Applied stochastics

with applications for security and cryptography

Problem set 2

1. Let $(X_0, X_1, \ldots)$ be an irreducible Markov chain over a state space $S = \{s_1, \ldots, s_t\}$ with transition a matrix P. Show that if there exists a state s_i such that $P_{ii} > 0$ then the chain is also aperiodic.

2. A well known know football player has a large house with two parking lots on each side of the house. He has 6 cars. Each day he leaves his house to buy bread in the morning. He goes to a randomly selected parking lot (equally likely) and drives a randomly selected car (if there is any left on the lot). When he comes back, he arrives at randomly selected parking lot and parks a car there.

   He gots angry (and remains angry for the whole day) when there is no car on the parking lot when he leaves.

   (a) find a Markov chain (state space, transition matrix) for the problem.

   (b) what is the (long-term) proportion of days that he is angry – check it for different initial distributions.

3. Consider the settings from Problem 2 with the following changes:

   (a) each parking lot has only 6 parking spaces,

   (b) he has $7/8/9/\ldots$ cars now,

   (c) he also gets angry when he has no place to park when he arrives.

4. A Markov chain can help modeling a spread of a virus. Assume that each day every person is either infected or susceptible. Assume that for each pair (i, j) of people they contact independently with probability p. If an infected person contacts a susceptible person, a susceptible person gets infected with probability q. Each person who gets infected sees symptoms after $d = 7$ days. Each person with symptoms stays at home until getting recovered (and then it becomes susceptible again). Probability that a given person recovers overnight is r. Let n be the size of the population.

   (a) Let m be the number of infected individuals. What is the distribution of new infections after one day?

   (b) Draw a transition graph that models the spread of a virus for $n = 2, 3, 4, \ldots$.

   (c) How the setting would change when another state is introduced: immune. A person after is recovered gets immunity: cannot be infected and does not infect.

   (d) Simulate the process for different parameters p, q, r.