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Critical path analysis in the network with fuzzy activity times

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Abstract

A natural generalization of the criticality notion in a network with fuzzy activity times is given. It consists in direct application of the extension principle of Zadeh to the notion of criticality of a path (an activity, an event) treated as a function of the activities duration times in the network. There are shown some relations between the notion of fuzzy criticality, introduced in the paper, and the notion of interval criticality (criticality in the network with interval activity times) proposed by the authors in another paper. Two methods of calculation of the path degree of criticality (according to the proposed concept of fuzzy criticality) are presented. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The critical path method (CPM), worked out at the beginning of the 1960s (see [10]), has become one of the tools that are most useful in practice and are applied in the planning and control of the realization of complex projects. First of all it consists in the identification of the so-called critical paths, critical activities and critical events in the network, which is the project model, assuming the earliest possible completion time of the whole project. By chance certain values useful for the decision maker such as: events and activities slacks, the earliest and the latest moments of the start and finish of the particular activities, etc., are calculated. What is essential in the CPM method is that the activities duration times are deterministic and known. In practice, of course, this assumption not always can be fulfilled with the satisfying accuracy. Therefore, as early as the paper [12] the method called in literature the PERT method has been suggested with the formulation of the not unique estimations of the activity times, and the conception of the random variable with the beta distribution has been used to model the activity times. In this method many simplifying assumptions have been taken, therefore it has been intensively developed in many directions also under assumptions that the probability distributions of activity times are different to the beta distribution. So far, in the literature, hundreds of papers have been dedicated to problems connected with the PERT method and research on this area is still carried on. Starting with the second part of the 1970s (see [4,18]) the other approach to the network project analysis, usually called the fuzzy PERT method or the fuzzy CPM, has been

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developed, in which it is suggested to use fuzzy numbers (sets) to model the activity times. The papers ([1–4,6,8,9,11,13–16,18–22]) set in references surely do not exhaust all the papers connected with the fuzzy PERT. In all these papers the approach to the subject is similar. The classic formulae used in the CPM as well as some dependencies true for a network with deterministic activity times are used where the common operations are replaced with the operations generalized on the fuzzy numbers. So modified formulae and dependencies are used to define many project characteristics, with the most important, such as degrees of criticality of the paths and activities, among them. As it has been noticed in [21] the properties, which are equivalent in the deterministic case and lead to the unique identification of the critical path, cannot play such a role if they are automatically transfered to the fuzzy case. As result there are obtained different definitions of the fuzzy critical path which give different estimations of the degree of criticality for the same path in the network.

In this paper we present another natural approach, that has not been considered in the literature so far, to the generalization of the criticality notion for the case of the network with fuzzy activities duration times. This generalization is made directly without using generalized arithmetic operations on fuzzy numbers. Namely, the criticality of the path (as well as an activity and an event) is treated, in this approach, as a two-value function depending on the path (activity, event) and activities durations times assumed in the network. This function takes one of two values *yes* or *no* depending on the given path (activity, event) is critical or non-critical with assumed crisp activities durations times. By direct generalization of this function (based on the extension principle of Zadeh) on fuzzy activities durations times the notion of the critical path (activity, event) as fuzzy set in the set of all the paths (activities, events) in the network is obtained (Section 4.2).

There are relations between the notion of fuzzy criticality (criticality in the network with fuzzy activity times) introduced here and the notion of interval criticality (criticality in the network with interval activity times) introduced in [5]. Therefore, in Section 3 we refer to the most important results obtained in [5].

In the paper (Section 4.3) we give too effective methods of computing the value of membership function of an arbitrary path in the fuzzy critical path (the path degree of criticality), using also some results obtained in [5] in this aim. Unfortunately, it is not possible to construct similar effective methods to calculate the activity (event) degree of criticality.

Before we pass on to the essential considerations of this paper in the next section let us remind the most important elements connected with the classic CPM method.

2. The CPM with crisp activity times

A network $S=\langle V,A,t\rangle$, being a project model, is given. V is a set of nodes (events) and $A\subset V\times V$ is a set of arcs (activities). The network S is a directed, compact, acyclic graph. The set $V=\{1,2,\ldots,n\}$ is labelled in such a way that the following condition holds: $(i,j)\in A \Rightarrow i<j$. By means of function t, $t:A\rightarrow\mathbb{R}^+$, the activity times in the network are determined, $t(i,j)\stackrel{\text{def}}{=}t_{ij}$ is a duration of activity $(i,j)\in A$.

The essence of the CPM method (from numerical point of view) are two recurrence formulae which are used to determine the earliest and the latest moments of occurring the events $i\in V$. Let us denote by $P(i)=\{k\in V\mid(k,i)\in A\}$ the set of predecessors and by $S(i)=\{k\in V\mid(i,k)\in A\}$ the set of successors of event $i\in V$, respectively. The earliest moments, T_i^{e}, of occurrence of the events $i\in V$ are determined by means of the following recurrence formula:

$$T_i^{\mathrm{e}}=\begin{cases}0 & \text{for } i=1,\\ \max_{k\in P(i)}(T_k^{\mathrm{e}}+t_{ki}) & \text{for } i>1.\end{cases} \tag{1}$$

And the latest moments, T_i^{l}, of occurrence of the events $i\in V$ can be found by use of the following formula:

$$T_i^{\mathrm{l}}=\begin{cases}T_n^{\mathrm{e}} & \text{for } i=n,\\ \min_{k\in S(i)}(T_k^{\mathrm{l}}-t_{ik}) & \text{for } i<n.\end{cases} \tag{2}$$

The times obtained by the use of (1) and (2) are applied to the calculation of slack times, $L_i=T_i^{\mathrm{l}}-T_i^{\mathrm{e}}$, for events $i\in V$ and slack times, $Z(i,j)=T_j^{\mathrm{l}}-T_i^{\mathrm{e}}-t_{ij}$, for activities $(i,j)\in A$. By means of such determined quantities the notions of criticality of an event and of an activity can be defined.

Definition 1. An activity $(i,j)\in A$ is critical if and only if $Z(i,j)=0$.

Definition 2. An event $i\in V$ is critical if and only if $L_i=0$.

Let us denote by P the set of all paths in S from node 1 to node n.

Definition 3. A path $p\in P$ is critical if and only if all activities belonging to p are critical.

The following theorems are obvious.

Theorem 1. *A path $p\in P$ is critical if and only if it is the longest path in the network S with the lengths of arcs equal to t_{ij}, $(i,j)\in A$. The length of this path is equal to T_n^{e}.*

Theorem 2. *An activity $(i,j)\in A$ (an event $i\in V$) is critical if and only if it belongs to a certain critical path $p\in P$.*

The problems of determining critical activities, events and paths are easy ones in a network with deterministic (crisp) durations of activities. If we remove from the network all the non-critical activities then we obtain the network in which all the paths leading from the initial node 1 to the end node n are critical ones (with the same length equal to T_n^{e}). Naturally, the number of these paths may be very great as it increases exponentially together with the network size extension. However, the network reduced to the set of critical activities is available in a time bounded by a polynomial in the size of the initial network.

3. The CPM with interval activity times

A network $S=\langle V,A,T\rangle$ is given. All assumptions on the network are the same as in the deterministic case except for function T, that is defined now in the following way: $T:A\to I(\mathbb{R}^+)$, where $I(\mathbb{R}^+)$ is the set of non-negative interval numbers (intervals with non-negative ends). Interval $T(i,j)\overset{\mathrm{def}}{=}T_{ij}=[\underline{t}_{ij},\bar{t}_{ij}]$ is a set of possible duration times of activity $(i,j)\in A$.

Definition 4. A path $p\in P$ is i-critical (interval critical) in S if and only if there exists a set of times t_{ij}, $t_{ij}\in[\underline{t}_{ij},\bar{t}_{ij}]$, $(i,j)\in A$, such that p is critical in the sense of Definition 3, after replacing the interval times T_{ij} with the exact values t_{ij}, $(i,j)\in A$.

Definition 5. An activity $(k,l)\in A$ (an event $k\in V$) is i-critical in the network S if and only if there exists a set of times t_{ij}, $t_{ij}\in[\underline{t}_{ij},\bar{t}_{ij}]$, $(i,j)\in A$, such that (k,l) (k) is critical in the sense of Definition 1 (Definition 2) in the network S, after replacing the interval times T_{ij} with the exact values t_{ij}, $(i,j)\in A$.

Theorem 3. *If a path $p\in P$ is i-critical in S, then all activities (also all events) contained in p are i-critical.*

Proof. It follows from Definitions 4 and 5, Theorem 2. □

Remark 1. It is easy to show that the reverse implication to that in Theorem 3 is not true, i.e. one can find an example in which a path consists of i-critical activities (events) and is not i-critical itself.

From Remark 1 it follows that in the case of interval activity times it is impossible, opposite to the case of crisp activity times, to construct the "reduced" network, by removing from the initial network S all non i-critical activities, in which any path leading from node 1 to node n would be i-critical.

The key theorem for further considerations is Theorem 4 which determines a necessary and sufficient condition of i-criticality of a given path $p\in P$.

Theorem 4. *A path $p\in P$ is i-critical in S if and only if it is critical in the sense of Definition 3 in the network S, in which the interval activity times $T_{ij}=[\underline{t}_{ij},\bar{t}_{ij}]$, $(i,j)\in A$, have been replaced with the exact values t_{ij} determined by means of the following formula:*

$$t_{ij}=\begin{cases}\bar{t}_{ij} & \text{if } (i,j)\in p,\\ \underline{t}_{ij} & \text{if } (i,j)\notin p.\end{cases}$$

Proof. Obvious. The theorem is a direct consequence of Definition 4. □

From Definition 4 and Theorem 4 follows that the problem of determining an arbitrary i-critical path and that of stating if a fixed path $p \in P$ is i-critical in a network S are easy and they can be solved in time bounded by a polynomial in the size of the network. In the first case it is enough to apply the classical CPM (Section 2) to the network S with any fixed values t_{ij} of activity times chosen from the corresponding interval duration times, $t_{ij} \in [\underline{t}_{ij}, \overline{t}_{ij}]$. And in the second case it is sufficient to apply the classical CPM to the network S, after replacing the interval times $[\underline{t}_{ij}, \overline{t}_{ij}]$ with the exact values t_{ij} determined as in Theorem 4. However, not all the problems concerning the i-criticality are so easy. Some of them have turned out to be hard ones. In [5] we have proved that the problem of asserting the i-criticality of a fixed activity (a fixed event) is NP-complete in the strong sense. We have also shown that the problem of determining K i-critical paths is NP-hard.

4. The CPM with fuzzy activity times

Before presentation of the main results, concerning the analysis of a network with fuzzy activity times, we remind the notions of a fuzzy number and a fuzzy number of L–R type.

4.1. A fuzzy number notion

A general definition of a fuzzy number is following.

Definition 6. Fuzzy number $\tilde{A}$ is a fuzzy set defined on the set of real numbers $\mathbb{R}$ characterized by means of a membership function $\mu_{\tilde{A}}(x)$, $\mu_{\tilde{A}}: \mathbb{R} \to [0, 1]$:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{for } x \leqslant a, \\ f_{\tilde{A}}(x) & \text{for } a \leqslant x \leqslant c, \\ 1 & \text{for } c \leqslant x \leqslant d, \\ g_{\tilde{A}}(x) & \text{for } d \leqslant x \leqslant b, \\ 0 & \text{for } x \geqslant b, \end{cases} \tag{3}$$

where $f_{\tilde{A}}$ and $g_{\tilde{A}}$ are continuous functions, $f_{\tilde{A}}$ is increasing (from 0 to 1), $g_{\tilde{A}}$ is decreasing (from 1 to 0). In special cases it may be $a = -\infty$ and (or) $b = +\infty$.

A concept of a fuzzy number of L–R type, introduced by Dubois and Prade [7] is very popular and convenient in applications.

Definition 7. A fuzzy number $\tilde{A}$ is called a number of L–R type if its membership function $\mu_{\tilde{A}}$ has the following form:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 & \text{for } x \in [\underline{a}, \overline{a}], \\ L\left(\dfrac{\underline{a} - x}{\alpha_A}\right) & \text{for } x \leqslant \underline{a}, \\ R\left(\dfrac{x - \overline{a}}{\beta_A}\right) & \text{for } x \geqslant \overline{a}, \end{cases} \tag{4}$$

where L and R are continuous non-increasing functions, defined on $[0, +\infty)$, strictly decreasing to zero in those subintervals of the interval $[0, +\infty)$ in which they are positive, and fulfilling the conditions $L(0) = R(0) = 1$. The parameters α_A and β_A are non-negative real numbers.

The functions L and R are called shape functions. They usually take one of the following forms:

linear:

$S(y) = \max(0, 1 - y)$,

exponential:

$S(y) = \mathrm{e}^{-py}$, $\quad p \geqslant 1$,

power:

$S(y) = \max(0, 1 - y^p)$, $\quad p \geqslant 1$,

rational:

$S(y) = 1/(1 + y^p)$, $\quad p \geqslant 1$,

exponential power:

$$S(y) = \mathrm{e}^{-y^p}, \quad p \geqslant 1. \tag{5}$$

We will use the following notation, introduced in [7], for a fuzzy number $\tilde{A}$ of L–R type:

$\tilde{A} = (\underline{a}, \overline{a}, \alpha_A, \beta_A)_{L\text{–}R}$.

Definition 8. Let $\tilde{A}$ be a fuzzy number. For each $\lambda \in (0, 1]$ the interval $\tilde{A}^{\lambda} = [\overline{a}^{\lambda}, \underline{a}^{\lambda}] = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \geqslant \lambda\}$ is called λ-cut of the fuzzy number $\tilde{A}$.

For a fuzzy number $\tilde{A}$ of L–R type (Definition 7) λ-cut $\tilde{A}^{\lambda}$, $\lambda \in (0,1]$, has the following form:

$$\tilde{A}^{\lambda} = [\underline{a} - L^{-1}(\lambda)\alpha_A, \overline{a} + R^{-1}(\lambda)\beta_A],$$

where L^{-1} (similarly R^{-1}) denotes the reverse function to L in this part of its domain in which it is positive.

The reverse functions to these listed in (5) are the following ones:

for linear:

$$S^{-1}(y) = 1 - y, \quad y \in (0,1],$$

for exponential:

$$S^{-1}(y) = -\frac{1}{p} \ln y, \quad y \in (0,1],$$

for power:

$$S^{-1}(y) = \sqrt[p]{1 - y}, \quad y \in (0,1],$$

for rational:

$$S^{-1}(y) = \sqrt[p]{\frac{1 - y}{y}}, \quad y \in (0,1],$$

for exponential power:

$$S^{-1}(y) = \sqrt[p]{- \ln y}, \quad y \in (0,1]. \tag{6}$$

4.2. A concept of f-criticality in a network with fuzzy activity times

A network $S = \langle V, A, \tilde{T} \rangle$ is given. All elements of this network are the same as in the deterministic case except for function $\tilde{T}$ which is now defined in the following way: $\tilde{T}: A \to F(\mathbb{R}^+)$, where $F(\mathbb{R}^+)$ is the set of non-negative fuzzy numbers. Fuzzy number $\tilde{T}(i,j) \stackrel{\text{def}}{=} \tilde{T}_{ij}$ determines imprecisely a duration time of activity $(i,j) \in A$. Membership function $\mu_{\tilde{T}_{ij}}$ generates a possibility distribution for the duration time of activity $(i,j) \in A$, i.e. value $\mu_{\tilde{T}_{ij}}(t_{ij})$ means a possibility degree of performance of activity (i,j) in time $t_{ij} \in \mathbb{R}^+$.

Definition 9. The fuzzy set $\tilde{P}$ in set P with the membership function $\mu_{\tilde{P}}: P \to [0,1]$ determined by formula

$$\mu_{\tilde{P}}(p) = \sup_{\substack{t_{ij} \in \mathbb{R}^+, (i,j) \in A \\ \text{and } p \text{ is critical with} \\ \text{activity times} \\ \text{equal to } t_{ij}, (i,j) \in A}} \min_{(i,j) \in A} \mu_{\tilde{T}_{ij}}(t_{ij}), \quad p \in P \tag{7}$$

is called the fuzzy critical (f-critical) path in S.

We say that a path p is f-critical with the degree $\mu_{\tilde{P}}(p)$. The value $\mu_{\tilde{P}}(p)$ stands for the path degree of criticality, possibility of the criticality of path p. To put it in another way, $\mu_{\tilde{P}}$ determines a possibility distribution of the criticality of the path in the set P which is generated by possibility distributions of activities duration times $\mu_{\tilde{T}_{ij}}$, $(i,j) \in A$ (generated according to extension principle of Zadeh – if the criticality, or lack of it, is treated as the activities duration times function in the network).

Definition 10. The fuzzy set $\tilde{A}$ ($\tilde{E}$) in set A (V) with the membership function determined by the formula

$$\mu_{\tilde{A}}(k,l) = \sup_{\substack{t_{ij} \in \mathbb{R}^+, (i,j) \in A \\ \text{and } (k,l) \text{ is critical with} \\ \text{activities duration times} \\ \text{equal to } t_{ij}, (i,j) \in A}} \min_{(i,j) \in A} \mu_{\tilde{T}_{ij}}(t_{ij}), \ (k,l) \in A, \tag{8}$$

$$\left(\mu_{\tilde{E}}(k) = \sup_{\substack{t_{ij} \in \mathbb{R}^+, (i,j) \in A \\ \text{and } k \text{ is critical with} \\ \text{activities duration times} \\ \text{equal to } t_{ij}, (i,j) \in A}} \min_{(i,j) \in A} \mu_{\tilde{T}_{ij}}(t_{ij}), \quad k \in V \right) \tag{9}$$

is called the fuzzy critical (f-critical) activity (event) in S.

The following theorem determines a relation between the notions of f- and i-criticality for a given path $p \in P$.

Theorem 5. *A path p is f-critical with a degree not less than $r \in (0,1]$, $p \in \tilde{P}^r$ ($\tilde{P}^r = \{p \in P \mid \mu_{\tilde{P}}(p) \geqslant$*

$r\}$), *if and only if* p *is i-critical in the network* S *with the interval activity times* $T_{ij} = \tilde{T}_{ij}^r = \{t_{ij} \in \mathbb{R}^+ \mid \mu_{\tilde{T}_{ij}}(t_{ij}) \geqslant r\}$, $(i,j) \in A$ ($\tilde{P}^r$ *and* $\tilde{T}_{ij}^r$ *denote r-cuts of fuzzy sets* $\tilde{P}$ *and* $\tilde{T}_{ij}$, *respectively*).

Proof. Directly from the definition of r-cut, Definitions 4 and 9. □

There exists the following relation between a criticality degree of a path and criticality degrees of activities forming this path.

Theorem 6. *For any path* $p \in P$ *the following relation holds*:

$$\mu_{\tilde{P}}(p) \leqslant \mu_{\tilde{A}}(k,l) \quad \text{for each } (k,l) \in p. \tag{10}$$

Proof. Let us take any $p \in P$. Assume that relation (10) is not true, i.e. there exists an activity $(k',l') \in p$, such that

$$\mu_{\tilde{P}}(p) > \mu_{\tilde{A}}(k',l'). \tag{11}$$

From (7) it follows that for any $\varepsilon > 0$ there exists a set of activity times t_{ij}^*, $(i,j) \in A$, with which p is a critical path in S and the following condition holds:

$$\min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}(t_{ij}^*) \geqslant \mu_{\tilde{P}}(p) - \varepsilon. \tag{12}$$

The criticality of p with activity times t_{ij}^*, $(i,j) \in A$, implies the criticality of activity (k',l') with these times (see Definition 3). Therefore, from (8) we obtain

$$\min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}(t_{ij}^*) \leqslant \mu_{\tilde{A}}(k',l'). \tag{13}$$

And finally, from (12) and (13) we get the sequence of inequalities

$$\mu_{\tilde{P}}(p) - \varepsilon \leqslant \min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}(t_{ij}^*) \leqslant \mu_{\tilde{A}}(k',l')$$

which is contrary to assumption (11). Thus, the theorem has been proved. □

A similar relation between a path and events is true.

Theorem 7. *For any path* $p \in P$ *the following relation holds*:

$$\mu_{\tilde{P}}(p) \leqslant \mu_{\tilde{E}}(k) \quad \text{for each } k \in p.$$

Proof. Similar to the proof of Theorem 6. □

The next two theorems provide a way of calculating the criticality degree of an activity (event) by means of criticality degrees of the paths crossing this activity (event).

Theorem 8. *The following equality is true*:

$$\mu_{\tilde{A}}(k,l) = \max_{p \in P(k,l)} \mu_{\tilde{P}}(p), \quad (k,l) \in A, \tag{14}$$

where

$$P(k,l) = \{p \mid p \in P \text{ and } (k,l) \in p\} \subseteq P.$$

Proof. Let us define for each $p \in P$ and $(k,l) \in A$ the following subsets of $\mathbb{R}^m$, where m is the number of activities in A:

$$TC(p) = \{(t_{ij})_{(i,j)\in A} \mid p \text{ is critical}\},$$

$$TC(k,l) = \{(t_{ij})_{(i,j)\in A} \mid (k,l) \text{ is critical}\}.$$

From Theorem 2 we get for any $(k,l) \in A$ the equality:

$$TC(k,l) = \bigcup_{p \in P(k,l)} TC(p). \tag{15}$$

Thus, using (8) and (7) we obtain the following sequence of equalities:

$$\begin{aligned}\mu_{\tilde{A}}(k,l) &= \sup_{(t_{ij})\in TC(k,l)} \min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}(t_{ij}) \\ &= \max_{p\in P(k,l)} \sup_{(t_{ij})\in TC(p)} \min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}(t_{ij}) \\ &= \max_{p\in P(k,l)} \mu_{\tilde{P}}(p),\end{aligned}$$

which completes the proof. □

We have a similar formula for $\mu_{\tilde{E}}(k)$ value.

Theorem 9. *The following equality is true*:

$$\mu_{\tilde{E}}(k) = \max_{p\in P(k)} \mu_{\tilde{P}}(p), \quad k \in V, \tag{16}$$

where

$$P(k) = \{p \mid p \in P \text{ and } k \in p\} \subseteq P.$$

Proof. Analogous to the proof of Theorem 8. □

4.3. Determining the path degree of criticality

In this section, we present two effective methods of determining the path degree of criticality. The first method enables to determine the path degree of criticality in a general case while the second one needs special assumptions on the membership functions of the fuzzy activity times $\tilde{T}_{ij}$, $(i,j)\in A$.

Before we pass on to the essentials we introduce a notion of *feasibility* of a value $\lambda\in(0,1]$ under a path $p\in P$ and then we formulate a theorem which we substantially use in the both methods.

Definition 11. The value $\lambda\in(0,1]$ is called feasible under the path $p\in P$ if and only if p is i-critical in the network S with interval activities duration times $T_{ij}=\tilde{T}^{\lambda}_{ij}$, where $\tilde{T}^{\lambda}_{ij}=[\underline{t}^{\lambda}_{ij},\bar{t}^{\lambda}_{ij}]$ are λ-cuts of fuzzy activity times $\tilde{T}_{ij}$, $(i,j)\in A$.

Theorem 10. *The following equality holds*:

$$\mu_{\tilde{P}}(p)=\sup\{\lambda \mid \lambda \text{ is a feasible value under the path } p\in P\}.$$

Proof. Obvious. It follows directly from the theorem which asserts that every fuzzy set can be decomposed according to its λ-cuts (see [17]). □

4.3.1. Calculation of the path degree of criticality in a general case

Now we present an algorithm for computing the criticality degree $\mu_{\tilde{P}}(p)$ of path $p\in P$ which is correct in a general case. We do not make any assumption on the fuzzy activity times in the network. The algorithm is based on the idea of bisection of the unit interval of possible values of λ to compute the maximal feasible (according to Definition 11) value λ under the path p. In this algorithm at each iteration k we test if the value $\lambda_k\in(0,1]$ is feasible under the path p. The testing can be reduced to applying the classical CPM method to the network S, after replacing the interval times $T_{ij}=\tilde{T}^{\lambda_k}_{ij}=[\underline{t}^{\lambda_k}_{ij},\bar{t}^{\lambda_k}_{ij}]$, $(i,j)\in A$, with the exact values t_{ij} determined as in Theorem 4.

Algorithm 1.

Step 1 *Assign* $k:=0$.

Step 2 *Test if* $\lambda=\varepsilon$ *is a feasible value under the path* p. *If it is not then assign* $\lambda_{\max}:=0$ *and go to step* 6.

Step 3 *Assign* $\lambda_k:=1$ *and test if* λ_k *is a feasible value under the path* p. *If it is then assign* $\lambda_{\max}:=1$ *and go to step* 6.

Step 4 *Assign* $k:=k+1$,

$$\lambda_k:=\begin{cases}\lambda_{k-1}+1/2^k & \text{if } \lambda_{k-1} \text{ is a feasible value},\\ \lambda_{k-1}-1/2^k & \text{if } \lambda_{k-1} \text{ is not a feasible value}.\end{cases}$$

Test if λ_k *is a feasible value under the path* p. *If it is then assign* $\lambda_{\max}:=\lambda_k$.

Step 5 *If* $k<K$ *then go to step* 4.

Step 6 *Assign* $\mu_{\tilde{P}}(p):=\lambda_{\max}$, *stop*.

The length K of the generated sequence in step 4 depends on the assumed accuracy of computation. If we want the absolute error of computation to be not greater than 10^{-N} then K has to fulfill the following condition

$$K\geqslant N/\log_{10} 2.$$

The value of ε used in step 2 should be positive and not greater than the absolute error of computation assumed.

4.3.2. Calculation of the path degree of criticality by the linear programming

Now we present another approach to the problem of determining the path degree of criticality showing that the problem can be reduced, under certain assumptions about membership functions of fuzzy activities duration times $\tilde{T}_{ij}$, to that of determining the optimal solution of a classical linear programming problem.

Let us return to the interval case considered in Section 3. Testing if path $p\in P$ is i-critical in network S with activity times equal to $T_{ij}=[\underline{t}_{ij},\bar{t}_{ij}]$ can be reduced to solution of the following system of linear equalities and inequalities:

$$\begin{aligned}&t_j-t_i-\bar{t}_{ij}=0, \quad (i,j)\in p,\\ &t_j-t_i-\underline{t}_{ij}\geqslant 0, \quad (i,j)\notin p,\\ &t_1=0,\\ &t_i\geqslant 0, \quad (i=2,\dots,n),\end{aligned} \tag{17}$$

where unknowns t_i denote moments of occurring the events $i\in V$ in network S.

From Theorem 4 it follows that path p is i-critical if and only if the system (17) has a solution.

Thus, the statement of $\lambda \in (0,1]$ feasibility under the path p (in the fuzzy case) can be reduced to testing if the following system of equalities and inequalities has a solution:

$$
\begin{aligned}
&t_j - t_i - \bar{t}_{ij}^{\lambda} = 0, \quad (i,j) \in p,\\
&t_j - t_i - \underline{t}_{ij}^{\lambda} \geqslant 0, \quad (i,j) \notin p,\\
&t_1 = 0,\\
&t_i \geqslant 0, \quad (i = 2,\dots,n),
\end{aligned}
\tag{18}
$$

where $\underline{t}_{ij}^{\lambda}$ and $\bar{t}_{ij}^{\lambda}$ are the ends of the interval $\tilde{T}_{ij}^{\lambda} = [\underline{t}_{ij}^{\lambda}, \bar{t}_{ij}^{\lambda}]$ (the λ-cut of $\tilde{T}_{ij}$).

Hence, determination of the criticality degree of path $p \in P$, $\mu_{\tilde{P}}(p)$, can be reduced, according to Theorem 10, to the following mathematical programming problem:

$$
\begin{aligned}
&\lambda \to \max,\\
&t_j - t_i - \bar{t}_{ij}^{\lambda} = 0, \quad (i,j) \in p,\\
&t_j - t_i - \underline{t}_{ij}^{\lambda} \geqslant 0, \quad (i,j) \notin p,\\
&t_1 = 0,\\
&t_i \geqslant 0, \quad (i = 2,\dots,n),
\end{aligned}
\tag{19}
$$

where $\lambda \in (0,1]$.

If $\lambda_{\max}$ is the optimal objective value of (19) then $\mu_{\tilde{P}}(p) = \lambda_{\max}$.

Let us assume that fuzzy activities duration times $\tilde{T}_{ij}$, $(i,j) \in A$, are given by means of fuzzy numbers of the same L–L type (see Definition 7), i.e. $\tilde{T}_{ij} = (\underline{t}_{ij}, \bar{t}_{ij}, \alpha_{ij}, \beta_{ij})_{L-L}$. In this case λ-cuts of a fuzzy number $\tilde{T}_{ij}$ have the form

$$\tilde{T}_{ij}^{\lambda} = [\underline{t}_{ij} - L^{-1}(\lambda)\alpha_{ij}, \bar{t}_{ij} + L^{-1}(\lambda)\beta_{ij}]$$

and problem (19) reduces itself to the following linear programming problem:

$$
\begin{aligned}
&\Theta \to \min\\
&t_j - t_i - \bar{t}_{ij} - \beta_{ij}\Theta = 0 \quad (i,j) \in p,\\
&t_j - t_i - \underline{t}_{ij} + \alpha_{ij}\Theta \geqslant 0 \quad (i,j) \notin p,\\
&t_1 = 0,\\
&t_i \geqslant 0 \quad (i = 2,\dots,n),
\end{aligned}
\tag{20}
$$

where $\Theta \in [\underline{\Theta}, \bar{\Theta})$, $\Theta = L^{-1}(\lambda)$, $\underline{\Theta} = L^{-1}(1)$, $\bar{\Theta} = L^{-1}(0)$. If $\Theta_{\min}$ is the optimal objective value of (20) then the path degree of criticality, $\mu_{\tilde{P}}(p)$, is equal to $L(\theta_{\min})$.

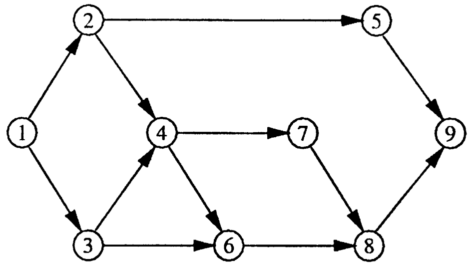

Fig. 1. The structure of a project in the Examples 1, 2.

4.4. Examples

Let us illustrate the utility of the methods of calculation of the path degrees of criticality with simple numerical examples.

Example 1. The network representing a structure of a project is given in Fig. 1. The activities duration times are fuzzy numbers of L_{ij}–R_{ij} type, $(i,j) \in A$:

$$
\begin{aligned}
&\tilde{T}_{12} = (1, 1.5, 1, 1)_{L_{12}-R_{12}}, && \tilde{T}_{13} = (2, 3, 0, 2)_{L_{13}-R_{13}},\\
&\tilde{T}_{24} = (0, 0, 0, 0)_{L_{24}-R_{24}}, && \tilde{T}_{25} = (2, 3, 1, 2)_{L_{25}-R_{25}},\\
&\tilde{T}_{34} = (0, 0, 0, 0)_{L_{34}-R_{34}}, && \tilde{T}_{36} = (6, 7, 0, 2)_{L_{36}-R_{36}},\\
&\tilde{T}_{46} = (5, 5, 1, 1)_{L_{46}-R_{46}}, && \tilde{T}_{47} = (9, 9, 1, 1)_{L_{47}-R_{47}},\\
&\tilde{T}_{59} = (8, 9, 2, 4)_{L_{59}-R_{59}}, && \tilde{T}_{68} = (4, 4, 2, 2)_{L_{68}-R_{68}},\\
&\tilde{T}_{78} = (3, 4, 2, 0)_{L_{78}-R_{78}}, && \tilde{T}_{89} = (6, 9, 2, 3)_{L_{89}-R_{89}},
\end{aligned}
$$

where $L_{12}(x) = L_{68}(x) = L_{89}(x) = \max(1 - x^2, 0)$, $L_{13}(x) = \mathrm{e}^{-x}$, $L_{24}(x) = L_{34}(x) = L_{46}(x) = L_{78}(x) = \max(0, 1 - x)$, $L_{25}(x) = L_{47}(x) = L_{59}(x) = \max(0, 1 - x^4)$, $L_{36}(x) = \mathrm{e}^{-x^2}$, $R_{12}(x) = R_{13}(x) = R_{24}(x) = \max(0, 1 - x)$, $R_{34}(x) = R_{36}(x) = R_{59}(x) = \max(1 - x^2, 0)$, $R_{46}(x) = R_{68}(x) = R_{78}(x) = \max(0, 1 - x^4)$, $R_{25}(x) = R_{47}(x) = \mathrm{e}^{-x}$, $R_{89}(x) = \mathrm{e}^{-x^2}$.

Applying Algorithm 1 we have obtained the results which are listed in Table 1. The path degrees of criticality have been computed with accuracy 10^{-4}.

Example 2. The network representing a structure of a project is similar to that in Example 1 (Fig. 1). The

Table 1
The path degrees of criticality in Example 1

$p \in P$	$\mu_{\tilde{P}}(p)$
1-2-5-9	0.6269
1-2-4-7-8-9	0.5001
1-2-4-6-8-9	0.3854
1-3-4-7-8-9	1
1-3-4-6-8-9	0.0001
1-3-6-8-9	0.9941

Table 2
The path degrees of criticality in Example 2

$p \in P$	$\mu_{\tilde{P}}(p)$	$\Theta_{\min}$
1-2-5-9	0.7024	0.5455
1-2-4-7-8-9	0.75	0.5
1-2-4-6-8-9	0.4375	0.75
1-3-4-7-8-9	1	0
1-3-4-6-8-9	0	1
1-3-6-8-9	0.9796	0.1429

activities duration times are fuzzy numbers of the same L–L type, where $L(x) = \max(0, 1 - x^2)$:

$$\tilde{T}_{12} = (1, 1.5, 1, 1)_{L-L}, \quad \tilde{T}_{13} = (2, 3, 0, 2)_{L-L},$$
$$\tilde{T}_{24} = (0, 0, 0, 0)_{L-L}, \quad \tilde{T}_{25} = (2, 3, 1, 2)_{L-L},$$
$$\tilde{T}_{34} = (0, 0, 0, 0)_{L-L}, \quad \tilde{T}_{36} = (6, 7, 0, 2)_{L-L},$$
$$\tilde{T}_{46} = (5, 5, 1, 1)_{L-L}, \quad \tilde{T}_{47} = (9, 9, 1, 1)_{L-L},$$
$$\tilde{T}_{59} = (8, 9, 2, 4)_{L-L}, \quad \tilde{T}_{68} = (4, 4, 2, 2)_{L-L},$$
$$\tilde{T}_{78} = (3, 4, 2, 0)_{L-L}, \quad \tilde{T}_{89} = (6, 9, 2, 3)_{L-L}.$$

In this case we have determined the degree of criticality solving for each paths $p \in P$ the linear programming problem (20). The results are listed in Table 2.

The following relation is valid between the values set in the second and third column in Table 2: $\mu_{\tilde{P}}(p) = 1 - \Theta_{\min}^2$.

5. Final remarks

The presented approach to the criticality concept in a network with fuzzy activity times is devoid of faults which are characteristic for the definitions of fuzzy criticality proposed till now in the literature. It is so owing to direct application of the extension principle of Zadeh to the classical criticality notion treated as a function of activity duration times in the network. Though Definition 9 of fuzzy critical path seems to be very complicated it is possible to calculate effectively the exact (corresponding to formula (7)) value of criticality degree for a given path. In the paper two efficient methods of calculation of the path degree of criticality (according to the proposed concept of fuzzy criticality) have been proposed. The first one is adapted to fuzzy activity times given in general form and the second one, based on linear programming, is true only for fuzzy activity times determined in the network by means of fuzzy numbers of the same L–L type. There is no such method for calculating the criticality degree of an activity (an event). This problem is NP-hard since it includes, as a very special case, the problem of asserting the i-criticality of a fixed activity (a fixed event) in a network which is, as shown in [5], NP-complete in the strong sense. The exact value of criticality degree of an activity (event) may be calculated efficiently only if the number of paths crossing the activity (event) is not great. In this case, it is enough to apply Theorem 8 (9) and calculate the activity (event) degree by criticality degrees of these paths.

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