On nonmeasurable unions

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Abstract. For a Polish space X and a σ-ideal I of subsets of X which has a Borel base we consider families $\mathcal{A}$ of sets in I with the union $\bigcup\mathcal{A}$ not in I. We determine several conditions on $\mathcal{A}$ which imply the existence of a subfamily $\mathcal{A}'$ of $\mathcal{A}$ whose union $\bigcup\mathcal{A}'$ is not in the σ-field generated by the Borel sets on X and I. Main examples are $X = \mathbb{R}$ and I being the ideal of sets of Lebesgue measure zero or the ideal of sets of the first category.

1. Introduction

It is known that for a partition $\mathcal{A}$ of the real line consisting of sets of Lebesgue measure zero, the union of some of these sets is Lebesgue nonmeasurable. Analogous result is known for the sets of the first category (the Lebesgue measurability is then replaced by having the Baire property). Actually, this result remains true, if, in the above statement, the real line is replaced by any Polish space, the σ-ideals of sets of Lebesgue measure zero, or of sets of the first category, are replaced by any σ-ideal I with a Borel base, and, instead of assuming that the family $\mathcal{A}$ of sets of an ideal is a partition of the space, we assume that $\mathcal{A}$ is point finite and its union is not in I. The conclusion says now that there exists a subfamily $\mathcal{A}'$ of $\mathcal{A}$ such that the union of its sets is not in the σ-algebra generated by the σ-algebra of Borel sets and I ([**3**],[**4**],[**10**]; compare also [**13**],[**14**]; the proof in [**4**] is probably the shortest and the simplest).

It is known that within ZFC is not possible to replace the assumption that the family $\mathcal{A}$ is point finite even by the one saying that $\mathcal{A}$ is point-countable (see [**11**]).

Thus we always have to enrich our hypothesis to get the same conclusion without point-finiteness of $\mathcal{A}$.

One way is set-theoretic. It is known that under some assumptions on I, which are independent of ZFC, in the case of both ideals of sets of Lebesgue measure zero and of sets of the first category, one can obtain the conclusion of the above theorem assuming about $\mathcal{A}$ only that it is a subfamily of I and that its union is not in I. In Section 3 of this article, under set-theoretic assumptions, two theorems stating the existence of a subfamily of $\mathcal{A}$ with a (strongly) nonmeasurable union are given (Theorems 3.1 and 3.2).

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If we want to stay within ZFC to get the same conclusion about the existence of a subfamily of $\mathcal{A}$ having the union that is not in the σ-algebra generated by the σ-algebra of Borel sets and I we must assume some regularity of $\mathcal{A}$ (of course, point-finiteness is one example of such an assumption) or/and some regularity of the sets in $\mathcal{A}$. In Section 4 we consider the case where $\mathcal{A}$ is point countable and the sets which are elements of $\mathcal{A}$ are countable. We prove in this case the existence of a subfamily of $\mathcal{A}$ having a non-measurable union (Theorem 4.1). We also consider the case where the elements of $\mathcal{A}$ are countable closed sets of uniformly bounded Cantor-Bendixson rank (thus, in particular, the case where all sets in $\mathcal{A}$ are finite is covered). Then, with no additional assumption about $\mathcal{A}$, the conclusion regarding the existence of a subfamily of $\mathcal{A}$ with nonmeasurable union also holds.

Another special family of sets of both Lebesgue measure zero and of the first category on the real line is considered in Section 5. Namely, we consider translates of the elements of the standard ternary Cantor set $\mathbf{C}$. It is a well known (and easy to prove) fact that the algebraic sum $\mathbf{C}+\mathbf{C}$ is equal to $[0,2]$. One can view $\mathbf{C}+\mathbf{C}$ as a union of some translates of $\mathbf{C}$, actually those determined by $\mathbf{C}$ itself. It turns out that in this case we are able to derive again the conclusion on the existence of $A \subseteq \mathbf{C}$ such that $A+\mathbf{C}$ is nonmeasurable (Theorem 5.9, Corollary 5.10, Remark 5.11). It seems very interesting how far this theorem can be generalized, as the methods we use seem not to allow for any substantial generalization and very general conjectures can be made here.

Some of the results of this paper state, actually, more than nonmeasurability of the union of a subfamily of $\mathcal{A}$. They say that the intersection of this union with any measurable set that is not in I is nonmeasurable (recall, the measurability is understood here in the sense of belonging to the σ-algebra generated by the family of Borel sets and I). It turns out that the same strong conclusion can be obtained for the ideal of the first Baire category sets under the assumption that $\mathcal{A}$ is a partition, but without assuming anything about the regularity of the elements of $\mathcal{A}$. This is the result of Section 6. We do not know if the point-finiteness is also sufficient to get this conclusion in the case of the ideal of sets of the first Baire category. We also do not know if the analogous theorem holds for the ideal of the Lebesgue measure zero sets on the real line.

2. Definitions and notations

The cardinality of a set A is denoted by $|A|$. Cardinal numbers will usually be denoted by κ and λ.

The symbols $[A]^{\leq\kappa}$ and $[A]^{<\kappa}$ denote the families of all subsets of A of cardinality not bigger than κ and smaller than κ, respectively. The sets of positive integers, rational numbers and the set of real numbers are denoted by $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively. If R is a binary relation then $R[X]$ denotes then set $\{y\colon (\exists x \in X)((x,y) \in R)\}$. An **ideal** of subsets of a set X is a family of subsets of X which is closed under finite unions and taking subsets and such that $[X]^{<\omega} \subseteq I$. A family of sets is a σ**-ideal** if it is an ideal and is closed under countable unions.

For a topological space T, by B_T we denote the family of Borel subsets of T. If I is an ideal of subsets of a set X and $\mathcal{S}$ is a field of subsets of X, then by $\mathcal{S}(I)$ we denote the field generated by $\mathcal{S} \cup I$. If I is a σ-ideal and $\mathcal{S}$ is a σ-field then $\mathcal{S}(I)$ is a σ-field, too. The σ-ideal of Lebesgue measure zero subsets of $\mathbb{R}$ will be denoted by $\mathbb{L}$ and the σ-ideal of sets of the first Baire category in $\mathbb{R}$ will be denoted by $\mathbb{K}$.

Then $B_{\mathbb{R}}(\mathbb{L})$ is the σ-field of Lebesgue measurable subsets of $\mathbb{R}$ and $B_{\mathbb{R}}(\mathbb{K})$ is the σ-field of subsets of $\mathbb{R}$ with the Baire property.

Definition 2.1. Suppose that I is an ideal of sets. Then

(1) $add(I) = \min\{|\mathcal{A}| \colon \mathcal{A} \subseteq I \wedge \bigcup \mathcal{A} \notin I\}$,
(2) $cov(I) = \min\{|\mathcal{A}| \colon \mathcal{A} \subseteq I \wedge \bigcup \mathcal{A} = \bigcup I\}$,
(3) $non(I) = \min\{|X| \colon X \subseteq \bigcup I \wedge X \notin I\}$.
(4) $cof(I) = \min\{|\mathcal{X}| \colon \mathcal{X} \subseteq I \wedge (\forall X \in I)(\exists Y \in \mathcal{X})(X \subseteq Y)\}$.

The following inequalities hold for every ideal: $add(I) \leq non(I) \leq cof(I)$ and $add(I) \leq cov(I) \leq cof(I)$. If the Continuum Hypothesis or Martin's Axiom holds, then $add(\mathbb{L}) = add(\mathbb{K}) = 2^{\aleph_0}$ (see [**15**]). The theory $\mathbf{ZFC} \cup \{\aleph_1 = non(\mathbb{L}), cov(\mathbb{L}) = \aleph_2 = 2^{\aleph_0}\}$ is relatively consistent, too (see [**2**], compare also [**1**]).

If I is an ideal of subsets of a topological space T then we say that the ideal I has a Borel base if for each set $X \in I$ there exists a set $Y \in B_T \cap I$ such that $X \subseteq Y$. The two classical ideals $\mathbb{K}$ and $\mathbb{L}$ have Borel bases. If an ideal I on a Polish topological space has a Borel base then $cof(I) \leq 2^{\aleph_0}$.

Definition 2.2. A pair (T, I) is a **Polish ideal space** if P is an uncountable Polish topological space and I is a σ- ideal of subsets of T with a Borel base.

The pairs $(\mathbb{R}, \mathbb{K})$, $(\mathbb{R}, \mathbb{L})$ and $(\mathbb{R}, [\mathbb{R}]^{\leq\omega})$ are examples of Polish ideal spaces. If $\mathcal{S}$ is a field of subsets of a set X, then by $\mathcal{S}^-$ we denote the family $\{A \in P(X) \colon P(A) \subseteq \mathcal{S}\}$ and by $\mathcal{S}^+$ we denote the family $\mathcal{S} \setminus \mathcal{S}^-$. The family $\mathcal{S}^-$ is an ideal. If $\mathcal{S}$ is a σ-field then $\mathcal{S}^-$ is a σ-ideal. If a σ-ideal I of subsets of a Polish space has a Borel base then $(B_T(I))^- = I$ (see [**6**]). In particular, the equality $(B_T)^- = [T]^{\leq\omega}$ holds for any Polish space T.

Definition 2.3. Let (T, I) be a Polish ideal space. Then we put

$$cov_H(I) = \min\{|\mathcal{A}| \colon \mathcal{A} \subseteq I \wedge \bigcup \mathcal{A} \in (\mathcal{B}_T(I))^+\}$$

and

$$cof(T, I) = \min\{|\mathcal{A}| \colon \mathcal{A} \subseteq (B_T(I))^+ \wedge (\forall X \in (B_T(I))^+)(\exists Y \in \mathcal{A})(Y \subseteq X)\}.$$

It is worth to remark that the following equalities hold: $cov(\mathbb{L}) = cov_H(\mathbb{L})$, $cov(\mathbb{K}) = cov_H(\mathbb{K})$. It is proved in [**7**] that if the quotient boolean algebra $B_T(I)/I$ satisfies c.c.c. then $cof(I) = cof(T, I)$. Therefore we have $cof(\mathbb{R}, \mathbb{L}) = cof(\mathbb{L})$ and $cof(\mathbb{R}, \mathbb{K}) = cof(\mathbb{K})$.

Definition 2.4. Let $\mathcal{S}$ be a field of subsets of a set X. A subset B of X is an **$\mathcal{S}$-Bernstein set** if for all $A \in \mathcal{S}^+$ both sets $A \cap B$ and $A \setminus B$ are nonempty.

If T is a Polish space then the notion of B_T-Bernstein set coincides with the classical notion of Bernstein set. If B is an $\mathcal{S}$-Bernstein set, $A \in \mathcal{S}$ and $A \subseteq B$ or $A \cap B = \emptyset$ then $A \in \mathcal{S}^-$. Thus, in the case when $\mathcal{S} = B_{\mathbb{R}}(\mathbb{L})$, the notion of $\mathcal{S}$-Bernstein set coincides with the notion of "saturated nonmeasurable set". The following property of the notion of "$\mathcal{S}$-Bernstein" follows immediately from the definition.

Lemma 2.5. *Suppose that $\mathcal{S}$ is a field of subsets of a set X and that $A \subseteq B \subseteq X$ are two $\mathcal{S}$-Bernstein sets. If $A \subseteq C \subseteq B$ then C is an $\mathcal{S}$-Bernstein set, too.*

LEMMA 2.6. *Suppose that $\mathcal{S}$ is a σ-algebra of subsets of a set T and $R \subseteq T \times T$ is a reflexive and symmetric relation such that*

$$(\forall A \in \mathcal{S}^+)(\forall X \subseteq T)(|X| < cof(\mathcal{S}^+, \mathcal{S}^-) \to A \setminus R[X] \neq \emptyset). \tag{2.1}$$

Then there exists a set $P \subseteq T$ such that both sets P and $R[P]$ are $\mathcal{S}$-Bernstein.

PROOF. Let $\kappa = cof(\mathcal{S}^+, \mathcal{S}^-)$ and let $(L_\alpha)_{\alpha<\kappa} \subseteq \mathcal{S}^+$ be a family of sets such that for each $A \in \mathcal{S}^+$ there exists an $\alpha < \kappa$ such that $L_\alpha \subseteq A$. Using a transfinite recursion of length κ we build two sequences $\{p_\alpha : \alpha < \kappa\}$ and $\{q_\alpha : \alpha < \kappa\}$ such that

(1) $p_\alpha \in L_\alpha \setminus R[\{p_\beta : \beta < \alpha\} \cup \{q_\beta : \beta < \alpha\}]$,
(2) $q_\alpha \in L_\alpha \setminus R[\{p_\beta : \beta \leq \alpha\} \cup \{q_\beta : \beta < \alpha\}]$.

Let $P = \{p_\alpha : \alpha < \kappa\}$ and $Q = \{q_\alpha : \alpha < \kappa\}$. Then $P \cap Q = \emptyset$ and for each $A \in \mathcal{S}^+$ the sets $P \cap A$ and $Q \cap A$ are nonempty. This implies that P is an $\mathcal{S}$-Bernstein set.

We claim that $R[P] \cap Q = \emptyset$. Suppose this is not the case. Let $\alpha, \beta < \kappa$ be ordinals such that $q_\alpha \in R[\{p_\beta\}]$. From condition (2) we deduce that $\alpha < \beta$. By symmetry of the relation R we get $p_\beta \in R[\{q_\alpha\}]$, and this contradicts condition (1). □

3. Summable families of sets

Suppose that N is a Lebesgue nonmeasurable subset of the real line. Then there exists a $B_{\mathbb{R}}(\mathbb{L})$-Bernstein set of the same cardinality as the cardinality of the set N. Therefore the least cardinality of a $B_{\mathbb{R}}(\mathbb{L})$-Bernstein set coincides with the least cardinality of a Lebesgue nonmeasurable set. The same observation holds for the ideal $\mathbb{K}$ and the σ-field of sets with the Baire property.

THEOREM 3.1. *Suppose that (T, I) is a Polish ideal space and that there exists a $B_T(I)$-Bernstein set of cardinality strictly less than $cov_H(I)$. If $\mathcal{A} \subseteq I$ and $\bigcup \mathcal{A} = T$ then there exists a subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcup \mathcal{B}$ is a $B_T(I)$-Bernstein set.*

PROOF. Let B be a $B_T(I)$-Bernstein set of cardinality strictly less than the number $cov_H(I)$. For each $b \in B$ we choose a set $A_b \in \mathcal{A}$ such that $b \in A_b$ and we put $\mathcal{B} = \{A_b : b \in B\}$. If $S \in B_T(I)$ and $S \cap \bigcup \mathcal{B} = \emptyset$ then $S \cap B = \emptyset$, so $S \in I$. On the other hand, if $S \in B_T(I) \setminus I$ and $S \subseteq \bigcup \mathcal{B}$ then $S = \bigcup\{S \cap A_b : b \in B\}$, which is impossible, since $|B| < cov_H(I)$. Therefore $\bigcup \mathcal{B}$ is a $B_T(I)$- Bernstein set. □

Suppose that $\mathcal{A}$ and $\mathcal{S}$ are two families of sets. We say (see [**6**]) that the family $\mathcal{A}$ is $\mathcal{S}$-summable if for every $\mathcal{A}' \subseteq \mathcal{A}$ we have $\bigcup \mathcal{A}' \in \mathcal{S}$. From Theorem 3.1 we easily deduce that if (T, I) is a Polish ideal space such that there exists a $B_T(I)$-Bernstein set of cardinality strictly less than $cov_H(I)$ and $\mathcal{A} \subseteq I$ is a $\mathcal{B}_T(I)$-summable family then $\bigcup \mathcal{A} \in I$.

THEOREM 3.2. *Suppose that (T, I) is a Polish ideal space and that $cov_H(I) = cof(T, I)$. Assume that $\mathcal{A} \subseteq I$, $\bigcup \mathcal{A} = T$ and $\bigcup\{A \in \mathcal{A} : t \in A\} \in I$ for each $t \in T$. Then there exists a subfamily $\mathcal{C} \subseteq \mathcal{A}$ such $\bigcup \mathcal{C}$ is a $B_T(I)$-Bernstein set.*

PROOF. Let

$$R = \{(x, y) : (\exists A \in \mathcal{A})(\{x, y\} \subseteq A)\}.$$

The relation R is reflexive and symmetric. For each $X \subset T$ we have $R[X] = \bigcup\{A \in \mathcal{A} : (\exists t \in X)(t \in A)\}$. Therefore if $|X| < cof(T, I)$ and $A \in (B_T(I))^+$ then $A \setminus R[X] \neq \emptyset$. Hence we may apply Lemma 2.6 and we get a set P such that both sets P and $R[P]$ are $B_T(I)$-Bernstein sets. For each $p \in P$ we choose a set $A_p \in \mathcal{A}$ such that $p \in A_p$ and we put $\mathcal{C} = \{A_p \colon p \in P\}$. Lemma 2.5 implies that $\bigcup \mathcal{C}$ is a $B_T(I)$-Bernstein set. □

4. Families of countable sets

We say that a family of sets $\mathcal{A}$ is **point-countable** if for every x the set $\{A \in \mathcal{A} \colon x \in A\}$ is countable.

Theorem 4.1. *Suppose that T is an uncountable Polish topological space, $\mathcal{A} \subseteq [T]^{\leq\omega}$ is point-countable and $\bigcup \mathcal{A} = T$. Then there exists a subfamily $\mathcal{B} \subseteq \mathcal{A}$ such $\bigcup \mathcal{B}$ is a Bernstein subset of T.*

Proof. Let $\mathcal{A} \subseteq [T]^{\leq\omega}$ be point countable. We define

$$R = \{(x, y) \colon (\exists A \in \mathcal{A})(\{x, y\} \subseteq A)\}.$$

Then the relation R is reflexive and symmetric. For every $X \subseteq T$ such that $|X| < |T| = 2^{\aleph_0}$ we have

$$|R[X]| = |\bigcup_{t \in X} \{A \in \mathcal{A} \colon t \in A\}| \leq |X| \cdot \aleph_0 < 2^{\aleph_0},$$

hence we can apply Lemma 2.6 to the ideal $I = [T]^{\leq\omega}$. We obtain a set P such that P and $R[P]$ are Bernstein sets. For each $p \in P$ we choose a set $A_p \in \mathcal{A}$ such that $p \in A_p$ and we put $\mathcal{C} = \{A_p \colon p \in P\}$. Then $P \subseteq \bigcup \mathcal{C} \subseteq R[P]$, so by Lemma 2.5, the set $\bigcup \mathcal{C}$ is a Bernstein set, too. □

For a subset X of a topological space T by X' we denote the set of accumulation points of X. Using transfinite induction on ordinal numbers we define $X^{(\alpha+1)} = (X^{(\alpha)})'$ and $X^{(\lambda)} = \bigcap\{X^{(\alpha)} \colon \alpha < \lambda\}$ for limit ordinals λ. If X is a closed subset of a Polish space then the sequence $(X^{(\alpha)})$ is decreasing and there exists an $\alpha < \omega_1$ such that $X^{(\alpha)} = X^{(\alpha+1)}$. The least such α is called the **Cantor-Bendixson rank** of X. If $X^{(\alpha)} = X^{(\alpha+1)}$ then $X^{(\alpha)}$ is a closed set without isolated points. Therefore, if X is a countable compact subset of a Polish space and α is its Cantor-Bendixson rank, then $X^{(\alpha)} = \emptyset$.

Lemma 4.2. *Suppose that (T, I) is a Polish ideal space and $\mathcal{C} = \{C_j \colon j \in J\}$ is a $B_T(I)$-summable family. Let $\mathcal{D} = \{D_j \colon j \in J\}$ be a family of subsets of T such that*

$$(\forall j \in J)((C_j)' \subseteq D_j \subseteq C_j \wedge |C_j \setminus D_j| \leq 1).$$

Then $\bigcup \mathcal{C} \setminus \bigcup \mathcal{D} \in I$ and $\mathcal{D}$ is a $B_T(I)$-summable family.

Proof. Let $\{B_n \colon n \in \mathbb{N}\}$ be an open base of the topological space T. For each $n \in \mathbb{N}$ we put

$$J_n = \{j \in J \colon C_j \cap B_n = C_j \setminus D_j\}.$$

Then $\bigcup_{j \in J_n} C_j \cap B_n \in I$ since otherwise there would exist a set $Z \subseteq J_n$ such that $\bigcup_{j \in Z} C_j \cap B_n \notin B_T(I)$. Hence $\bigcup_{j \in J_n} C_j \setminus \bigcup_{j \in J_n} D_j \in I$. Moreover $J = \bigcup\{J_n \colon n \in \mathbb{N}\}$. Therefore

$$\bigcup_{j \in J} C_j \setminus \bigcup_{j \in J} D_j \subseteq \bigcup_n (\bigcup_{j \in J_n} C_j \setminus \bigcup_{j \in J_n} D_j) \in I.$$

The family $\{D_j \colon j \in J\}$ is $B_T(I)$-summable because for each $Z \subseteq J$ we have $\bigcup_{j \in Z} C_j \setminus \bigcup_{j \in Z} D_j \in I$. □

LEMMA 4.3. *Suppose that (T,I) is a Polish ideal space and that $\mathcal{C}=\{C_j\colon j\in J\}$ is a $B_T(I)$-summable family of closed sets. Then $\bigcup\{C_j\colon j\in J\}\setminus\bigcup\{(C_j)'\colon j\in J\}\in I$ and $\{(C_j)'\colon j\in J\}$ is a $B_T(I)$-summable family.*

PROOF. Let us fix $j\in J$. The set C_j is closed, hence $(C_j)'\subseteq C_j$. The set $C_j\setminus(C_j)'$ is countable. Let us fix an enumeration $(p_{j,n})_{n<n_j}$ of $C_j\setminus(C_j)'$ where $n_j\le\omega$. Let $C_{j,n}=C_j\setminus\{p_{j,k}\colon k<n\wedge k<n_j\}$. For each $n\in\omega$ we put $\mathcal{C}_n=\{C_{j,n}\colon j\in J\}$. Then $(\bigcup\mathcal{C}_n)_{n<\omega}$ is a decreasing sequence of sets. Lemma 4.2 implies that $\bigcup\mathcal{C}_n\setminus\bigcup\mathcal{C}_{n+1}\in I$ for each $n<\omega$. Therefore $\bigcup\{C_j\colon j\in J\}\setminus\bigcup\{(C_j)'\colon j\in J\}\in I$. From this we obtain the summability of $\{(C_j)'\colon j\in J\}$. $\square$

THEOREM 4.4. *Let (T,I) be a Polish ideal space. Suppose that $\mathcal{A}$ is a $B_T(I)$-summable family of closed countable subsets of T such that $(\exists\alpha<\omega_1)(\forall A\in\mathcal{A})(A^{(\alpha)}=\emptyset)$. Then $\bigcup\mathcal{A}\in I$.*

PROOF. Let $C=\bigcup\mathcal{A}$. We put $\mathcal{A}^{(\beta)}=\{A^{(\beta)}\colon A\in\mathcal{A}\}$ for $\beta\le\alpha$. By induction on $\beta\le\alpha$, using Lemma 4.3, we prove that $C\setminus\bigcup\mathcal{A}^{(\beta)}\in I$ and that $\mathcal{A}^{(\beta)}$ is a $B_T(I)$-summable family of sets. But $\mathcal{A}^{(\alpha)}=\emptyset$, hence $C\in I$. $\square$

COROLLARY 4.5. *Let (T,I) be a Polish ideal space. Suppose that $\mathcal{A}$ is a $B_T(I)$-summable family of closed countable subsets of T. Moreover, suppose that $cov_H(I)>\omega_1$. Then $\bigcup\mathcal{A}\in I$.*

PROOF. For each $\alpha<\omega_1$ we put $\mathcal{A}_\alpha=\{A\in\mathcal{A}\colon A^{(\alpha)}=\emptyset\}$. Then $\mathcal{A}_\alpha$ is a $B_T(I)$-summable family of sets and, thus, Theorem 4.4 implies that $\bigcup\mathcal{A}_\alpha\in I$ for each $\alpha<\omega_1$. Since $\bigcup\mathcal{A}=\bigcup_{\alpha<\omega_1}\bigcup\mathcal{A}_\alpha$, $\bigcup\mathcal{A}\in B_T(I)$ and $cov_H(I)>\omega_1$, we deduce that $\bigcup\mathcal{A}\in I$. $\square$

5. Translations of Cantor set

If A and B are subsets of a group $(G,+)$, then by $A+B$ we denote the algebraic sum $\{a+b\colon a\in A\wedge b\in B\}$. The standard ternary Cantor subset of the interval $[0,1]$ is the set

$$\mathbf{C}=\{\sum_{i\in\mathbb{N}}\frac{a_i}{3^i}\colon a\in\{0,2\}^{\mathbb{N}}\}.$$

It is well known that $\mathbf{C}+\mathbf{C}=[0,2]$. G. Gruenhage showed that for each $X\subseteq\mathbb{R}$, if $|X|<2^{\aleph_0}$, then $\mathbf{C}+X\neq\mathbb{R}$. A generalization of this result was proved in [**8**]. An easy modification of the proof from [**8**] shows, that if $X\subseteq\mathbb{R}$, $|X|<2^{\aleph_0}$ and $A\subseteq\mathbb{R}$ is a set of positive Lebesgue measure, then $A\setminus(\mathbf{C}+X)\neq\emptyset$.

DEFINITION 5.1. Let $\mathcal{S}$ be a field of subsets of a group $(G,+)$. We say that a set $B\subset G$ is a **Gruenhage set** for $\mathcal{S}$ if $B\in\mathcal{S}^-$ and

$$(\forall A\in\mathcal{S}^+)(\forall X\subset G)(|X|<|G|\rightarrow A\setminus(B+X)\neq\emptyset).$$

If I is an ideal of subsets of a group $(G,+)$, then we say that I is **invariant** if for each $A\in I$ and $x\in G$ the set $A+x$ also belongs to I. For a set $A\subseteq G$ we put $-A=\{-a\colon a\in A\}$.

THEOREM 5.2. *Suppose that I is an invariant ideal with a Borel base of subsets of an abelian Polish group $(G,+)$. Let $C\subseteq G$ be such that $C\cup -C$ is a Gruenhage set for $B_G(I)$. Then there exists a set $P\subseteq G$ such that the algebraic sum $C+P$ is a $B_G(I)$-Bernstein set.*

Proof. Since I is an invariant ideal, we may assume that the neutral element of the group $(G,+)$ belongs to the set C. Let

$$R = \{(x,y) \in G^2 \colon y \in C + x \vee x \in C + y\}.$$

Then $R[x] = (C \cup (-C)) + x$ and R is reflexive and symmetric, so the condition (2.1) of the Lemma 2.6 holds. Therefore there exists a set P such that P and $R[P]$ are $B_G(I)$-Bernstein sets. Note that $P \subseteq P + C \subseteq R[P]$, so by Lemma 2.5, the set $C + P$ is also a $B_G(I)$-Bernstein set. □

For $a \in \{0,1,2\}^{\mathbb{N}}$, we put

$$s(a) = \sum_{n \in \mathbb{N}} \frac{a_n}{3^n}.$$

Let Γ be the set of all those $s(a)$ where $a \in \{0,1,2\}^{\mathbb{N}}$ and a has only zeros from some point on. Of course, Γ is a countable dense subset of $[0,1)$.

Let us consider the group $S = ([0,1), \oplus)$, where $\oplus$ stands for the addition modulo 1. For each $x \in [0,1)$ we fix a sequence $a^x \in \{0,1,2\}^{\mathbb{N}}$ such that $x = s(a^x)$.

Definition 5.3. Let $\sigma, \eta \in \{0,1,2\}^2$. For each $x \in [0,1)$ we define

$$T_{\sigma,\eta}(x) = s(b),$$

where

$$(b_{2k-1}, b_{2k}) = \begin{cases} \sigma & \text{if } (a^x_{2k-1}, a^x_{2k}) = \eta, \\ \eta & \text{if } (a^x_{2k-1}, a^x_{2k}) = \sigma, \\ (a^x_{2k-1}, a^x_{2k}) & \text{otherwise.} \end{cases}$$

For $\sigma \in \{0,1,2\}^2$, σ_1 denotes the first element of σ and σ_2 denotes the second element of σ, i.e. $\sigma = (\sigma_1, \sigma_2)$. The first three lemmas have standard and immediate proofs. We omit these proofs.

Lemma 5.4. *Suppose that A is a Lebesgue measurable subset of $[0,1)$ and $\sigma, \eta \in \{0,1,2\}^2$. Then $\lambda(T_{\sigma,\eta}(A)) = \lambda(A)$.*

Lemma 5.5. *Suppose that $E \subseteq [0,1)$, D is a dense subset of $[0,1)$ and $E \oplus D \subseteq E$. Then $\lambda(E) = 0$ or $\lambda^*(E) = 1$.*

Definition 5.6. Let U be an ultrafilter of subsets of $\mathbb{N}$ and let $\sigma \in \{0,1,2\}^2$. Then

$$A^U_\sigma = \{s(a) \colon a \in \{0,1,2\}^{\mathbb{N}} \wedge \{k \in \mathbb{N} \colon a_{2k-1} = \sigma_1 \wedge a_{2k} = \sigma_2\} \in U\}.$$

Lemma 5.7. *Suppose that U is a non-principal ultrafilter of subsets of $\mathbb{N}$ and that $\sigma \in \{0,1,2\}^2$. Then $\Gamma \oplus A^U_\sigma \subseteq A^U_\sigma$.*

Lemma 5.8. *Let U be a non-principal ultrafilter of subsets of $\mathbb{N}$ and let $\sigma \in \{0,1,2\}^2$. Then the set A^U_σ is nonmeasurable and $\lambda^*(A^U_\sigma) = 1$.*

Proof. The maximality of the filter U implies that

$$[0,1) = \bigcup \{A^U_\sigma \colon \sigma \in \{0,1,2\}^2\}.$$

Moreover, if $\sigma, \eta \in \{0,1,2\}^2$ and $\sigma \neq \eta$, then $A^U_\sigma \cap A^U_\eta \subseteq \Gamma$ (where the only case when $A^U_\sigma \cap A^U_\eta \neq \emptyset$ is $\sigma = (0,0)$ and $\eta = (2,2)$ or conversely) and $|T_{\sigma,\eta}(A^U_\sigma) \triangle A^U_\eta| \leq \aleph_0$ (where $T_{\sigma,\eta}(A^U_\sigma) \neq A^U_\eta$ may happen only when $\sigma = (0,0)$ or $\sigma = (2,2)$). Therefore if there exists σ such that A^U_σ is measurable, then, by Lemma 5.4, for each $\eta \in \{0,1,2\}^2$ the set A^U_η is measurable. But this, by Lemmas 5.5 and 5.7, implies that $\lambda(A^U_\eta) = 0$ for each $\eta \in \{0,1,2\}^2$, which is impossible. $\square$

Note that in the group S the role of the standard ternary Cantor set $\mathbf{C}$ is played by the set $\mathbb{C} = \mathbf{C} \setminus \{1\}$. The inspiration for our proof of the next theorem was Sierpiński's classical result about nonmeasurability of ultrafilters (see [**16**]).

Theorem 5.9. *There exists a set $A \subseteq \mathbb{C}$ such that $A \oplus \mathbb{C}$ is a Lebesgue nonmeasurable subset of* $[0,1)$.

Proof. Let us recall that $\mathbb{C} = \{s(a) \colon a \in \{0,2\}^{\mathbb{N}}\} \setminus \{1\}$. Let us fix a nonprincipal ultrafilter U of subsets of the set $\mathbb{N}$. Let $A = A^U_{(0,0)} \cap \mathbb{C}$. Let $D_1 = A^U_{(0,0)}$ and

$$D_7 = \bigcup \{A^U_\sigma \colon \sigma \in \{(0,0),(0,1),(0,2),(1,0),(2,0),(2,1),(2,2)\}\}.$$

We claim that $D_1 \subseteq A \oplus \mathbb{C} \subseteq D_7$.

First we shall prove the inclusion $D_1 \subseteq A \oplus \mathbb{C}$. Thus suppose that $x \in D_1$. Let $a \in \{0,1,2\}^{\mathbb{N}}$ be a sequence such that $s(a) = x$ and $I = \{k \in \mathbb{N} \colon a_{2k-1} = a_{2k} = 0\} \in U$. Since $\mathbb{C} \oplus \mathbb{C} = [0,1)$ there are $b, c \in \{0,2\}^{\mathbb{N}}$ such that $x = s(b) \oplus s(c)$. As for $i \in I$ we have $a_{2i-1} = 0 \wedge a_{2i} = 0$, it is easy to check that for $i \in I$ we are in one of the following five situations:

(1) $(b_{2i-1}, b_{2i}) = (2,2)$ and $(c_{2i-1}, c_{2i}) = (0,0)$,
(2) $(b_{2i-1}, b_{2i}) = (2,0)$ and $(c_{2i-1}, c_{2i}) = (0,2)$,
(3) $(b_{2i-1}, b_{2i}) = (0,2)$ and $(c_{2i-1}, c_{2i}) = (2,0)$,
(4) $(b_{2i-1}, b_{2i}) = (0,0)$ and $(c_{2i-1}, c_{2i}) = (2,2)$,
(5) $(b_{2i-1}, b_{2i}) = (0,0)$ and $(c_{2i-1}, c_{2i}) = (0,0)$.

We define now two sequences $d, e \in \{0,2\}^{\mathbb{N}}$. If $i \in \mathbb{N} \setminus I$, then we put $d_{2i-1} = b_{2i-1}$, $d_{2i} = b_{2i}$, $e_{2i-1} = c_{2i-1}$ and $e_{2i} = c_{2i}$. Suppose now that $i \in I$. If (4) or (5) holds then we also put $d_{2i-1} = b_{2i-1}$, $d_{2i} = b_{2i}$, $e_{2i-1} = c_{2i-1}$ and $e_{2i} = c_{2i}$. If (1) or (2) or (3) holds then we put $d_{2i-1} = 0$, $d_{2i} = 0$, $e_{2i-1} = 2$ and $e_{2i} = 2$. Then $x = s(a) = s(d) + s(e)$ and $s(d) \in A$, $s(e) \in \mathbb{C}$ and the first inclusion is proved.

We shall show now that $A \oplus \mathbb{C} \subseteq D_7$. Let $u \in A$, $v \in \mathbb{C}$ and let $a \in \{0,1,2\}^{\mathbb{N}}$ be a sequence such that $s(a) = u \oplus v$. Since $\Gamma \subseteq D_7$, we may assume that $s(a) \notin \Gamma$. Let $b \in \{0,1,2\}^{\mathbb{N}}$ and $c \in \{0,2\}^{\mathbb{N}}$ be such sequences that $I = \{i \in \mathbb{N} \colon b_{2i-1} = 0 \wedge b_{2i} = 0\} \in U$, $u = s(b)$ and $v = s(c)$. We shall check all possible configurations of pairs (a_{2i-1}, a_{2i}) for $i \in I$.

(1) if $(c_{2i-1}, c_{2i}) = (0,0)$, then $(a_{2i-1}, a_{2i}) \in \{(0,0),(0,1)\}$,
(2) if $(c_{2i-1}, c_{2i}) = (2,2)$, then $(a_{2i-1}, a_{2i}) \in \{(2,2),(0,0)\}$,
(3) if $(c_{2i-1}, c_{2i}) = (2,0)$, then $(a_{2i-1}, a_{2i}) \in \{(2,0),(2,1)\}$,
(4) if $(c_{2i-1}, c_{2i}) = (0,2)$, then $(a_{2i-1}, a_{2i}) \in \{(0,2),(1,0)\}$.

Hence $(a_{2i-1}, a_{2i}) \in \{(0,0),(0,1),(0,2),(1,0),(2,0),(2,1),(2,2)\}$ for each $i \in I$ and, therefore, the inclusions $D_1 \subseteq A \oplus \mathbb{C} \subseteq D_7$ are proved.

From Lemmas 5.5, 5.7 and 5.8 we get $\lambda^*(D_1) = \lambda^*(D_7) = 1$. The same lemma implies that $\lambda^*(D_1^c) = \lambda^*(D_7^c) = 1$, therefore $\lambda_*(D_1) = \lambda_*(D_7) = 0$. Hence, from the inclusions $D_1 \subseteq A \oplus \mathbb{C} \subseteq D_7$ we have just proved, we infer that $\lambda^*(A \oplus \mathbb{C}) = 1$ and $\lambda_*(A \oplus \mathbb{C}) = 0$. $\square$

Corollary 5.10. *Let $\mathbf{C}$ denote the standard ternary Cantor subset of* $[0,1]$. *Then there exists a set $A \subseteq \mathbf{C}$ such that $A + \mathbf{C}$ is a nonmeasurable subset of the real line.*

PROOF. The only difference between this corollary and Theorem 5.9 is that in Theorem 5.9 we considered subsets of $([0,1),\oplus)$ instead of those of the real line. Note that

$$A \oplus \mathbb{C} = ((A + \mathbf{C}) \cap [0,1)) \cup ((A + \mathbf{C}) \cap [1,2) + \{-1\})$$

hence measurability of $A + \mathbf{C}$ implies measurability of $A \oplus \mathbb{C}$. □

REMARK 5.11. A similar result may be proved for the Baire property. Namely, for the same set A which we constructed in the proof of Theorem 5.9 one can prove that $A + \mathbf{C}$ does not have the Baire property.

Conjecture. Suppose that P is a closed subset of the real line $\mathbb{R}$ such that $\lambda(P) = 0$ and $\lambda(P + P) > 0$. Then there exists a set $A \subseteq P$ such that $A + P$ is a Lebesgue nonmeasurable set.

Theorem 3.1 implies that this conjecture is consistent with ZFC. It is, of course, quite easy to formulate far-reaching generalizations of this conjecture.

6. The ideal of meager sets

Let us recall that an ideal I of subsets of a set X is a c.c.c. ideal if for every family $\mathcal{A} \subseteq P(X) \setminus I$ such that $(\forall A, B \in \mathcal{A})(A = B \vee A \cap B \in I)$ we have $|\mathcal{A}| \leq \aleph_0$. We also say that an ideal I is **μ-additive** if $add(I) \geq \mu$.

DEFINITION 6.1. Let κ, λ, μ, ν be cardinal numbers. The relation $(\kappa\colon \lambda, \mu) \to \nu$ holds if for every family $\mathcal{R}$ of μ-additive ideals on κ such that $|\mathcal{R}| = \lambda$ there exists a family $\{X_\alpha\}_{\alpha<\nu}$ such that

(1) $(\forall \alpha < \nu)(X_\alpha \in P(\kappa) \setminus \bigcup \mathcal{R})$,
(2) $(\forall \alpha < \beta < \nu)(X_\alpha \cap X_\beta \in \bigcap \mathcal{R})$.

THEOREM 6.2 (Alaoglu, Erdös). $(\forall \kappa)\,(((\kappa\colon \omega, \omega_1) \to \omega_1) \leftrightarrow ((\kappa\colon 1, \omega_1) \to \omega_1))$.

We will need the following version of Theorem 6.2, which follows easily from the proof of Theorem 6.2 from [**17**]:

LEMMA 6.3. *Assume that $\{I_n\}_{n\in\omega}$ is a family of σ-additive ideals on κ which are not c.c.c. Then there exists a family $\{X_\alpha\}_{\alpha<\omega_1} \subseteq P(\kappa)$ such that*

(1) $(\forall \alpha < \omega_1)(\forall n \in \omega)(X_\alpha \notin I_n)$
(2) $(\forall \alpha, \beta < \omega_1)(\alpha \neq \beta \to X_\alpha \cap X_\beta = \emptyset)$.

Let T be an uncountable Polish topological space. In this section the ideal $\mathbb{K}_T$ is denoted by $\mathcal{K}$. The family of all open subsets of T is denoted by $\mathcal{O}_T$. For two subsets $A, B \subseteq T$ we write $A \subseteq^* B$ if $A \setminus B \in \mathcal{K}$ and $A =^* B$ if $A \subseteq^* B$ and $B \subseteq^* A$.

DEFINITION 6.4. Let $N \subseteq X \subseteq T$. We say that the set N is **completely non-Baire** in X if

$$(\forall A \in B_T)(A \cap X \notin \mathcal{K} \to (A \cap N \notin \mathcal{K} \,\wedge\, A \cap (X \setminus N) \notin \mathcal{K})).$$

Notice, that a set N is completely non-Baire in X if for every open base $\mathcal{S}$ of the topological space T we have

$$(\forall U \in \mathcal{S})(U \cap X \notin \mathcal{K} \to (U \cap N \notin \mathcal{K} \,\wedge\, U \cap (X \setminus N) \notin \mathcal{K})).$$

LEMMA 6.5. *Suppose that for every family $\mathcal{Q} \subseteq \mathcal{K}$ of pairwise disjoint sets such that $\bigcup \mathcal{Q} \notin \mathcal{K}$ there exists a subfamily $\mathcal{Q}' \subseteq \mathcal{Q}$ such that $\bigcup \mathcal{Q}' \notin \mathcal{K}$ and*

$$(\forall U \in \mathcal{O}_T)(U \cap \bigcup \mathcal{Q} \notin \mathcal{K} \to U \cap \bigcup \mathcal{Q} \not\subseteq^* \bigcup \mathcal{Q}').$$

Then for every family $\mathcal{P} \subseteq \mathcal{K}$ of pairwise disjoint sets such that $\bigcup \mathcal{P} \notin \mathcal{K}$ there exists a subfamily $\mathcal{P}' \subseteq \mathcal{P}$ such that the union $\bigcup \mathcal{P}'$ is completely non-Baire in $\bigcup \mathcal{P}$.

PROOF. Let $\mathcal{P} = \{P_\alpha\}_{\alpha<\kappa} \subseteq \mathcal{K}$ be a family of pairwise disjoint sets such that $\bigcup_{\alpha<\kappa} P_\alpha \notin \mathcal{K}$. Let C be a minimal open set with respect to the relation $\subseteq^*$ such that $\bigcup_{\alpha<\kappa} P_\alpha \subseteq^* C$. For each nonempty open set $U \subseteq C$ we put

$$I_U = \{X \subseteq \kappa \colon \bigcup_{\alpha \in X} P_\alpha \cap U \in \mathcal{K}\}.$$

Then I_U is a proper σ-ideal. Notice that if $\emptyset \neq V \subseteq U \subseteq C$ are open sets and the ideal I_U satisfies c.c.c then the ideal I_V is a c.c.c. ideal, too. Moreover, if $\{U_n \colon n \in \omega\}$ is a family of open nonempty subsets of C and for each $n \in \omega$ the ideal I_{U_n} is a c.c.c. ideal, then $I_{\bigcup_n U_n}$ is a c.c.c. ideal, too.

Let $\mathcal{S}$ be a maximal family of pairwise disjoint nonempty open subsets of C such that I_U is a c.c.c ideal for each $U \in \mathcal{S}$. Let $A = \bigcup \mathcal{S}$ and let B be an open set such that $A \cup B =^* C$ and $A \cap B = \emptyset$. Then I_A is a c.c.c. ideal and I_U is not a c.c.c ideal for each nonempty open set $U \subseteq B$.

Suppose that $B \neq \emptyset$. Let $\{U_n\}_{n\in\omega}$ be a countable base of open subsets of B. We put $I_n = I_{U_n}$. By assumption, I_n is not a c.c.c. ideal. By Lemma 6.3 we get a disjoint family $\{X_\beta\}_{\beta<\omega_1}$ of subsets of κ such that for every $\beta < \omega_1$ the set $(\bigcup_{\alpha\in X_\beta} P_\alpha) \cap B$ is completely non-Baire in $(\bigcup_{\alpha<\kappa} P_\alpha) \cap B$. Since I_A is a c.c.c. ideal, we can find $\beta_0 < \beta_1 < \omega_1$ such that $(\bigcup_{\alpha\in X_{\beta_0}} P_\alpha) \cap A \in \mathcal{K}$ and $(\bigcup_{\alpha\in X_{\beta_1}} P_\alpha) \cap A \in \mathcal{K}$.

If $B = \emptyset$ then we put $X_{\beta_0} = X_{\beta_1} = \emptyset$.

Notice that if $A = \emptyset$ then the set $\bigcup\{P_\alpha \colon \alpha \in X_{\beta_0}\}$ is a completely non-Baire in $\bigcup_{\alpha<\kappa} P_\alpha$. Hence we may assume that $A \neq \emptyset$.

We define now by transfinite recursion on ordinal number ξ some sequence $(Y_\xi)_\xi$ of pairwise disjoint subsets of κ. Suppose we have already defined sets $(Y_\zeta)_{\zeta<\xi}$. Let us put $T_\xi = \kappa \setminus (X_{\beta_0} \cup X_{\beta_1} \cup \bigcup_{\zeta<\xi} Y_\zeta)$. If $(\bigcup_{\alpha\in T_\xi} P_\alpha) \cap A \in \mathcal{K}$ then we terminate our construction. If $(\bigcup_{\alpha\in T_\xi} P_\alpha) \cap A \notin \mathcal{K}$ then we choose a set Y_ξ with the following three properties:

(1) $Y_\xi \subseteq T_\xi$,
(2) $\bigcup_{\alpha\in Y_\xi} P_\alpha \cap A \notin \mathcal{K}$,
(3) $(\forall U \in \mathcal{O}_T)(U \cap \bigcup_{\alpha\in T_\xi} P_\alpha \cap A \notin \mathcal{K} \to U \cap \bigcup_{\alpha\in T_\xi} P_\alpha \cap A \not\subseteq^* \bigcup_{\alpha\in Y_\xi} P_\alpha)$.

Since I_A is a c.c.c. ideal, our construction must break after some $\lambda < \omega_1$ steps. For each nonempty open subset U of A we define

$$G_U = \{\xi < \lambda \colon (\bigcup_{\alpha\in Y_\xi} P_\alpha) \cap U \notin \mathcal{K}\}.$$

Notice that if $\emptyset \neq U \subseteq A$ then $U \cap \bigcup_{\alpha<\lambda} \bigcup_{\alpha\in Y_\xi} P_\alpha \notin \mathcal{K}$, so $G_U \neq \emptyset$. We claim that G_U is infinite. Suppose otherwise and let $G_U = \{\alpha_0, \ldots, \alpha_n\}$, $\alpha_0 < \alpha_1 < \ldots < \alpha_n$. Then

$$U \cap \bigcup \mathcal{P} =^* U \cap \bigcup_{\xi\leq\alpha_n} \bigcup_{\alpha\in Y_\xi} P_\alpha,$$

whence

$$U \cap (\bigcup \mathcal{P} \setminus \bigcup_{\xi<\alpha_n} \bigcup_{\alpha \in Y_\xi} P_\alpha) \subseteq^* \bigcup_{\alpha \in Y_{\alpha_n}} P_\alpha,$$

and thus

$$U \cap \bigcup_{\alpha \in T_{\alpha_n}} P_\alpha \subseteq^* \bigcup_{\alpha \in Y_{\alpha_n}} P_\alpha,$$

which contradicts the choice of Y_{α_n}.

Let us fix $\{V_n\}_{n\in\omega}$ a countable base of open subsets of A. Then for every $n \in \omega$ the set G_{V_n} is infinite. So we can find an injection $\varphi\colon \omega \to \lambda$ such that

$$(\forall n \in \omega)(\bigcup\{P_\alpha \colon \alpha \in Y_{\varphi(2n)}\} \cap V_n \notin \mathcal{K} \wedge \bigcup\{P_\alpha \colon \alpha \in Y_{\varphi(2n+1)}\} \cap V_n \notin \mathcal{K}).$$

Let $X = X_{\beta_0} \cup \bigcup\{Y_{\varphi(2n)} \colon n \in \omega\}$. Then the set $\bigcup_{\alpha\in X} P_\alpha$ is completely non-Baire in $\bigcup_{\alpha<\kappa} P_\alpha$. □

The Cohen Boolean algebra is the complete Boolean algebra $B_{\mathbb{R}}/\mathcal{K}_{\mathbb{R}}$.

Lemma 6.6. *Assume that there exists a pairwise disjoint family $\{P_\alpha\}_{\alpha<\kappa} \subseteq \mathcal{K}$ such that $\bigcup_{\alpha<\kappa} P_\alpha \notin \mathcal{K}$ and such that for every set $X \subseteq \kappa$ either $\bigcup_{\alpha\in X} P_\alpha \in \mathcal{K}$ or there exists an open set U such that $U \cap \bigcup_{\alpha<\kappa} P_\alpha \notin \mathcal{K}$ and $U \cap \bigcup_{\alpha<\kappa} P_\alpha \subseteq^* \bigcup_{\alpha\in X} P_\alpha$. Let $I = \{X \subseteq \kappa \colon \bigcup_{\alpha\in X} P_\alpha \in \mathcal{K}\}$. Then the Boolean algebra $P(\kappa)/I$ is isomorphic to the Cohen algebra.*

Proof. Let C be an open set, minimal with respect to the relation $\subseteq^*$, such that $\bigcup_{\alpha<\kappa} P_\alpha \subseteq^* C$. Let $\{U_n\}_{n\in\omega}$ be a base of open subsets of C and let us put

$$\mathcal{B}_n = \{[X] \in P(\kappa)/I \colon U_n \cap \bigcup_{\alpha<\kappa} P_\alpha \subseteq^* \bigcup_{\alpha\in X} P_\alpha\}.$$

Then $P(\kappa)/I \setminus \{0\} = \bigcup_n \mathcal{B}_n$. Hence $P(\kappa)/I$ is a c.c.c. Boolean algebra. Therefore for each $n \in \omega$ there exists a family $\{X^n_k \colon k \in \omega\}$ such that

$$\prod \mathcal{B}_n = \prod\{[X^n_k] \colon k \in \omega\} = [\bigcap\{X^n_k \colon k \in \omega\}].$$

Let $Y_n = \bigcap_k X^n_k$. Since $\mathcal{K}$ is a σ-ideal we have $[Y_n] \in \mathcal{B}_n$. Thus $\{[Y_n] \colon n \in \omega\}$ is a countable dense subset of the algebra $P(\kappa)/I$.

Finally let us observe that the Boolean algebra $P(\kappa)/I$ has no atom. Namely, assume that $[X]$ is an atom. Then for every $Y \subseteq X$ either $[Y]$ is zero or $[Y] = [X]$. Since $\mathcal{K}$ is σ-additive we would obtain a σ-complete ultrafilter on $|X|$. But $|X| \le 2^\omega$, which is impossible. □

We will need the following theorem from [**12**]:

Theorem 6.7 (Gitik, Shelah). *If I is a σ-ideal on κ, then $P(\kappa)/I$ is not isomorphic to the Cohen algebra.*

Theorem 6.8. *Let $\mathcal{P}$ be a pairwise disjoint family of meager sets such that $\bigcup \mathcal{P} \notin \mathcal{K}$. Then there exists a subfamily $\mathcal{P}' \subseteq \mathcal{P}$ such that $\bigcup \mathcal{P}'$ is completely non-Baire in $\bigcup \mathcal{P}$.*

Proof. If the hypothesis of Lemma 6.5 is satisfied, then its conclusion gives us the conclusion of Theorem. If the hypothesis of Lemma 6.5 is not satisfied, then by Lemma 6.6, we find a σ-ideal I on the cardinal number $|\mathcal{P}|$ such that $P(|\mathcal{P}|)/I$ is isomorphic to the Cohen algebra, which contradicts Theorem 6.7. □

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