eigenvector with the eigenvalue α, but it can belong to E_α only if

$$\beta_1 \boldsymbol{e_{m+1}} + \ldots + \beta_1 \boldsymbol{e_n} = \mathbf{0}$$

(because otherwise this vector is linearly independent of the basis of E_α). Because vectors $\boldsymbol{e_{m+1}}, \ldots, \boldsymbol{e_n}$ are linearly independent we infer that the coefficients $\beta_1, \ldots, \beta_{n-m}$ are all equal to zero. This shows that indeed the vectors $(f - \alpha \cdot \mathrm{id})(\boldsymbol{e_m})$,...,$(f - \alpha \cdot \mathrm{id})(\boldsymbol{e_n})$ are linearly independent, and hence

$$\dim((f - \alpha \cdot \mathrm{id})[V]) = n - m$$

. From the last inequality the conclusion follows.

$\square$

Problem 3. Show an example of a linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, such that the space $\mathbb{R}^2$ has no basis consisting of eigenvectors of f.

Problem 4. Let V be a linear space. Let $f : V \to V$ be a linear mapping and let α be an eigenvalue of the mapping f. Let $p(x) - a_n x^n + \ldots + a_1 x + a_0$ be an arbitrary polynomial. Show that $p(\alpha)$ is an eigenvalue of the mapping $p(f)$, where $p(f) := a_n \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}} + \ldots + a_1 f + a_0 \mathrm{id}$.

Let V be a linear space of finite dimension n. Let $f : V \to V$ be a linear mapping represented in some basis by the matrix

$$F = \begin{bmatrix} a_{11} & \dots & a_{1,n} \\ \vdots & \dots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix}.$$

we call the sum of the elements on diagonal the *trace* of f (and of F) and denote it by

$$\mathrm{tr}(f) = \sum_{i=1}^{n} a_{i,i}.$$

Problem 5. Show that the trace of f does not depend on the choice of a basis in which we represent f by a matrix. In other words, of F' represents f in another basis the sum of its diagonal elkements is the same as for F.

Problem 6. Let V be a linear space of finite dimension n. Show that a linear mapping f of V into itself has n different eigenvalues if and only if it can be represented in some basis by a matrix of the form $[a_{i,j}]$, where a $a_{i,i} \neq 0$, and $a_{i,j} = 0$ if $i \neq j$, and $a_{i,i} \neq a_{j,j}$ if $i \neq j$.