Criticality analysis in activity networks under incomplete information

Stefan Chanas
Institute of Industrial
Engineering and Management,
Technical University of Wrocław,
Wybrzeże Wyspiańskiego 27,
50-370 Wrocław, Poland,
chanas@ioz.pwr.wroc.pl

Didier Dubois
I.R.I.T.,
Université Paul Sabatier,
118 route Narbonne,
31062 Toulouse Cedex 4, France,
dubois@irit.fr

Paweł Zieliński
Institute of Mathematics,
Technical University of Opole,
Luboszycka 3,
45-036 Opole, Poland,
pziel@po.opole.pl

Abstract

A review of the main results obtained by the authors concerning analysis of networks with imprecise activity duration times, with a stress on the analysis of criticality, is presented.

Keywords: Scheduling, Possibility and necessity measure, Criticality, Fuzzy PERT.

1 Introduction

The notions of the criticality of a path and an activity were born with the development of the Critical Path Method (Kelley [9]). An activity is critical if and only if its earliest and latest starting times are equal and that critical activities form critical paths. So finding the critical activities yields the critical paths. When the durations of activities are ill-known and modeled by intervals (and fuzzy intervals), it is not longer true that these results are valid. Namely, floats can no longer be recovered from the intervals containing earliest and latest starting times, and critical paths may no longer exist. Such obtained imprecise quantities do not lead to the unique identification of the critical path. Several works tried to cope with this problem. Here, we only mention some selected existing approaches to defining the criticality notion. McCahon and Lee [10], McCahon [11] propose to go back to standard critical path methods via defuzzification of the fuzzy activity times and to compute fuzzy slack times of activities from the fuzzy starting times obtained by the forward and backward recursions, but these fuzzy variables are interactive so that what is obtained is only a rough imprecise approximation of the fuzzy range of the actual float of the activity. Such a computation makes sense only if the fuzzy due-date and the fuzzy release date of the projects are prescribed independently of each other (Dubois and Prade [5]). Rommelfanger [13] suggests substitutes to the fuzzy subtraction, so as to improve the situation, but these techniques remain ad hoc. Nasution [12] resorts to symbolic computations on the variable processing times. However this technique is unwieldy and highly combinatorial. Another way of approaching the criticality analysis of activities is to directly check if a path or an activity is critical, which in the fuzzy case is a matter of degree. Kamburowski [8] tries and computes a criticality index for path and activities directly. The criticality of an activity is obtained from two fuzzy evaluations of starting times of an activity: one using activities that precede it, the other from the activities that take place after, then comparing to the maximal fuzzy length of paths. For checking the extent is a path is critical, one computes the height of the intersection of the fuzzy length of this path and the fuzzy completion time. It is clear that if this intersection is empty, then the path is not critical. However when the height of this intersection can be maximal for paths that are surely not critical. Of course, the presented approaches do not exhaust all approaches to defining the criticality which have been appeared in the literature so far.

Actually, a correct solution to the whole problem of critical path analysis under fuzzy uncertainty cannot be reached by mending existing algorithms. It requires a mathematically clean statement of the problem in the setting of possibility theory. This step was taken by Buckley [1]. He states the problem of computing fuzzy latest starting times of activities and their floats in a correct way, points out its difficulty without proposing a solving method.

This paper is devoted to review some results, obtained by the authors recently concerning the possibilistic criticality analysis in networks with imprecise activity duration times.

First, we consider the case when the imprecise activity times are defined in the network by means of intervals and next we generalize the obtained results to the case of the network with fuzzy activity times.

2 Networks with interval activity times

Assume that a network $S = < V, A >$ ($|V| = n$, $|A| = m$), being a project activity-on-arc model, is given. V is the set of nodes (events) and $A \subset V \times V$ is the set of arcs (activities). The network S is a directed, connected, acyclic graph with one initial node and one end node. Activity duration times are determined by means of intervals $T_{ij} = [\underline{t}_{ij}, \overline{t}_{ij}]$, $(i, j) \in A$.

Let us denote by T a configuration of activity duration times in S, $T = (t_{ij})_{(i,j)\in A}$, where t_{ij} is an exact value from the interval activity time T_{ij}. P is the set of all paths in S from the initial node to the end node.

We define four criticality notions. The first two have been proposed in Chanas and Zieliński [3] [4] and the last two in Dubois [6] and Fargier et al. [7].

A path $p \in P$ (resp. an activity $(i, j) \in A$) is possibly critical in S if and only if there exists a configuration of times T such that p (resp. (i, j)) is critical in S in the usual sense. The next notion is complementary to the first one. A path $p \in P$ (resp. an activity $(i, j) \in A$) is necessarily non-critical in S if and only if for each configuration of times T, p (resp. (i, j)) is not critical in S in the usual sense. A path $p \in P$ (resp. an activity $(i, j) \in A$) is necessarily critical in S if and only if for each configuration of times T, p (resp. (i, j)) is critical in S in the usual sense. And the last notion, which is complementary to the third one. A path $p \in P$ (resp. an activity $(i, j) \in A$) is possibly non-critical in S if and only if there exists a configuration of times T such that p (resp. (i, j)) is not critical in S in the usual sense. It is worth noticing that on the basis of the possible criticality of path (resp. activity) we also have an information about its necessary non-criticality. Similarly, on the basis of the necessary criticality of path (resp. activity) we also have an information about its possible non-criticality.

The notion of possible criticality (or necessary non-criticality) is thoroughly investigated by Chanas and Zieliński [3] [4]. It has been shown there that the problem of determining an arbitrary possibly critical path and that of asserting if a fixed path $p \in P$ is possibly critical in the network S are easy and they can be solved in time bounded by a polynomial in the size of the network. However, not all the problems concerning the possible criticality are easy. Chanas and Zieliński [3] have proved that the problem of asserting the possible criticality of a fixed activity is NP-complete in the strong sense.

Similarly as concerns the notion of necessarily criticality (or possible non-criticality), both the problem of asserting whether a given path is necessarily critical and the problem of determining an arbitrary necessarily critical path (more exactly, a subnetwork covering all the necessarily critical paths) are easy. We propose (Chanas, Dubois and Zieliński [2]) corresponding solution algorithms. Unfortunately, the problem of evaluating whether a given isolated activity is necessarily critical, case when there is no a necessarily critical path in the network (it is not very rarely), does not seem to be such. We can only formulate conjecture that this problem is hard. The question of proving this fact is still open. We give the lemma (Chanas et al. [2]), which confirm the combinatorial nature of the problem and support mentioned conjecture. The lemma suggests an algorithm, unfortunately ineffective, for determining all necessarily critical activities. It requires to check so-called extreme configurations (T such that $t_{ij} = \underline{t}_{ij}$ or $t_{ij} = \overline{t}_{ij}$). There are 2^m such configurations. In Chanas et al. [2], there have been shown relationships between necessarily critical paths and activities in the network and additionally formulated conditions, which in some situations allow evaluating the necessary criticality of activities also in case when there is no necessarily critical path. However, they do not cover all the possible situations, so do not solve the problem definitively. The obtained results allow to construct an algorithm, which determines an approximation set of all necessarily critical activities. When a necessarily critical path exists, in this case, the problem is easy to solve: all the necessarily critical activities belong then to the necessarily critical paths, which are easy to determine.

In Fargier et al. [7] the mentioned hard problems have been solved in the case of series-parallel networks, that is there have been proposed polynomial algorithms for asserting the possible or necessary criticality of a fixed activity.

3 Networks with fuzzy activity times

Now we focus on the fuzzy case. All the elements of the network S are the same as in the interval case except for activity duration times, which are determined by means of fuzzy numbers $\tilde{T}_{ij}$, $(i,j) \in A$, which imprecisely determine duration times of activities $(i,j) \in A$. $\tilde{T}_{ij}$ expresses uncertainty connected with the ill-known activity duration time modeled by this number. It generates possibility and necessity functions for sets of values containing the unknown activity duration. More formally, we say that the assertion of the form "T_{ij} is $\tilde{T}_{ij}$", where T_{ij} is a variable and $\tilde{T}_{ij}$ is a fuzzy number, generates the possibility distribution of T_{ij} with respect to the following formula (see Zadeh [14]):

$$\mathrm{Poss}(T_{ij} = t_{ij}) = \mu_{\tilde{T}_{ij}}(t_{ij}),\ \ t_{ij} \in \mathbb{R}_+.$$

The same assertion induces necessity measure for T_{ij} in the following way:

$$\mathrm{Nec}(T_{ij} \in [a,b]) = 1 - \inf\{\lambda \,|\, \tilde{T}_{ij}^{\lambda} \subseteq [a,b]\},\ [a,b] \in I(\mathbb{R}),$$

where $I(\mathbb{R})$ is the set of all intervals and $\tilde{T}_{ij}^{\lambda}$ is λ-cut, i.e. $\tilde{T}_{ij}^{\lambda} = \{t_{ij} \in \mathbb{R}^+ \,|\, \mu_{\tilde{T}_{ij}}(t_{ij}) \geq \lambda\} = [\underline{t}_{ij}^{\lambda}, \overline{t}_{ij}^{\lambda}]$, $\lambda \in (0,1]$.

Let T be a configuration of activity duration times in the network with activity times $t_{ij} \in \mathbb{R}_+$, $(i,j) \in A$. The (joint) possibility distribution over configurations, induced by the $\tilde{T}_{ij}$'s is $\pi(T) = \min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}(t_{ij})$, $T \in \mathbb{R}_+^m$.

We introduce the criticality notions for the network with fuzzy activity duration times. The first notion has been proposed in Chanas and Zieliński [4]. The possibility that a path $p \in P$ (resp. an activity $(i,j) \in A$) is critical is determined by the following formula:

$$\begin{aligned} \mathrm{Poss}(p \text{ is critical}) &= \sup_{T:p \text{ is critical in } T} \pi(T) \\ (\mathrm{Poss}((i,j) \text{ is critical}) &= \sup_{T:(i,j) \text{ is critical in } T} \pi(T)). \end{aligned}$$

The following formula determines the possibility that a path $p \in P$ (resp. an activity $(i,j) \in A$) is not critical:

$$\begin{aligned} \mathrm{Poss}(p \text{ is not critical}) &= \sup_{T:p \text{ is not critical in } T} \pi(T) \\ (\mathrm{Poss}((i,j) \text{ is not critical}) &= \sup_{\substack{T:(i,j) \text{ is not critical} \\ \text{in } T}} \pi(T)). \end{aligned}$$

And the next notion has been proposed by Dubois [6] and Fargier et al. [7]. The necessity that a path $p \in P$ (resp. an activity $(i,j) \in A$) is critical is determined by the following formula:

$$\begin{aligned} \mathrm{Nec}(p \text{ is critical}) &= 1 - \mathrm{Poss}(p \text{ is not critical}) \\ &= \inf_{\substack{T:p \text{ is not} \\ \text{critical in } T}} (1 - \pi(T)) \\ (\mathrm{Nec}((i,j) \text{ is critical}) &= \inf_{\substack{T:(i,j) \text{ is not} \\ \text{critical in } T}} (1 - \pi(T))). \end{aligned}$$

The following formula determines the necessity that a path $p \in P$ (resp. an activity $(i,j) \in A$) is not critical:

$$\begin{aligned} \mathrm{Nec}(p \text{ is not critical}) &= 1 - \mathrm{Poss}(p \text{ is critical}) \\ &= \inf_{\substack{T:p \text{ is} \\ \text{critical in } T}} (1 - \pi(T)) \\ (\mathrm{Nec}((i,j) \text{ is not critical}) &= \inf_{\substack{T:(i,j) \text{ is} \\ \text{critical in } T}} (1 - \pi(T))). \end{aligned}$$

In Chanas and Zieliński [4] two effective methods of calculating the value of index $\mathrm{Poss}(p \text{ is critical})$ have been presented. The first one is adapted to fuzzy activity times given in a general form and the second one, based on linear programming, is valid only for fuzzy activity times determined by fuzzy numbers of the same $L-L$ type. Naturally, those methods may be also applied to calculation of the index $\mathrm{Nec}(p \text{ is not critical})$. There is no a such method for calculating the value of index $\mathrm{Poss}((i,j) \text{ is critical})$ (or $\mathrm{Nec}((i,j) \text{ is not critical})$). This problem is *NP*-hard since it includes, as a very special case, the problem of asserting the possible criticality of an activity, which is *NP*-complete in the strong sense. We present (Chanas, Dubois and Zieliński [2]) two effective methods of determining the necessity degree that a path is critical, $\mathrm{Nec}(p \text{ is critical})$. The first method enables to determine the necessity degree in a general case while the second one, based on linear programming approach, needs special assumptions on the membership functions of the fuzzy activity times $\tilde{T}_{ij}$, $(i,j) \in A$, i.e. activity times should be determined by means of fuzzy numbers of the same $L-L$ type. Those methods may be also applied to calculation of the possibility degree that a path is not critical, $\mathrm{Poss}(p \text{ is not critical})$. We also propose (Chanas et al. [2]) the algorithm for determining paths with maximal necessity degree of criticality in the network. We have not been able to give such algorithms for computing the necessity degree of criticality of a fixed activity, $\mathrm{Nec}((i,j)$ is critical). Generally the problem does not seem to be easy, because it is more general than the problem, in the interval case, of asserting if an activity is necessarily critical. However, in some situations we may evaluate the degrees of necessary criticality of certain activities by using information provided by the algorithm, which determines paths with maximal necessity degree of criticality in the network. One should stress that the problems of evaluation of the possibility and necessity degree that activity is critical have been solved by Fargier et al. [7] in the of case series-parallel networks.

4 Conclusions

In conclusion, it is important to stress that the presence of activities with ill-known (interval-valued or fuzzy) durations completely questions the validity of standard criticality analysis: not only does the concept of criticality splits into two indices of possible and necessary criticality, but the notion of critical activity as belonging to a critical activity must be given up. A necessarily critical path seldom exists in practice while isolated necessarily critical activities may be found more frequently. However it turns out that deciding if an activity is necessarily and possibly critical is a hard problem.

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