Modeling

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Literature

[1] H.P. Williams, *Model building in mathematical programming*, John Wiley and Sons, 1993.

[2] F. Plastria, Formulating logical implications in combinatorial optimisation *European Journal of Operational Research* 140 (2002) 338-353.

A large part of the lecture has been prepared on the basis of the book [1].

http://www.im.pwr.wroc.pl/ ˜pziel/lectures/toulouse/html

A simple production planning problem

Example: A store has requested a manufacturer to produce pants and sports jackets. The manufacturer has 750 m^2 of cotton textile and 1,000m^2 of polyester. Every pair of pants (1 unit) needs 1 m^2 of cotton and 2 m^2 of polyester. Every jacket needs 1.5 m^2 of cotton and 1 m^2 of polyester. The price of the pants is fixed at $50 and the jacket, $40. What is the number of pants and jackets that the manufacturer must give to the stores so that these items obtain a maximum sale?

$$50x_1 + 40x_2 \to \max$$

$$\begin{aligned}
&x_1 + 1.5x_2 \leq 750 && \text{(cotton)} \\
&2x_1 + x_2 \leq 1000 && \text{(polyester)} \\
&x_1, x_2 \geq 0.
\end{aligned}$$

Mix problem

Example: A drug company produces a drug from two ingredients. Each ingredient contains the same three antibiotics in different proportions. One gram of ingredient 1 contributes 3 units, and ingredient 2 contributes 1 unit of antibiotic 1; the drug requires 6 units. At least 4 units of antibiotic 2 are required, and the ingredients each contribute 1 unit per gram. At least 12 units of antibiotic 3 are required; a gram of ingredient 1 contributes 2 units, and a gram of ingredient 2 contributes 6 units. The cost for a gram of ingredient 1 is \$80 and the cost for a gram of ingredient 2 is \$50. The company wants to determine the number of grams of each ingredient that must go into the drug in order to meet the antibiotic requirements at minimum cost.

Mix problem

$$\begin{array}{rcrcl} 80x_1 & + & 50x_2 & \longrightarrow & \min \ \text{(cost, \$)} \\ 3x_1 & + & x_2 & \geq 6 & \text{(antibiotic 1)} \\ x_1 & + & x_2 & \geq 4 & \text{(antibiotic 2)} \\ 2x_1 & + & 6x_2 & \geq 12 & \text{(antibiotic 3)} \\ & & x_1, x_2 & \geq 0 & \end{array}$$

Building integer programming models

In mathematical programming models, integer variables are used for different purposes:

- to model quantities that are integer in their nature, for instance: *the number of cars (aircrafts) produced, the number of employees, etc.*,
- to model logical conditions:
  if a new product is developed, then a new plant must be constructed,
- to model nonlinear dependences: for instance *fixed costs for building a warehouse*,
- to express certain states of continuous variables in linear programming models.
- ⋮

Binary variables - 0-1 variables

Suppose, we want to model activities:

- to build a plant,
- to undertake an advertising campaign,
- to develop a new product.

In each above case, we have to make YES-NO, GO-NO-GO decision. We introduce a binary variable x_j:

$$x_j = \begin{cases} 1 & \text{if the } j\text{-th decision is made,} \\ 0 & \text{otherwise} \end{cases}$$

Suppose that at most one of the above three activities can be performed:

$$x_1 + x_2 + x_3 \leq 3.$$

Integer variables

However, in some situation, variables may take different integer values:

$$\gamma = \begin{cases} 0 & \text{no warehouse is built} \\ 1 & \text{a warehouse of type A is built} \\ 2 & \text{a warehouse of type B is built} \end{cases}$$

Indicator variables

To express certain states of continuous variables Indicator variables are used.
Let δ be binary variable that helps to distinguish between two states of continuous variable x - the state, when $x = 0$ and state, when $x > 0$.
We introduce the following constraint that enforces: $\delta = 1$, when $x > 0$

$$x - M\delta \leq 0, \tag{1}$$

where M is an upper bound on values of x
Constraints (1) models the following implication:

$$x > 0 \Rightarrow \delta = 1. \tag{2}$$

Indicator variables

The opposite implication

$$x = 0 \Rightarrow \delta = 0 \tag{3}$$

or its equivalent form:

$$\delta = 1 \Rightarrow x > 0 \tag{4}$$

can not be expressed by a constraint. A slightly modified form implication can be applied

$$\delta = 1 \Rightarrow x > m, \tag{5}$$

where m is the minimal threshold value such that: if $x < m$ then value of x can be regarded as a zero. Thus, (5) can be expressed:

$$x - m\delta \geq 0. \tag{6}$$

Indicator variables

A problem with fixed costs: Let x be the amount of product produced. C_1 is unit cost of producing the product, C_2 are fixed costs of production. The total cost (TC) is equal to

$$TC(x) = \begin{cases} 0 & \text{if } x = 0. \\ C_1 x + C_2 & \text{if } x > 0. \end{cases}$$

The TC is not linear function.
To linearize TC, we introduce indicator variable δ such that $x > 0 \Rightarrow \delta = 1$, in consequence the constraint $x - M\delta \leq 0$, and we get

$$TC(x) = C_1 x + C_2\delta.$$

In this case, we do need introduce the implication $x = 0 \Rightarrow \delta = 0$, since it holds in an optimal solution (the minimization of objective function $TC(x)$).

Indicator variables

A mix problem: Let x_A i x_B be the variables that represent the percentage of components A and B in a mixture, respectively. Additionally, apart from other constraints in the problem that can be expressed in linear form, there is the following constraint:
"If the mixture contains component A then component B must be contained in the mixture'.
We introduce indicator variable δ such that: $x_A > 0 \Rightarrow \delta = 1$, i.e. the constraint

$$x_A - \delta \leq 0. \tag{7}$$

Here $M = 1$, since $x_A \leq 1$. Furthermore, we need to introduce the constraint

$$\delta = 1 \Rightarrow x_B > 0,$$

which can be modeled

$$x_B - 0.01\delta \geq 0, \tag{8}$$

where m is the threshold value (here $m = 0.01$). If the value of x_B is below m then it is assumed that component B is not present in the mixture.

Constraint feasibility "$\leq$"

Checking if a given constraint is satisfied. Consider the constraint:

$$\sum_j a_j x_j \leq b.$$

The implication

$$\delta = 1 \Rightarrow \sum_j a_j x_j \leq b$$

can be represented by the constraint:

$$\sum_j a_j x_j + M\delta \leq M + b,$$

where M is an upper bound on $\sum_j a_j x_j - b$.

Constraint feasibility "$\leq$"

The opposite implication

$$\sum_j a_j x_j \leq b \Rightarrow \delta = 1,$$

which can be expressed in the form

$$\delta = 0 \Rightarrow \sum_j a_j x_j > b \tag{9}$$

is modeled as follows: inequality

$$\sum_j a_j x_j > b$$

we rewrite it (as in (5))

$$\sum_j a_j x_j \geq b + \epsilon.$$

Constraint feasibility "$\leq$"

Thus, implication (9) ($\delta = 0 \Rightarrow \sum_j a_j x_j > b$) is written

$$\delta = 0 \Rightarrow -\sum_j a_j x_j + b + \epsilon \leq 0. \tag{10}$$

Now, the condition (10) is represented by the constraint

$$\sum_j a_j x_j - (m - \epsilon)\delta \geq b + \epsilon,$$

where m is an lower bound on values of $\sum_j a_j x_j - b$. ϵ is a small positive value. Exceeding it makes the constraint unsatisfied.

Constraint feasibility "≥"

Checking if a given constraint with '≥" is satisfied. Consider the constraint:

$$\sum_j a_j x_j \geq b$$

We associate a indicator variable δ with the above constraint (δ indicates if the constraint is satisfied or not satisfied). Hence

$$\begin{aligned} \sum_j a_j x_j + m\delta &\geq m + b \\ \sum_j a_j x_j - (M + \epsilon)\delta &\leq b - \epsilon, \end{aligned}$$

where m i M are, respectively, lower and upper bounds on $\sum_j a_j x_j - b$.

Constraint feasibility "="

Checking if a given constraint with '=" is satisfied. Consider the constraint:

$$\sum_j a_j x_j = b$$

We associate a indicator variable δ with the above constraint (δ indicates if the constraint is satisfied or not satisfied).

$$\begin{aligned}
\sum_j a_j x_j + M\delta &\leq M + b, \\
\sum_j a_j x_j + m\delta &\geq m + b, \\
\sum_j a_j x_j - (m - \epsilon)\delta^{'} &\geq b + \epsilon, \\
\sum_j a_j x_j - (M + \epsilon)\delta^{''} &\leq b - \epsilon,
\end{aligned}$$

$$\delta^{'} + \delta^{''} - \delta \leq 1.$$

Constraint feasibility

Example: We are given the inequality

$$2x_1 + 3x_2 \leq 1,$$

where x_1, x_2 are integer numbers not greater than 1. In order to indicate that the constraint is satisfied, we need to introduce the conditions:

$$\delta = 1 \Rightarrow 2x_1 + 3x_2 \leq 1,$$
$$2x_1 + 3x_2 \leq 1 \Rightarrow \delta = 1.$$

Setting $M = 4$, $m = -1$ i $\epsilon = 0.1$, we get the following constraints represented the conditions

$$\begin{aligned} 2x_1 + 3x_2 + 4\delta &\leq 5 \\ 2x_1 + 3x_2 + 1.1\delta &\geq 1.1 \end{aligned}$$

Logical constraints

Let X_i be the proposition

Component i is in the mixture,

where $i \in \{A, B, C\}$, then

$$X_A \Rightarrow (X_B \vee X_C)$$

means the proposition

If component A is in the mixture, then B or C or both are in the mixture

We write the above proposition as

$$(X_A \Rightarrow X_B) \vee (X_A \Rightarrow X_C)$$

Logical constraints

Recalling the known facts:

$$
\begin{aligned}
\sim\sim P &\equiv P, \\
P \Rightarrow Q &\equiv \sim P \vee Q, \\
P \Rightarrow Q \wedge R &\equiv (P \Rightarrow Q) \wedge (P \Rightarrow R), \\
P \Rightarrow Q \vee R &\equiv (P \Rightarrow Q) \vee (P \Rightarrow R), \\
P \wedge Q \Rightarrow R &\equiv (P \Rightarrow R) \vee (Q \Rightarrow R), \\
P \vee Q \Rightarrow R &\equiv (P \Rightarrow R) \wedge (Q \Rightarrow R), \\
\sim (P \vee Q) &\equiv \sim P \wedge \sim Q, \\
\sim (P \wedge Q) &\equiv \sim P \vee \sim Q.
\end{aligned}
$$

Logical constraints

Let X_i means the proposition "$\delta_i = 1$", whereδ_i is indicator variable. Then, we have the following equivalent conditions:

$$\begin{aligned} X_1 \vee X_2 &\equiv \delta_1 + \delta_2 \geq 1, \\ X_1 \wedge X_2 &\equiv \delta_1 = 1, \delta_2 = 1, \\ \sim X_1 &\equiv \delta_1 = 0 (\text{or } 1 - \delta_1 = 1), \\ X_1 \Rightarrow X_2 &\equiv \delta_1 - \delta_2 \leq 0, \\ X_1 \Leftrightarrow X_2 &\equiv \delta_1 - \delta_2 = 0. \end{aligned}$$

Logical constraints

Example: If products A or B (both) are produced, then at least one product from products C, D or E will have to be produced. Let X_i means the proposition:

$$\text{Product } i \text{ is produced}, \; i \in \{A, B, C, D, E\}$$

The following condition is included to a model:

$$(X_A \vee X_B) \Rightarrow (X_C \vee X_D \vee X_E).$$

Let δ_i be the indicator variable such that:

$$\delta_i = 1 \Leftrightarrow \text{the } i\text{-th product is produced}$$

and

$$\delta = 1 \text{ if the proposition } X_A \vee X_B \text{ is true.}$$

Logical constraints

Proposition $X_A \vee X_B$ is represented by the following inequality

$$\delta_A + \delta_B \geq 1,$$

and proposition $X_C \vee X_D \vee X_E$ by the following inequality

$$\delta_C + \delta_D + \delta_E \geq 1,$$

We write the condition:

$$\delta_A + \delta_B \geq 1 \Rightarrow \delta = 1,$$

which is enforced by the constraint

$$\delta_A + \delta_B - 2\delta \leq 0.$$

And the condition

$$\delta = 1 \Rightarrow \delta_C + \delta_D + \delta_E \geq 1,$$

which is enforced by the constraint

$$-\delta_C - \delta_D - \delta_E + \delta \leq 0.$$

Logical constraints

Implication $(X_A \vee X_B) \Rightarrow (X_C \vee X_D \vee X_E)$ can be replaced

$$(X_A \Rightarrow (X_C \vee X_D \vee X_E) \wedge (X_B \Rightarrow (X_C \vee X_D \vee X_E)$$

and can be expressed by the following system of inequalities:

$$\begin{aligned} -\delta_C - \delta_D - \delta_E + \delta &\leq 0 \\ \delta_A - \delta &\leq 0 \\ \delta_B - \delta &\leq 0. \end{aligned}$$

Both ways of modeling are correct.

The product of binary variables

If there is the product of two binary variables $\delta_1\delta_2$ in a model, then we can linearize it in the following way:

- we replace $\delta_1\delta_2$ with binary variable δ_3,
- we enforce the logical condition
$$\delta_3 = 1 \Leftrightarrow (\delta_1 = 1) \wedge (\delta_2 = 1)$$
by adding the following constraints:
$$\begin{aligned} -\delta_1 + \delta_3 &\leq 0 \\ -\delta_2 + \delta_3 &\leq 0 \\ \delta_1 + \delta_2 - \delta_3 &\leq 1. \end{aligned}$$

Constraint $\delta_1\delta_2 = 0$ represents the condition:
$$\delta_1 = 0 \vee \delta_2 = 0.$$

The product of more than two binary variables can be successively reduced to the product of two binary variables.

The product of binary variables

If there is the product of continuous variable x and binary variable δ, $x\delta$, then we can linearize it in the following way:

- we replace $x\delta$ with continuous variable y,
- we enforce the logical conditions

$$
\begin{aligned}
\delta = 0 \quad &\Rightarrow \quad y = 0, \\
\delta = 1 \quad &\Rightarrow \quad y = x
\end{aligned}
$$

by including the constraints:

$$
\begin{aligned}
y - M\delta &\leq 0, \\
-x + y &\leq 0, \\
x - y + M\delta &\leq M,
\end{aligned}
$$

where M is upper bound on the values of x (and of y).

Modeling bounded set of values

Suppose x_i takes the values from the following set:

$$\{a_1, \ldots, a_m\}.$$

In order to model this situation, we introduce binary variables δ_j, $j = 1, \ldots, m$ and the constraints:

$$\begin{aligned} \sum_{j=1}^{m} a_j \delta_j &= x, \\ \sum_{j=1}^{m} \delta_j &= 1. \end{aligned}$$

Modeling bounded set of values

Example:(Building warehouse) Suppose that we wish to make decision about the size of a warehouse. Obviously, the sizes depend on costs:

size	cost
10	100
20	180
40	320
60	450
80	600

Using binary variables δ_i, we model the size and the cost of building:

$$\begin{aligned} COST &\equiv 100\delta_1 + 180\delta_2 + 320\delta_3 + 450\delta_4 + 600\delta_5 \\ SIZE &\equiv 10\delta_1 + 20\delta_2 + 40\delta_3 + 60\delta_4 + 80\delta_5. \end{aligned}$$

We include the constraint:

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = 1.$$

A piecewise linear objective function

- A piecewise linear function can be modeled by binary variables.
- A function is given by ordered pairs $(a_i, f(a_i))$. We wish to compute the value of $f(x)$.
- We introduce binary variables δ_i, in order to indicate interval $a_i \leq x \leq a_{i+1}$ that x belongs
- To compute the value of the function, we take linear combination $\sum_{i=1}^{k} \lambda_i f(a_i)$.
- The above method can be applied if at most two adjacent λ_i i λ_{i+1} are positive. They correspond to interval bounds a_i, a_{i+1}.

Minimizing a piecewise linear objective function

A model for minimizing a piecewise linear objective function:

$$\min \sum_{i=1}^{k} \lambda_i f(a_i)$$

$$\begin{array}{l}
\sum_{i=1}^{k} \lambda_i = 1, \\
\lambda_1 \leq \delta_1, \\
\lambda_i \leq \delta_{i-1} + \delta_i, \quad i = 2, \ldots k-1, \\
\lambda_k \leq \delta_{k-1}, \\
\sum_{i=1}^{k-1} \delta_i = 1, \\
\lambda_i \geq 0.
\end{array}$$

Alternative Constraints

Assume that at least one, but not necessary the all of the conditions:

$$R_1, R_2, \ldots, R_N.$$

must be satisfied. One can express this as follows:

$$R_1 \vee R_2 \vee \cdots \vee R_N,$$

where R_i is a condition.

"The *i*-th constraint is satisfied".

Alternative Constraints

We introduce N indicator variables δ_i associated with the fulfillment of the conditions R_i, $i = 1, \ldots, N$:

$$\delta_i = 1 \Rightarrow R_i.$$

If R_i jest an inequality of the form $\sum_j a_j x_j \leq b$, then we include the condition:

$$\sum_j a_j x_j + M\delta \leq M + b. \quad (11)$$

If R_i jest an inequality of the form $\sum_j a_j x_j \geq b$, then we include the condition:

$$\sum_j a_j x_j + m\delta \geq m + b. \quad (12)$$

For inequalities (11) and (12), we append the constraint:

$$\delta_1 + \delta_2 + \cdots + \delta_N \geq 1.$$

Alternative Constraints

Assume that we want to express the condition:
"at least k conditions $R_1, R_2, \ldots, R_N$ must be satisfied".
The above condition can be modeled by

$$\delta_1 + \cdots + \delta_N \geq k.$$

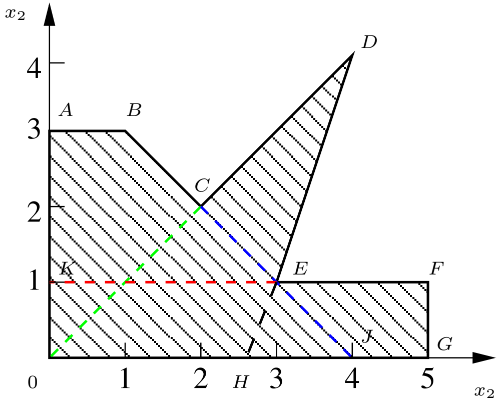
The condition:
"at most k conditions $R_1, R_2, \ldots, R_N$ must be satisfied".
can be modeled by

$$R_i \Rightarrow \delta_i = 1,$$
$$\delta_1 + \cdots + \delta_N \leq k.$$

Modeling nonconvex regions (sets)

The application of alternative constraints

Consider the following nonconvex region (ABCDEFGO).



The above region can be treaded as union of convex regions ABJO, ODH i KFGO.

Modeling nonconvex regions

Region ABJO is determined by the following constraints

$$x_2 \leq 3, \quad (13)$$

$$x_1 + x_2 \leq 4. \quad (14)$$

Region ODH is determined by the following constraints

$$-x_1 + x_2 \leq 0, \quad (15)$$

$$3x_1 - x_2 \leq 8. \quad (16)$$

Region KFGO is determined by the following constraints

$$x_2 \leq 1, \quad (17)$$

$$x_1 \leq 5. \quad (18)$$

Modeling nonconvex regions

We introduce indicator variables: $\delta_1, \delta_2, \delta_3$

$$\begin{aligned}
\delta_1 = 1 &\quad \Rightarrow \quad (x_2 \leq 3) \wedge (x_1 + x_2 \leq 4), \\
\delta_2 = 1 &\quad \Rightarrow \quad (-x_1 + x_2 \leq 0) \wedge (3x_1 - x_2 \leq 8), \\
\delta_3 = 1 &\quad \Rightarrow \quad (x_2 \leq 1) \wedge (x_1 \leq 5).
\end{aligned}$$

The above implications are modeled by the constraints:

$$\begin{aligned}
x_2 + \delta_1 &\leq 4, \\
x_1 + x_2 + 5\delta_1 &\leq 9, \\
-x_1 + x_2 + 4\delta_2 &\leq 4, \\
3x_1 - x_2 + 7\delta_2 &\leq 15, \\
x_2 + 3\delta_3 &\leq 4, \\
x_1 &\leq 5.
\end{aligned}$$

We need to include also the condition (constraint) that (13) and (14) or (15) and (16) or (17) and (18) are satisfied

$$\delta_1 + \delta_2 + \delta_3 \geq 1.$$

Restricting the number of variables

Suppose, we wish to restrict the number of variables (integer and continuous) that take positive values in a feasible solution.
For instance, we wish to restrict the number of components in a mixture or we wish to restrict an assortment of products produced.
In order to restrict the number of variables $x_1, x_2, \ldots, x_n$ to k, we introduce indicator variables δ_i associated with x_i

$$x_i > 0 \Rightarrow \delta_i = 1 \quad i = 1, \ldots, n.$$

The above implication is modeled by

$$x_i - M_i\delta_i \leq 0 \quad i = 1, \ldots, n,$$

M_i is an upper bound on values of x_i. We also include the constraint:

$$\delta_1 + \delta_2 + \cdots + \delta_n \leq k.$$

Resource limits having discrete values

Suppose that a linear programming model has the constraint, which limits a resource: $\sum_j a_j x_j \leq b_0$.
and the resource limit can be increased successively only by certain discrete values $b_1, b_2, \ldots, b_n$ at certain costs

$$\text{COST} = \begin{cases} 0 & \text{for } i = 0 \\ c_i & \text{otherwise} \end{cases}$$

where $c_1 < c_2 < \cdots < c_n$. This situation can be modeled by introducing binary variables δ_i that represent the resource increase

$$\sum_j a_j x_j \leq b_0\delta_0 + b_1\delta_1 + \cdots + b_n\delta_n.$$

We have to add to an objective function the expression:

$$c_0\delta_0 + c_1\delta_1 + \cdots + c_n\delta_n.$$

max – max objective functions

Consider the following objective function:

$$\max\left(\max_i\left(\sum_j a_{ij}x_j\right)\right)$$

where a set of feasible solution is determined by linear constraints.
We model this objective function by alternative constraints

$$\max z$$

subject to

$$\sum_j a_{1j}x_j - z = 0 \vee \sum_j a_{2j}x_j - z = 0 \vee \cdots$$

The set cover problem

The set cover problem is: given a set of elements $E = \{e_1, e_2, \ldots, e_n\}$ and a set of m subsets of E, $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ with costs $c_1, c_2, \ldots, c_m$.
Find a least cost collection C of sets from $\mathcal{S}$ such that C, covers all elements in E. That is, $\bigcup_{S_i \in C} S_i = E$.
Example:

$$E = \{1, 2, 3, 4, 5\},$$

and

$$\mathcal{S} = \{\{1, 2\}, \{1, 3, 5\}, \{2, 4, 5\}, \{3\}, \{1\}, \{4, 5\}\}.$$

Assume that $c_i = 1$, $i = 1, \ldots, m$. A collection C (feasible solution, cover) that covers E is

$$C = \{\{1, 2\}, \{1, 3, 5\}, \{2, 4, 5\}\}.$$

The set cover problem - a model

Binary variables δ_i, $i = 1, \ldots, 6$:

$$\delta_i = \begin{cases} 1 & \text{if the } i\text{-th subset } \mathcal{S} \text{ belongs to a cover} \\ 0 & \text{otherwise.} \end{cases}$$

The following constraints ensure that each element $i \in E$ must be covered:

$$\begin{array}{ccccccccccccl} & & & & \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 \to \min & & & & & & & & \\ \delta_1 & + & \delta_2 & & & & & & \delta_5 & & & \geq & 1 \text{ element 1} \\ \delta_1 & & & + & \delta_3 & & & & & & & \geq & 1 \text{ element 2} \\ & & \delta_2 & & & + & \delta_4 & & & & & \geq & 1 \\ & & & & \delta_3 & & & & & + & \delta_6 & \geq & 1 \\ & & \delta_2 & + & \delta_3 & & & & & + & \delta_6 & \geq & 1 \text{ element 6} \end{array}$$

The set packing problem

The set packing problem is: given a set of elements $E = \{e_1, e_2, \ldots, e_n\}$ and a set of m subsets of E, $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ with weights $w_1, w_2, \ldots, w_m$.
Find collection C of mutually disjoint sets from $\mathcal{S}$ whose weight is maximal.

Example:

$$E = \{1, 2, 3, 4, 5, 6\},$$

and

$$\mathcal{S} = \{\{1, 2, 5\}, \{1, 3\}, \{2, 4\}, \{3, 6\}, \{2, 3, 6\}\}.$$

Assume that $w_i = 1$, $i = 1, \ldots, m$. A collection C (feasible solution) is

$$C = \{\{1, 2, 5\}, \{3, 6\}\}.$$

The set packing problem

Binary variables δ_i, $i = 1, \ldots, 5$:

$$\delta_i = \begin{cases} 1 & \text{if the } i\text{-the subset } \mathcal{S} \text{belongs to } C \\ 0 & \text{otherwise.} \end{cases}$$

The following constraints ensure that each element i belongs to at most one subset of E

$$\begin{array}{ccccccccccccl} & & & & \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 \rightarrow \max \\ \delta_1 & + & \delta_2 & & & & & & & \leq & 1 & \text{element 1} \\ \delta_1 & & & + & \delta_3 & & & + & \delta_5 & \leq & 1 & \text{element 2} \\ & & \delta_2 & & & + & \delta_4 & + & \delta_5 & \leq & 1 \\ & & & & \delta_3 & & & & & \leq & 1 \\ \delta_1 & & & & & & & & & \leq & 1 \\ & & & & & + & \delta_4 & + & \delta_5 & \leq & 1 & \text{element 6} \end{array}$$

Generalized assignment problem

The generalized assignment problem consists in assigning $|I|$ "objects" to $|J|$ "boxes". We wish to assign each object to exactly one box; if assigned to box j, object i consumes a_{ij} units of a given "resource" in that box. The total amount of resource available in the jth box is d_j. This generic problem arises in a variety of problem contexts.

Example Machine scheduling: the objects are jobs, the boxes are machines; a_{ij} is the processing time of job i on machine j and d_j is the total amount of time available on machine j.

Generalized assignment problem - a model

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

$$\sum_{j \in J} x_{ij} = 1 \qquad \text{for } i \in I$$

$$\sum_{i \in I} a_{ij} x_{ij} \le d_j \qquad \text{for } j \in J$$

$$x_{ij} \in \{0, 1\} \qquad i \in I, j \in J$$

.

$x_{ij} = 1$ if object i is assigned to box j; $x_{ij} = 0$ otherwise.

Facility location problem

$$\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{j\in J} F_j y_j \leftarrow \min$$

$$\sum_{j\in J} x_{ij} = 1 \qquad \text{for } i \in I$$

$$\sum_{i\in I} d_i x_{ij} \leq K_j y_j \qquad \text{for } j \in J$$

$$0 \leq x_{ij} \leq 1 \qquad i \in I, j \in J$$

$$y_j \in \{0,1\} \qquad j \in J$$

I - the set of customers
J - the set of potential facility (warehouse) locations used to supply to the customers
y_j - the binary variable indicates whether or not we choose to locate a facility at location j
x_{ij} - the fraction of the demand of customer i that we satisfy from facility j
d_i - the demand of customer i
c_{ij} - the cost (transportation cost) of satisfying all of the ith customer's demand from facility j
F_j - the fixed cost of opening (leasing) a facility of size K_j at location j