ADVANCED TOPICS IN ALGEBRA

LECTURE 3

(lecture and problems to solve)
2020/21

CYCLIC SPACES

Let V be a linear space and $f : V \to V$ be a linear mapping. Let $\boldsymbol{v} \in V$. Let

$$\phi(x) = x^k + a_{k-1}x^{k-1} + \ldots + a_1 x + a_0$$

be the annnihilator of $\boldsymbol{v} \in V$.

Theorem 16. For V, f, $\boldsymbol{v}$ and ϕ as above, the vectors

$$\boldsymbol{v}, f(\boldsymbol{v}), \ldots, f^{k-1}(\boldsymbol{v})$$

are linearly independent.

Proof. If some nonzero linear combination of these vectors equaled to zero

$$b_0\boldsymbol{v} + b_1 f(\boldsymbol{v}) + \ldots b_{p-1}f^{p-1}(\boldsymbol{v}) + b_p f^p(\boldsymbol{v}) = 0,$$

where $p \leqslant k-1$ and $b_p \neq 0$, then also

$$\frac{b_0}{b_p}\boldsymbol{v} + \frac{b_1}{b_p}f(\boldsymbol{v}) + \ldots \frac{b_{p-1}}{b_p}f^{p-1}(\boldsymbol{v}) + f^p(\boldsymbol{v}) = 0,$$

but this would contradict the fact that $\phi(x)$ is the annihilator of $\boldsymbol{v}$.

□

The space

$$\Gamma(f, \boldsymbol{v}) := \operatorname{span}(\boldsymbol{v}, f(\boldsymbol{v}), \ldots, f^{k-1}(\boldsymbol{v}))$$

is called the *cyclic space generated by* $\boldsymbol{v}$. The vector $\boldsymbol{v}$ is called its *generator.* The vectors $\boldsymbol{v}, f(\boldsymbol{v}), \ldots, f^{k-1}(\boldsymbol{v})$ are called the *cyclic base.*

Problem 1. Find the matrix of f in the cyclic base.

Problem 2. Prove that the characteristic polynomial for the mapping f is equal to the annihilator of the space multiplied by $(-1)^{\dim(\Gamma(f,\boldsymbol{v})}$.

Theorem 17. Let $\xi(x)$ be any polynomial and let $w \in \Gamma(f, \boldsymbol{v})$. Then $\xi(f)(w) \in \Gamma(f, \boldsymbol{v})$.

Proof. We can express $\xi(x)$ as

$$\xi(x) = D(x)\phi(x) + r(x),$$

where $D(x)$ is some polynomial and $r(x)$ has degree smaller than k. Then

$$\xi(f)(\boldsymbol{v}) = D(f)(\phi(f)(\boldsymbol{v})) + r(f)(\boldsymbol{v}) = D(f)(\mathbf{0}) + r(f)(\boldsymbol{v}) = r(f)(\boldsymbol{v}) \in \Gamma(f, \boldsymbol{v}).$$

□

Theorem 18. The space $\Gamma(f, \boldsymbol{v})$ is invariant (with respect to f).

Proof. Let $\boldsymbol{w} \in \Gamma(f, \boldsymbol{v})$. We have $\boldsymbol{w} = \xi(f)(\boldsymbol{v})$ for some polynomial $\xi(x)$. In virtue of Theorem 17 (justifying the last inclusion below) we have:

$$f(\boldsymbol{w}) = f(\xi(f)(\boldsymbol{v})) = (\vartheta)(f)(\boldsymbol{v}) \in \Gamma(f, \boldsymbol{v}),$$

where $\vartheta(x) = x\xi(x)$.

□

Theorem 19. The annihilator $\phi(x)$ of a vector $\boldsymbol{v}$ is the annihilator $\phi_{\Gamma(f,\boldsymbol{v})}$ of the whole cyclic space $\Gamma(f, \boldsymbol{v})$

Proof. Let $\boldsymbol{u} \in \Gamma(f, \boldsymbol{v})$. Then

$$\boldsymbol{u} = \xi(f)(\boldsymbol{v})$$

for some polynomial $\xi(x)$ (one can choose $\xi(x)$ of degree smaller than $\deg(\phi(x))$, but this is irrelevant here). We have

$$\phi(f)(\xi(f)(\boldsymbol{u})) = \xi(f)(\phi(f)(\boldsymbol{u})) = \xi(f)(\mathbf{0}) = \mathbf{0}.$$

□

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Theorem 20. Let V be a linear space and $f : V \to V$ be a linear mapping. Let $W < V$ be a linear subspace of V which is invariant with respect to f. Then there exists $\boldsymbol{v} \in W$ such that $W = \Gamma(f, \boldsymbol{v})$ (in other words W is cyclic) if and oly if $\dim(W) = \deg(\phi_W)$, where ϕ_W is the annihilator of W.

Proof. If $W = \Gamma(f, \boldsymbol{v})$, then $\dim(W) = \dim(\phi_W)$ by the definition of a cyclic space $\Gamma(f, \boldsymbol{v})$ and Theorem 19.

Let us prove the opposite implication. Thus assume that $\dim(W) = \deg(\phi_W)$ and recall that W is invariant. By Theorem 15 there exists a vector $\boldsymbol{v}$ such that its annihilator is the annihilator of W. But then the space $\Gamma(f, \boldsymbol{v}) < W$ has dimension $\deg(\phi_W) = \dim(W)$, and therefore $\Gamma(f, \boldsymbol{v}) = W$.

□

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Theorem 21. Let W be a cyclic space and let it be a direct sum of invariant subspaces $W_1, W_2 < W$. Then W_1 and W_2 are also cyclic spaces.

Proof. Let ϕ_W be the annihilator of the space W. Let $\phi_1(x) = \phi_{W_1}(x), \phi_2(x) = \phi_{W_2}(x)$ be the annihilators of the spaces W_1, W_2, respectively. By Theorem 14

$$\phi(x) = LCM(\phi_1(x), \phi_2(x)).$$

By Problem 6.2 (Problem 6, Lecture 2)

$$\deg(\phi_1(x)) \leqslant \dim(W_1)$$

and

$$\deg(\phi_2(x)) \leqslant \dim(W_2).$$

If, say,

$$\deg(\phi_1(x)) < \dim(W_1),$$

then

$$\deg(\phi(x)) \leqslant \deg(\phi_1(x)) + \deg(\phi_2(x)) < \dim(W_1) + \dim(W_2) = \dim(W),$$

but this is not true because, by Theorem 20, $\deg(\phi(x)) = \dim(W)$. Thus

$$\deg(\phi_1(x)) = \dim(W_1),$$

and, again by Theorem 20, W_1 is cyclic. In the same way we see that W_2 is cyclic. □

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Theorem 22. Let $\phi_W(x)$ be the annihilator of a cyclic space $W = \Gamma(f, \boldsymbol{v})$. If the annihilator $\phi_W(x)$ of W is a product of two relatively prime polynomials

$$\phi_W(x) = \phi_1(x) \cdot \phi_2(x)$$

then

$$W = \Gamma(f, \phi_1(\boldsymbol{v})) \oplus \Gamma(f, \phi_2(\boldsymbol{v}))$$

and the annihilators of the spaces $\Gamma(f, \phi_1(\boldsymbol{v}))$, $\Gamma(f, \phi_2(\boldsymbol{v}))$ are $\phi_2(x)$, $\phi_1(x)$, respectively.

Proof. Because the spaces $\Gamma(f, \phi_1(f)(\boldsymbol{v}))$ and $\Gamma(f, \phi_2(f)(\boldsymbol{v}))$ consist of some vectors of the form $\xi(f)(\boldsymbol{v})$ by Theorem 17 they are subspaces of the space W.

Note that the annihilator of the vector $\phi_1(f)(\boldsymbol{v})$ is $\phi_2(x)$. Indeed, let us temporarily call this annihilator $\psi(x)$; because

$$\phi_2(f)(\phi_1(f)(\boldsymbol{v})) = (\phi_1 \cdot \phi_2)(f)(\boldsymbol{v}) = \mathbf{0},$$

$\psi(x)$ must divide $\phi_2(x)$. If it is not $\phi_2(x)$, then it has degree smaller than $\phi_2(x)$ and

$$\psi(f)(\phi_1(f)(\boldsymbol{v})) = \mathbf{0}.$$

Hence $\phi_2(x)\phi_1(x) | \psi(x)\phi_2(x)$ which implies $\phi_2(x)|\psi(x)$ and thus $\psi(x) = \phi_2(x)$.

Analogously, the annihilator of the vector $\phi_2(f)(\boldsymbol{v})$ is $\phi_1(x)$.

We shall show that the only common element of the spaces $\Gamma(f, \phi_1(f)(\boldsymbol{v}))$ and $\Gamma(f, \phi_2(f)(\boldsymbol{v}))$ is $\mathbf{0}$. Let $\boldsymbol{w} \in \Gamma(f, \phi_1(f)(\boldsymbol{v})) \cap \Gamma(f, \phi_2(f)(\boldsymbol{v}))$. Then

$$\phi_1(f)(\boldsymbol{w}) = \mathbf{0}$$

and

$$\phi_2(f)(\boldsymbol{w}) = \mathbf{0}.$$

Hence the annihilator of $\boldsymbol{w}$ divides both $\phi_1(x)$ and $\phi_2(x)$, which implies that this annihilator is a constant and consequently $\boldsymbol{w} = \mathbf{0}$.

It remains to show that

$$\dim(\Gamma(f, \phi_1(f)(\boldsymbol{v}))) + \dim(\Gamma(f, \phi_2(f)(\boldsymbol{v}))) = \dim(\Gamma(f, \boldsymbol{v})).$$

But this is obvious now, because

$$\dim(\Gamma(f, \phi_1(f)(\boldsymbol{v}))) = \deg(\phi_2(x)),$$
$$\dim(\Gamma(f, \phi_2(f)(\boldsymbol{v}))) = \deg(\phi_1(x)),$$
$$\dim(\Gamma(f, \boldsymbol{v})) = \deg(\phi_1(x) \cdot \phi_2(x))$$

and

$$\deg(\phi_1(x)) + \deg(\phi_2(x)) = \deg(\phi_1(x) \cdot \phi_2(x)).$$

□

The following theorem is a straightforward (but important) generalization of Theorem 22.

Theorem 23. Let $\phi_W(x)$ be the annihilator of a cyclic space $W = \Gamma(f, \boldsymbol{v})$. If the annihilator $\phi_W(x)$ of W is equal to $\psi_1^{k_1}(x) \cdot \psi_2^{k_2}(x) \cdot \ldots \cdot \psi_m^{k_m}(x)$, where the polynomials $\psi_1(x), \ldots, \psi_m(x)$ are relatively prime then

$$W = \Gamma(f, \xi_1(f)(\boldsymbol{v})) \oplus \Gamma(f, \xi_2(f)(\boldsymbol{v})) \oplus \ldots \oplus \Gamma(f, \xi_m(f)(\boldsymbol{v})),$$

where

$$\xi_i(x) = \prod_{j \neq i} \psi_j^{k_j}(x).$$

Proof. It is easy to prove the theorem by induction with respect to m using Theorem 22.

Firstly, set $m = 2$. Then the conclusion follows from Theorem 22.

Now assume that the conclusion holds for some $m - 1 \geqslant 2$. And assume the hypothesis of the theorem for m. Then consider two polynomials

$$\phi_1(x) = \psi_1^{k_1}(x) \cdot \psi_2^{k_2}(x) \cdot \ldots \cdot \psi_{m-1}^{k_{m-1}}(x) \text{ and } \phi_1(x) = \psi_m^{k_m}(x).$$

Again by Theorem 22, we know that

$$W = \Gamma(f, \phi_1(\boldsymbol{v})) \oplus \Gamma(f, \phi_2(\boldsymbol{v})),$$

and the annihilators of the spaces $\Gamma(f, \phi_1(\boldsymbol{v}))$, $\Gamma(f, \phi_2(\boldsymbol{v}))$ are $\phi_2(x)$, $\phi_1(x)$, respectively. Now we are applying the inductive hypothesis to the space $\Gamma(f, \phi_2(\boldsymbol{v}))$.

□

Problem 3. Find a mapping $f : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\mathbb{R}^4$ becomes a cyclic space

$$\mathbb{R}^4 = \Gamma(f, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}),$$

where $\phi(x) = (x - 2)^2(x - 1)^2$ is the annihilator of the space and the standard base is the cyclic base. Find the matrix of f. Find the representation of $\mathbb{R}^4$ from Theorem 22 as a direct sum of two cyclic spaces whose annihilators are powers of prime polynomials. Find the cyclic bases of the summand spaces of the form given in Theorem 22. Write a matrix of f in the basis which is the union of these bases.

Problem 4. Is any system of n independent vectors in $\mathbb{R}^n$ a cyclic base in $\mathbb{R}^n$ for some mapping f?

Problem 5. Let $W < V$ and W be invariant with respect to a linear mapping $f : V \to V$. If $W = W_1 \oplus W_2$, W_1, W_2 are cyclic, and the annihilators of W_1 and W_2 are relatively prime, then W is cyclic.

Problem 6. Consider the space of polynomials of degree $< n$ with real coefficients. Is it cyclic with respect to the mapping $\phi(x) \mapsto \phi(x + 1)$? (Mostowski, Stark)

Hint. Prove, using induction (with respect to k), that for each $k \in \mathbb{N}$ and for each $i \in \mathbb{N}$ the polynomials

$$(x+i)^k,\ (x+i+1)^k,\ \ldots,(x+i+k-1)^k$$

are linearly independent. Then use Theorem 20.