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Theory and Methodology

On the equivalence of two optimization methods for fuzzy linear programming problems

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Abstract

The paper analyses the linear programming problem with fuzzy coefficients in the objective function. The set of nondominated (ND) solutions with respect to an assumed fuzzy preference relation, according to Orlovsky's concept, is supposed to be the solution of the problem. Special attention is paid to unfuzzy nondominated (UND) solutions (the solutions which are nondominated to the degree one). The main results of the paper are sufficient conditions on a fuzzy preference relation allowing to reduce the problem of determining UND solutions to that of determining the optimal solutions of a classical linear programming problem. These solutions can thus be determined by means of classical linear programming methods. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the paper the linear programming problem is considered in which the objective function coefficients are given in an imprecise way, by means of fuzzy numbers. The other coefficients are known precisely. In this case, for each feasible solution the corresponding objective function value, constituting an estimate of its utility, is also a fuzzy number. There are many concepts of the solution of this problem (see e.g., Refs. [3,13]). One of them was presented by Orlovsky [12], in which he makes use of the approach to the decision making (the choice of the "best" solution) which he had earlier proposed in Ref. [11] and which bases itself on a fuzzy preference relation defined on the set of feasible solutions. In this paper this concept is assumed to be the valid definition of the solution of the problem. Let a fuzzy preference relation be defined on the set of fuzzy numbers, a relation which determines a fuzzy order on this set. According to Orlovsky's concept, the solution of the problem is a fuzzy set of those feasible solutions which are nondominated with respect to this fuzzy

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preference relation. The value of the membership function of this set for a given feasible solution determines the degree to which we can be sure that the fuzzy objective function value corresponding to this solution is not dominated by the objective function values corresponding to other feasible solutions of the problem. It can be then recommended to the decision maker to choose as the definitive decision this feasible solution for which the membership function in the set of nondominated (ND) solutions attains a maximal value. From the practical point of view those solutions are interesting, which are nondominated to the degree one – if such solutions exist, of course. The set of the solutions which are nondominated to the degree one has been called by Orlovsky the set of unfuzzy nondominated (UND) solutions.

In the general case the determination of an UND solution with respect to a given fuzzy preference relation can be reduced to the determination, in the set of feasible solutions, of the saddle point of the membership function of a fuzzy relation of strict preference, linked to the original relation. For the problem reduced in this way, the existing algorithms determining the saddle point are very labour-consuming, which sometimes ruins their practical usefulness.

In the paper we give some conditions whose fulfilment makes it possible to reduce a linear programming problem with fuzzy coefficients in the objective function to a classical (usual) linear programming problem. The fuzzy coefficients in the objective function of the original problem are replaced with their real equivalents, being the values of a ranking function, which considerably simplifies the problem and allows to use classical algorithms of linear programming. The optimal solutions of the linear programming problem defined in this way are UND solutions of the original problem. In Ref. [4] we have obtained equally effective ways of determining UND solutions but only for specific six fuzzy preference relations known from the literature. In this paper, however, we answer a more general question: which conditions should be fulfilled by any fuzzy preference relation so that a ranking function exists and the problem of determining UND solutions with respect to this relation can be reduced to that of determining the optimal solutions of a classical linear programming problem? We cover a broader class of fuzzy preference relations. The conditions proved in this paper are a generalization, for the continuous case, of the result obtained by Kołodziejczyk [10] for the case of the discrete linear problem with fuzzy coefficients in the objective function with a finite set of feasible solutions.

2. Selected notions from the fuzzy numbers arithmetic

We will remind here only those notions linked to the fuzzy numbers arithmetic which we will use further on in the paper.

Definition 1. Fuzzy number A is a fuzzy set defined on the set of real numbers $\mathbb{R}$ characterized by means of a membership function $\mu_A(x)$, $\mu_A : \mathbb{R} \to [0,1]$, which is upper semicontinuous and which fulfils the following conditions:
1. normality, i.e. $\sup_{x\in\mathbb{R}}\mu_A(x) = 1$;
2. convexity, i.e. for any $x, y \in \mathbb{R}$, $\lambda \in [0,1]$ it holds $\mu_A(\lambda x + (1-\lambda)y) \geqslant \min\{\mu_A(x), \mu_A(y)\}$.

In the paper we will use extended operations of the addition of two fuzzy numbers and of the multiplication of a fuzzy number with a scalar. Let us remind the definitions of those operations, which are a consequence of the extension principle of Zadeh.

Definition 2. Let A, B be fuzzy numbers and $r \in \mathbb{R}$. Then:
1. $A + B$ is a fuzzy number with the membership function
$$\mu_{A+B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}, \quad x, y, z \in \mathbb{R},$$
2. for $r \neq 0$ rA is a fuzzy number with the membership function
$$\mu_{rA}(z) = \mu_A(z/r), \quad z \in \mathbb{R},$$
3. for $r = 0$ rA is zero, i.e. $\mu_{rA}(z) = 1$ for $z = 0$ and $\mu_{rA}(z) = 0$ for $z \neq 0$.

Dubois and Prade [7] proposed a representation of the fuzzy number which makes it considerably easier to perform the operations defined above. It is a so called $L-R$ form of a fuzzy number.

Definition 3. Fuzzy number A is called a number of the $L-R$ type if its membership function μ_A has the following form:

$$\mu_A(x)=\begin{cases}1 & \text{for } x\in[\underline{a},\overline{a}],\\ L\left(\frac{\underline{a}-x}{\alpha}\right) & \text{for } x\leqslant\underline{a},\\ R\left(\frac{x-\overline{a}}{\beta}\right) & \text{for } x\geqslant\overline{a},\end{cases} \tag{1}$$

where L and R are continuous nonincreasing functions, defined on $[0,\infty)$, strictly decreasing to zero in those subintervals of the interval $[0,\infty)$ in which they are positive, and fulfilling the conditions $L(0)=R(0)=1$. The parameters α and β are non-negative real numbers.

The functions L and R are called shape functions. Examples of such functions are $\max(0,1-x)$, e^{-x^p}, $\max(0,1-x^p)$, $x\in[0,\infty)$, $p\geqslant 1$. The following notation of the fuzzy number of the $L-R$ type with the membership function (1) will be assumed:

$$A=(\underline{a},\overline{a},\alpha,\beta)_{L-R}.$$

The operations defined in Definition 2 preserve the fuzzy numbers type. The following equalities hold:

$$\begin{aligned}&(\underline{a},\overline{a},\alpha_A,\beta_A)_{L-R}+(\underline{b},\overline{b},\alpha_B,\beta_B)_{L-R}\\ &\quad=(\underline{a}+\underline{b},\overline{a}+\overline{b},\alpha_A+\alpha_B,\beta_A+\beta_B)_{L-R},\end{aligned}$$

$$r(\underline{a},\overline{a},\alpha_A,\beta_A)_{L-R}=(r\underline{a},r\overline{a},r\alpha_A,r\beta_A)_{L-R}\quad\text{for } r\geqslant 0.$$

3. Formulation of the problem

The linear programming problem with fuzzy coefficients in the objective function, abbreviated as LPPFCO, can be formulated in the following way:

$$F(\boldsymbol{x})=\sum_{j=1}^{n}C_jx_j\to\max, \tag{2}$$

$$\boldsymbol{x}\in X\colon\begin{cases}\sum_{j=1}^{n}a_{ij}x_j\leqslant b_i, & i=1,\ldots,m,\\ x_j\geqslant 0, & j=1,\ldots,n,\end{cases}$$

where C_j, $j=1,\ldots,n$, are fuzzy numbers.

In the objective function of problem (2) we use of course the extended operations of the addition of fuzzy numbers and of the multiplication of a fuzzy number with a scalar, which are defined in Definition 2. From the properties of these operations it follows that the objective function $F(\boldsymbol{x})$ is a fuzzy number.

4. Orlovsky's concept

We will present now a definition of the solution of problem (2) based on the concept of the choice of the best solution with respect to a fuzzy preference relation (defined on the set of feasible solutions), presented by Orlovsky [11].

Let us denote with $\mathrm{FN}(\mathbb{R})$ the set of all fuzzy numbers. Let us assume that on this set there is defined a fuzzy preference (order) relation $\mathscr{R}$ with the membership function $\mu_{\mathscr{R}}:\mathrm{FN}(\mathbb{R})\times\mathrm{FN}(\mathbb{R})\to[0,1]$. The value $\mu_{\mathscr{R}}(A,B)$, $A,B\in\mathrm{FN}(\mathbb{R})$ denotes the degree to which the fuzzy number A is preferred (greater) with respect to the number B.

Definition 4. The fuzzy solution of problem (2) is the fuzzy set ND of nondominated elements in X with respect to a fuzzy relation $\mathscr{R}$, i.e. the fuzzy set ND with the following membership function:

$$\begin{aligned}\mu_{\mathrm{ND}}(\boldsymbol{x})&=\inf_{\boldsymbol{y}\in X}[1-\mu_{\mathscr{R}^s}(F(\boldsymbol{y}),F(\boldsymbol{x}))]\\ &=1-\sup_{\boldsymbol{y}\in X}\mu_{\mathscr{R}^s}(F(\boldsymbol{y}),\ F(\boldsymbol{x})),\end{aligned} \tag{3}$$

where $\mathscr{R}^s$ is a fuzzy relation of strict preference linked to the relation $\mathscr{R}$ with the membership function defined in the following way:

$$\mu_{\mathscr{R}^s}(A,B)=\max\{0,\mu_{\mathscr{R}}(A,B)-\mu_{\mathscr{R}}(B,A)\}. \tag{4}$$

The value of the membership function $\mu_{\mathrm{ND}}(\boldsymbol{x})$ denotes the degree to which the vector $\boldsymbol{x}$ is not dominated by another vector from the set X (the truth degree of the statement "$\boldsymbol{x}$ is not dominated by any other vector $\boldsymbol{y}\in X$"). The best solution seems to be the solution with the maximal (greatest) degree of the membership in the fuzzy set of ND solutions.

Definition 5. A crisp optimal solution of problem (2) is a vector $\boldsymbol{x} \in X$ for which the function $\mu_{\mathrm{ND}}(\boldsymbol{x})$ takes on the greatest value (the solution which is to the greatest degree nondominated by other solutions with respect to the relation $\mathscr{R}$), i.e. the solution of the following mathematical programming problem: $\max_{\boldsymbol{y}\in X}\mu_{\mathrm{ND}}(\boldsymbol{y})$.

The following definition is of essential importance for the reasoning in this paper:

Definition 6. If for a maximally ND solution $\boldsymbol{x}$ (i.e. for an optimal solution according to Definition 5) the value of the membership function $\mu_{\mathrm{ND}}(\boldsymbol{x})$ equals 1, then $\boldsymbol{x}$ is called an UND solution. The set of UND solutions is denoted as UND.

The UND solutions – if they exist, i.e. if UND $\neq \emptyset$ – seem to be of a high practical significance. With respect to them there is no doubt that they are the best ones, because the statement saying that they are not dominated by any other solution from the set of feasible solutions X is true to the degree one.

It seems that a certain remark would be adequate in this place: the concept of the ND solution according to Orlovsky cannot be identified with the concept of the ND solution (optimal according to Pareto) used in the vector optimization. However, there is a certain intuitive similarity between the two. In both cases the solution of the problem is a solution with respect to which there are no strictly preferred solutions. In the case of the Orlovsky approach such solutions are selected on the basis of a fuzzy relation of strict preference, and in the vector optimization on the basis of a certain ordering relation for vectors.

A direct consequence of formula (3) and Definitions 5 and 6 is the following theorem, which we will use in Section 5 of the paper.

Theorem 1. *A vector $\boldsymbol{x} \in X$ is an UND solution with respect to a fuzzy relation $\mathscr{R}$, i.e. $\boldsymbol{x} \in$ UND, if and only if for each $\boldsymbol{y} \in X$ the following formula holds*:

$$\mu_{\mathscr{R}^s}(F(\boldsymbol{y}), F(\boldsymbol{x})) = 0.$$

5. Reduction of the LPPFCO problem to the classical linear programming problem

Let us present a sufficient condition assuring that the problem of determining an UND solution of the LPPFCO problem, according to Definition 6, can be reduced to that of determining the optimal solutions of a classical linear programming problem. But first of all let us remind the theorem on the separating hyperplane (see Ref. [8]).

Theorem 2. *If two convex sets $\aleph_1$ and $\aleph_2$ have in common at the most some boundary points, then there exists a hyperplane*

$$f(\boldsymbol{x}) = a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \boldsymbol{a} \neq \boldsymbol{0}$$

which separates $\aleph_1$ and $\aleph_2$ in the following sense:

$$f(\boldsymbol{x}) \geqslant b \ \forall_{\boldsymbol{x}\in\aleph_1} \text{ and } f(\boldsymbol{y}) \leqslant b \ \forall_{\boldsymbol{y}\in\aleph_2}$$

or

$$f(\boldsymbol{x}) \geqslant f(\boldsymbol{y}) \ \forall_{\boldsymbol{x}\in\aleph_1, \boldsymbol{y}\in\aleph_2}.$$

We will now prove a lemma which we will make an essential use of in the proof of Lemma 2.

Lemma 1. *If $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous and anti-reflexive function (i.e. such that $\phi(\boldsymbol{x},\boldsymbol{x}) = 0$ for each $(\boldsymbol{x},\boldsymbol{x}) \in \mathbb{R}^n \times \mathbb{R}^n$) and the set of couples $\aleph = \{(\boldsymbol{x},\boldsymbol{y}){:}\, \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n, \phi(\boldsymbol{x},\boldsymbol{y}) > 0\} \subset \mathbb{R}^{2n}$ is convex, then there exists a vector $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a} \neq \boldsymbol{0}$ such that*

$$\begin{aligned} &a_1x_1 + a_2x_2 + \cdots + a_nx_n \\ &\quad > a_1y_1 + a_2y_2 + \cdots + a_ny_n \ \forall_{(\boldsymbol{x},\boldsymbol{y})\in\aleph}. \end{aligned}$$

Proof. From the continuity of function ϕ it follows that the set $\aleph$ is open.

We build the set $\aleph_1 = \{\boldsymbol{x}-\boldsymbol{y}{:}\, \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n, (\boldsymbol{x},\boldsymbol{y}) \in \aleph\} \subset \mathbb{R}^n$. The set is open (this follows from the openness of the set $\aleph$). We will show that it is also convex. Let us take two elements from $\aleph_1$, $\boldsymbol{x}'-\boldsymbol{y}' \in \aleph_1$ and $\boldsymbol{x}''-\boldsymbol{y}'' \in \aleph_1$. Then for each $\lambda \in [0,1]$,

$$\begin{aligned} &\lambda(\boldsymbol{x}'-\boldsymbol{y}') + (1-\lambda)(\boldsymbol{x}''-\boldsymbol{y}'') \\ &\quad = (\lambda\boldsymbol{x}' + (1-\lambda)\boldsymbol{x}'') - (\lambda\boldsymbol{y}' + (1-\lambda)\boldsymbol{y}'') \in \aleph_1, \end{aligned}$$

because $(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{x}'', \lambda\boldsymbol{y}' + (1-\lambda)\boldsymbol{y}'') \in \aleph$, which follows from the convexity of $\aleph$. Thus $\aleph_1$ is convex. Let us take a one-element set $\{\mathbf{0}\} \subset \mathbb{R}^n$, where $\mathbf{0}$ is the zero vector. It is of course convex. The intersection of the sets $\{\mathbf{0}\}$ and $\aleph_1$ is empty, because if it held $\{\mathbf{0}\} \subset \aleph_1$, then such vectors would exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ that $\boldsymbol{x} - \boldsymbol{y} = \mathbf{0}$ (i.e. $\boldsymbol{x} = \boldsymbol{y}$) and $\phi(\boldsymbol{x}, \boldsymbol{y}) > 0$, which would contradict the anti-reflexiveness of the function ϕ ($\phi(\boldsymbol{x}, \boldsymbol{x}) = 0$). Hence, the sets $\aleph_1$ and $\{\mathbf{0}\}$ are disjoint. As they are also convex, Theorem 2 implies that there exists a function

$$f(\boldsymbol{z}) = a_1 z_1 + a_2 z_2 + \cdots + a_n z_n,$$

$\boldsymbol{a} \neq \mathbf{0}$ such that

$$f(\boldsymbol{x} - \boldsymbol{y}) \geqslant f(\mathbf{0}) = 0 \ \forall_{\boldsymbol{x}-\boldsymbol{y} \in \aleph_1}. \tag{5}$$

Only boundary points of the set $\aleph_1$ can belong to the hyperplane $f(\boldsymbol{z}) = 0$ separating the sets $\aleph_1$ and $\{\mathbf{0}\}$. As the set $\aleph_1$ is open, it does not contain such points. Therefore the inequality (5) is fulfilled with a strict relation, i.e.

$$f(\boldsymbol{x} - \boldsymbol{y}) > f(\mathbf{0}) = 0 \ \forall_{\boldsymbol{x}-\boldsymbol{y} \in \aleph_1}. \tag{6}$$

As f is a linear function, condition (6) can be written as

$$f(\boldsymbol{x}) - f(\boldsymbol{y}) > f(\mathbf{0}) = 0 \ \forall_{(\boldsymbol{x},\boldsymbol{y}) \in \aleph}.$$

This implies

$$f(\boldsymbol{x}) > f(\boldsymbol{y}) \ \forall_{(\boldsymbol{x},\boldsymbol{y}) \in \aleph},$$

which completes the proof of the lemma. □

Let us denote with $\tilde{\mathscr{R}}$ a fuzzy preference relation generated by $\mathscr{R}$, determining in the set $\mathbb{R}^n$ a fuzzy order, whose membership function is defined in the following way:

$$\mu_{\tilde{\mathscr{R}}}(\boldsymbol{x}, \boldsymbol{y}) = \mu_{\mathscr{R}}\left(\sum_{j=1}^{n} C_j x_j, \sum_{j=1}^{n} C_j y_j\right),$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $C_j \in \mathrm{FN}(\mathbb{R})$, $j = 1, \ldots, n$.

Lemma 2. *If for the fuzzy preference relation $\tilde{\mathscr{R}}$ the membership function $\mu_{\tilde{\mathscr{R}}^s}$ of the fuzzy strict preference relation $\tilde{\mathscr{R}}^s$ generated (implied) by $\tilde{\mathscr{R}}$ according to formula (4) is continuous function and the set of couples*

$$\aleph = \{(\boldsymbol{x}, \boldsymbol{y}) \colon \mu_{\tilde{\mathscr{R}}^s}(\boldsymbol{x}, \boldsymbol{y}) > 0\} \subset \mathbb{R}^{2n}$$

is convex then there exists a ranking function $r \colon \mathrm{FN}(\mathbb{R}) \Rightarrow \mathbb{R}$ such that:

$$\mu_{\tilde{\mathscr{R}}^s}(\boldsymbol{x}, \boldsymbol{y}) = \mu_{\mathscr{R}^s}\left(\sum_{j=1}^{n} C_j x_j, \sum_{j=1}^{n} C_j y_j\right) > 0 \Rightarrow \sum_{j=1}^{n} r(C_j) x_j > \sum_{j=1}^{n} r(C_j) y_j, \tag{7}$$

or, equivalently,

$$\sum_{j=1}^{n} r(C_j) x_j \leqslant \sum_{j=1}^{n} r(C_j) y_j \Rightarrow \mu_{\mathscr{R}^s}\left(\sum_{j=1}^{n} C_j x_j, \sum_{j=1}^{n} C_j y_j\right) = 0, \tag{8}$$

where C_j are fuzzy numbers, $C_j \in \mathrm{FN}(\mathbb{R})$, $j = 1, \ldots, n$.

Proof. Because of the convexity of the set $\aleph$, the continuity of the relation $\mu_{\tilde{\mathscr{R}}^s}$ and its anti-reflexiveness, which follows from the definition of relation $\mu_{\tilde{\mathscr{R}}^s}$, Lemma 1 implies the existence of a vector $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a} \neq \mathbf{0}$ such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n > a_1 y_1 + a_2 y_2 + \cdots + a_n y_n \ \forall_{(\boldsymbol{x},\boldsymbol{y}) \in \aleph}. \tag{9}$$

Having made the following assignment:

$$r(C_j) = a_j \ j = 1, \ldots, n,$$

we get implication (7) from inequality (9). □

The continuity condition of function $\mu_{\tilde{\mathscr{R}}^s}$ in Lemma 2 (like the continuity condition of function ϕ in Lemma 1) can be replaced by a weaker condition of the openness of set $\aleph$. Thus the main results of the paper expressed in Lemma 2 and the essential Theorem 3 presented further on are true for a wider class of preference relations $\mathscr{R}$ – which is used in the initial problem to compare fuzzy values of the objective function. However, in practice most preference relations known from the literature satisfy the continuity condition; apart from it seems easier to verify this condition than to prove the openness of set $\aleph$.

Let us now present an answer to the question formulated in the introduction. The following theorem gives a sufficient condition assuring that a ranking function exists and the problem of determining the UND solutions of problem LPPFCO can be reduced to that of determining the optimal solutions of a linear programming problem.

Theorem 3. *Let X be a bounded set of feasible solutions of problem (2). If a fuzzy preference preference $\tilde{\mathscr{R}}$ fulfils the assumptions of Lemma 2, then a ranking function exists $r: \mathrm{FN}(\mathbb{R}) \to \mathbb{R}$ such that*

$$\sum_{j=1}^{n} r(C_j)x_j^* = \max_{\boldsymbol{x}\in X} \sum_{j=1}^{n} r(C_j)x_j \Rightarrow \boldsymbol{x}^* \in \mathrm{UND}. \quad (10)$$

Proof. From Lemma 2 follows the existence of a ranking function $r: \mathrm{FN}(\mathbb{R}) \to \mathbb{R}$ such that

$$\sum_{j=1}^{n} r(C_j)x_j \leqslant \sum_{j=1}^{n} r(C_i)y_j \Rightarrow \mu_{\mathscr{R}^s}\left(\sum_{j=1}^{n} C_j x_j, \sum_{j=1}^{n} C_j y_j\right) = \mu_{\mathscr{R}^s}(F(\boldsymbol{x}), F(\boldsymbol{y})) = 0. \quad (11)$$

We will show that this function fulfils condition (10). Let $\boldsymbol{x}^* \in X$ be a vector such that

$$\sum_{j=1}^{n} r(C_j)x_j^* = \max_{\boldsymbol{x}\in X} \sum_{j=1}^{n} r(C_j)x_j. \quad (12)$$

Such a vector exists, because the set of feasible solutions of problem (2), $X \subset \mathbb{R}^n$, is bounded and closed, and thus compact, and $\sum_{j=1}^{n} r(C_j)x_j$ is a continuous function. Eq. (12) implies that for each $\boldsymbol{y} \in X$ the following inequality is fulfilled:

$$\sum_{j=1}^{n} r(C_j)y_j \leqslant \sum_{j=1}^{n} r(C_j)x_j^*. \quad (13)$$

Making use of Eq. (13) and condition (11) we obtain

$$\mu_{\mathscr{R}^s}(F(\boldsymbol{y}), F(\boldsymbol{x}^*)) = 0, \ \boldsymbol{y} \in X. \quad (14)$$

Condition (14) and Theorem 1 imply that $\boldsymbol{x}^* \in \mathrm{UND}$. □

There may of course exist several ranking functions fulfilling condition (10) in Theorem 3. If we have one such mapping, we can identify other ones by carrying out a sensitivity analysis of the linear programming problem from Eq. (10), with respect to the changes in the objective function coefficients.

6. Computational example

Before we give an example illustrating the usefulness of Theorem 3, we will present several proposals of fuzzy preference relations and the ranking functions for those relations. These are of course not the only definitions of fuzzy preference relations which have been proposed so far. A systematic survey and classification of various methods of comparing fuzzy numbers can be found in the paper of Chen and Hwang [5].

Baas and Kwakernaak [1] have suggested fuzzy preference relation $\mathscr{R}_1$ with the following membership function:

$$\mu_{\mathscr{R}_1}(A,B) = \sup_{x \geqslant y} \min\{\mu_A(x), \mu_B(y)\}, \quad x, y \in \mathbb{R}, \quad A, B \in F(\mathbb{R}).$$

Kołodziejczyk [9] defined another fuzzy preference relation $\mathscr{R}_2$, a more complicated one, which has the following membership function:

$$\mu_{\mathscr{R}_2}(A,B) = [d(A^L \vee B^L, B^L) + d(A^P \vee B^P, B^P)]/[d(A^L, B^L) + d(A^P, B^P)],$$

where $d(A,B) \overset{\mathrm{df}}{=} \int_{\mathbb{R}} \mid \mu_A(x) - \mu_B(x) \mid \mathrm{d}x$ is the Hamming distance between the fuzzy numbers A and B, it is also assumed that $\int_{\mathbb{R}} \mu_A(x)\mathrm{d}\,x < +\infty$ and $\int_{\mathbb{R}} \mu_B(x)\mathrm{d}\,x < +\infty$, $\vee$ is the operation of the maximum of two fuzzy numbers, i.e. $\mu_{A\vee B}(t) = \sup_{t=\max(x,y)} \min\,\{\mu_A(x), \mu_B(y)\}$ for $t \in \mathbb{R}$ and

$$\mu_{A^L}(x) \overset{\mathrm{df}}{=} \begin{cases} \mu_A(x) & \text{for } x \leqslant z, \\ 1 & \text{for } x \geqslant z, \end{cases}$$

$$\mu_{A^P}(x) \overset{\mathrm{df}}{=} \begin{cases} \mu_A(x) & \text{for } x \geqslant z, \\ 1 & \text{for } x \leqslant z, \end{cases}$$

where z is a number fulfilling the condition $\mu_A(z) = 1$.

Delgado et al. [6] proposed the relation $\mathscr{R}_3$ whose membership function has the following form:

$$\mu_{\mathscr{R}_3}(A,B) = 1 - \sup_{x\in\mathbb{R}} \{\mu_{A^{\lambda,L}}(x) t \mu_B(x)\},$$
$$A, B \in F(\mathbb{R}),$$

where

$$\mu_{A^{\lambda,L}}(x) = [(1+\lambda)\mu_{A^L}(x)]/[1+\lambda\mu_{A^L}(x)],$$
$$\lambda \in (-1, 0],$$

t is a t-norm (see Ref. [14]). Relation $\mathscr{R}_3$ is anti-reflexive and anti-symmetric, thus the following relationship holds: $\mathscr{R}_3 = \mathscr{R}_3^s$.

The values of the membership functions of relations $\mathscr{R}_i$, $i = 1, 2, 3$, denote the truth degree of the statement $A \geqslant B$.

It can be shown (we omit the proof because of its arithmetical character) that the fuzzy relations of strict preference $\mathscr{R}_i^s$, $i = 1, 2, 3$, linked to the presented fuzzy relations $\mathscr{R}_i$, $i = 1, 2, 3$, fulfil the assumptions of Theorem 3. Thus, from Theorem 3 it follows that for the relations $\mathscr{R}_i^s$, $i = 1, 2, 3$, there exist ranking functions, r_i, $i = 1, 2, 3$, respectively, which allow us to reduce the problem of determining the ND solutions of problem LPPFCO to the determination of the optimal solutions of a linear programming problem. In order to build up such ranking functions, it is enough to indicate such r_i for which the implication (7) (Lemma 2) is fulfilled. For the relation $\mathscr{R}_2^s$, $i = 2$, the function r_2 fulfilling implication (7) is equal, as shown in Ref. [2], to the following ranking function suggested by Yager [15]:

$$r_2(A) \stackrel{\text{df}}{=} \frac{1}{2}\int_0^1 (\underline{a}(\alpha) + \overline{a}(\alpha))\, \mathrm{d}\alpha,$$

where

$$\underline{a}(\alpha) = \inf_{x\in\mathbb{R}}\{x: \mu_A(x) \geqslant \alpha\},$$

$$\overline{a}(\alpha) = \sup_{x\in\mathbb{R}}\{x: \mu_A(x) \geqslant \alpha\}.$$

$\underline{a}(\alpha)$ and $\overline{a}(\alpha)$ are the ends of the interval constituting the α-level of the fuzzy number A, i.e. $A^\alpha = \{x: \mu_A(x) \geqslant \alpha\}$. For the relations $\mathscr{R}_i^s$, $i = 1, 3$, functions fulfilling implication (7) take on, which is easy to notice, the form $r_i(A) \stackrel{\text{df}}{=} \overline{a}(1)$, $i = 1, 3$, where $\overline{a}(1)$ is the right end of the interval of the modal values of the fuzzy number A, i.e. $A^1 = \{x: \mu_A(x) = 1\}$.

Let us pass on to the announced computational example. Let a linear programming problem with fuzzy coefficients in the objective function be given:

$$\begin{array}{rcrcl} F(\boldsymbol{x}) = C_1x_1 & + & C_2x_2 & \rightarrow & \max \\ 2x_1 & + & 3x_2 & \leqslant & 10 \\ 3x_1 & + & 2x_2 & \leqslant & 15 \\ & & x_1, x_2 & \geqslant & 0, \end{array} \tag{15}$$

where $C_1 = (3, 5, 8, 6)_{L-R}$, $C_2 = (1, 2, 1, 6)_{L-R}$ are fuzzy numbers of the $L - R$ type (see Definition 3) with the shape functions $L(y) = \mathrm{e}^{-y}$ and $R(y) = \max(0, 1 - y^2)$.

In order to obtain an UND solution of problem (15) with respect to relation $\mathscr{R}_i$, $i = 1, 3$, it is necessary, as follows from Theorem 3, to solve an adequate linear programming problem:

$$\begin{array}{rcrcl} 5x_1 & + & 2x_2 & \rightarrow & \max \\ 2x_1 & + & 3x_2 & \leqslant & 10 \\ 3x_1 & + & 2x_2 & \leqslant & 15 \\ & & x_1, x_2 & \geqslant & 0, \end{array} \tag{16}$$

where $r_i(C_1) = 5$, $r_i(C_2) = 2$, $i = 1, 3$, are the values of the ranking function r_i, $i = 1, 3$, which correspond to the fuzzy coefficients C_1, C_2 of the original problem. The solution of problem (16) is the vector $\boldsymbol{x}^* = (5, 0)$ and it is an UND solution of the original problem with respect to the relation $\mathscr{R}_i$, $i = 1, 3$.

Similarly, in order to obtain an UND solution of problem (15) with respect to relation $\mathscr{R}_2$, we have to solve the adequate linear programming problem:

$$\begin{array}{rcrcl} 2x_1 & + & 3x_2 & \rightarrow & \max \\ 2x_1 & + & 3x_2 & \leqslant & 10 \\ 3x_1 & + & 2x_2 & \leqslant & 15 \\ & & x_1, x_2 & \geqslant & 0, \end{array} \tag{17}$$

where

$$r_2(C_1) = \frac{1}{2}\int_0^1 (8 + 8\ln\alpha + 6\sqrt{1-\alpha})\, \mathrm{d}\alpha = 2,$$

$$r_2(C_2) = \frac{1}{2}\int_0^1 (3 + \ln\alpha + 6\sqrt{1-\alpha})\, \mathrm{d}\alpha = 3$$

are the values of the ranking function r_2, which correspond to the fuzzy coefficients C_1, C_2 of the original problem. The solution of problem (17) is the vector $\boldsymbol{x}^* = (0, 3.333)$, which is an UND solution of problem (15) with respect to the relation $\mathscr{R}_2$.

7. Conclusions

The main result of the paper are the conditions which should be fulfilled by the strict fuzzy preference relation assumed in the definition of the solution, which ensure that the problem of determining UND solutions of problem LPPFCO can be reduced to that of determining the optimal solutions of a linear programming problem, with the coefficients in the objective function being the values of a ranking function which exists for this relation. These conditions are a consequence of the theorem we have proven, concerning the existence of a so called ranking function (Theorem 3). Theorem 3 is a generalization, for the continuous case, of the result obtained by Kołodziejczyk [10] for the case of the discrete problem with a finite set of feasible solutions. For some relations, this theorem allows to use the existing classical methods of linear programming in order to determine the UND solutions with respect to those relations and to avoid the necessity to develop specific algorithms.

The practical usefulness of Theorem 3 was illustrated by the computational example, where we determined the UND solutions with respect to three fuzzy preference relations known from the literature.

The example illustrates also the fact that the conditions which have to be satisfied by a fuzzy preference relation for the transition from the fuzzy problem to the usual one (according to Theorem 3) to be possible, do not seem to be ones that would considerably limit the presented approach – although they are not always easy to verify. Most relations that are important in the literature (both such simple ones as the relation of Baas and Kwakernaak and more complicated ones, as e.g. the relation of Kołodziejczyk) fulfil these conditions.

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