Unfuzzy non dominated solutions in the linear programming problem with fuzzy coefficients in the objective function

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Abstract:

In the paper the linear programming problem with fuzzy coefficients in the objective function is analyzed. It is assumed that the solution of the problem is the set of non dominated solutions, according to the concept of Orlovsky, with respect to a preference relation defined on the set of fuzzy numbers. The main result of the paper is a characterization, useful from the practical point of view, of the set of unfuzzy non dominated solutions with respect to six different fuzzy preference relations. In each case it has been shown that the unfuzzy non dominated solutions coincide either with the feasible solutions of a linear inequalities system or with the optimal solutions of a classical linear programming problem. They can thus be determined by means of classical linear programming methods.

Moreover, in the paper several properties of the considered relations are given. They are needed in the proofs of the main theorems of the paper, but are interesting per se and can be used also for other purposes.

Keywords:

linear programming, fuzzy programming, fuzzy preference relation, non dominated solution.

1 Introduction

In the paper we consider the linear programming problem in which the coefficients in the objective function are given in the imprecise form of fuzzy numbers. In this case the objective function value for a given feasible solution, a value constituting an evaluation of this solution, is also a fuzzy number. For this reason various concepts of the solution of such a problem are possible. It is not the aim of this paper to present a full survey of them.

For example, those presented in [2] and [8] are among the newer ones. However, those two papers do not of course exhaust all proposals of the solution of the problem put forward in the literature so far. Among the older ones there is a very interesting concept presented by Orlovsky in [7], in which the author makes use of an approach to the decision making (selection of the best solution), based on a fuzzy preference relation determined on the set of feasible solutions, which he himself proposed considerably earlier in [6]. It is the concept of Orlovsky we assume in this paper to be the valid definition of the solution of the problem. Let us suppose that there is a fuzzy preference relation defined on the set of fuzzy numbers and determining a fuzzy order in this set. According to the concept of Orlovsky, the solution of the problem is the fuzzy set of non dominated solutions in the set of feasible solutions. For a given feasible solution, the value of the membership function of this set means the degree to which one can be sure that the fuzzy objective function value corresponding to this solution is not dominated by the objective function values for the other solutions with respect to the given preference relation. The decision maker can be recommended to select as his final choice the solution that has the maximal (greatest) membership degree in the fuzzy set of non dominated solutions. Very interesting from the practical point of view are those from among feasible solutions that are non dominated to the degree one — of course if with the assumed fuzzy preference relation such solutions exist. In the terminology introduced by Orlovsky the set of such solutions is called the set of unfuzzy non dominated solutions.

In the paper we give a characterization, useful from the practical point of view, of the set of unfuzzy non dominated solutions of the linear programming problem with fuzzy coefficients in the objective function for six different fuzzy preference relations considered in [1]. As it has been shown, in each case the unfuzzy non dominated solutions are either the feasible solutions of a linear inequalities system or optimal solutions of a certain linear programming problem with crisp coefficients. They can thus be determined by means of classical linear programming methods.

The structure of this paper is as follows: in section 2 we remind, for the convenience of the reader, some basic notions from the arithmetic of fuzzy interval numbers used in the paper. In section 3 we formulate the linear programming problem with fuzzy coefficients in the objective function and give the definitions of the solution of this problem according to the Orlovsky concept, emphasizing the concept of unfuzzy non dominated solutions, which in this paper constitutes the main object of consideration. In the next section we quote from [1] the definitions of six fuzzy ordering relations on the set of fuzzy interval numbers, proving at the same time several new properties of theirs, used further in the paper. In section 5, the essential one of the paper, we prove a series of theorems characterizing the sets of unfuzzy non dominated solutions with respect to the fuzzy preference relations described in the previous section. To put it another way, these theorems determine constructive necessary and sufficient conditions for a given feasible solution of the problem to be an unfuzzy non dominated solution according to the considered fuzzy preference relations. We finish the paper with a short summary.

2 Selected notions and symbols of the arithmetic of fuzzy interval numbers

We remind here only those concepts and elements of notation from the arithmetic of fuzzy interval numbers which we use in the paper.

Definition 1. *A fuzzy number of the $L-R$ type is defined as a fuzzy interval A (a fuzzy set in the space of real numbers) with the membership function of the following form:*

$$\mu_A(z) = \begin{cases} 1 & for \quad z \in [\underline{a}, \overline{a}], \\ L\left(\frac{\underline{a}-z}{\alpha}\right) & for \quad z \leq \underline{a}, \\ R\left(\frac{z-\overline{a}}{\beta}\right) & for \quad z \geq \overline{a}, \end{cases} \tag{1}$$

where L (also R) is a so called shape function, i.e. a function fulfilling the conditions: L is a continuous non-increasing function on the half-line $[0, \infty)$, $L(0) = 1$ and L is strictly decreasing on this part of the half-line $[0, \infty)$, on which it is positive. The parameters α and β are non-negative real numbers.

We will use the following notation — introduced by Dubois and Prade [3] for a fuzzy number A of the $L - R$ type with the membership function given by formula (1):

$$A = (\underline{a}, \overline{a}, \alpha, \beta)_{L-R}. \tag{2}$$

In the paper we will use the generalized (according to the extension principle of Zadeh) operations of addition of two fuzzy numbers and of multiplication of a fuzzy number by a scalar. We remind the definition of these operations.

Definition 2. *Let A and B be two fuzzy numbers and r a non-negative real number. Then:*

(a) $A + B$ is a fuzzy number with membership function

$$\mu_{A+B}(z) = \sup_{\substack{x,y \\ x+y=z}} min\{\mu_A(x), \mu_B(y)\}, \qquad z \in \mathcal{R}, \tag{3}$$

(b) for $r > 0$ rA is a fuzzy number with membership function

$$\mu_{rA}(z) = \mu_A(z/r), \qquad z \in \mathcal{R}, \tag{4}$$

(c) for $r = 0$ rA is equal to zero, i.e. $\mu_{rA}(z) = 1$ for $z = 0$ and $\mu_{rA}(z) = 0$ for $z \neq 0$.

When A and B are fuzzy numbers of the same type $L - R$ and r is a non-negative real number ($A = (\underline{a}, \overline{a}, \alpha_A, \beta_A)_{L-R}$, $B = (\underline{b}, \overline{b}, \alpha_B, \beta_B)_{L-R}$), $A + B$ and rA are also fuzzy numbers of this type:

$$A + B = (\underline{a} + \underline{b}, \overline{a} + \overline{b}, \alpha_A + \alpha_B, \beta_A + \beta_B)_{L-R}, \tag{5}$$

$$rA = (r\underline{a}, r\overline{a}, r\alpha_A, r\beta_A)_{L-R}. \tag{6}$$

3 Formulation of the problem

We consider the following linear programming problem with fuzzy coefficients in the objective function:

$$F(x) = \sum_{j=1}^{n} C_j x_j \longrightarrow max,$$

$$x \in X : \begin{cases} \sum_{j=1}^{n} a_{ij} x_j = b_i, & i = 1, \ldots, m, \\ x_j \geq 0, & j = 1, \ldots, n, \end{cases} \tag{7}$$

where $C_j = (\underline{c}_j, \overline{c}_j, \alpha_j, \beta_j)_{L-R}$, $j = 1, \ldots, n$, are fuzzy numbers of the same type $L - R$.

The operations used in the objective function of problem (7) are of course the generalized operations determined in definition 2: the addition of two fuzzy numbers and the multiplication of a fuzzy number by a scalar. From the properties of these operations (see (5) and (6)) and from the fact that $x \geq 0$ it follows that $F(x)$ is a fuzzy number of the type $L - R$:

$$\begin{aligned} F(x) &= (\underline{f}(x), \overline{f}(x), \alpha_{F(x)}, \beta_{F(x)})_{L-R}, \\ where: & \\ \underline{f}(x) &= \sum_{j=1}^{n} \underline{c}_j x_j, \qquad \overline{f}(x) = \sum_{j=1}^{n} \overline{c}_j x_j, \\ \alpha_{F(x)} &= \sum_{j=1}^{n} \alpha_j x_j, \qquad \beta_{F(x)} = \sum_{j=1}^{n} \beta_j x_j. \end{aligned} \tag{8}$$

Now we will formulate a definition of the solution of problem (7) which is an direct consequence of the Orlovsky approach the decision making on the basis of a fuzzy preference relation [6] (see also [7]).

Let us denote with $FN(L, R)$ the set of all fuzzy numbers of the type $L - R$ (L and R are shape functions identical to those from problem (7)). Let us assume that there is a fuzzy preference (ordering) relation R defined on this set with the membership function $\mu_R : FN(L, R)^2 \to [0, 1]$. The value $\mu_R(A, B)$, $A, B \in FN(L, R)$, stands for the degree to which a fuzzy number A is preferred (greater) with respect to number B.

Definition 3. *Fuzzy solution of problem (7) is the fuzzy set ND of the non dominated elements in X with respect to the fuzzy relation R, i.e. the fuzzy set ND with the following membership function:*

$$\mu_{ND}(x) = \inf_{y \in X} [1 - \mu_{R^s}(F(y), F(x))] = 1 - \sup_{y \in X} \mu_{R^s}(F(y), F(x)), \tag{9}$$

where R^s is the fuzzy relation of strict preference linked to relation R. According to the Orlovsky concept, the membership function of the relation R^s has the following form:

$$\mu_{R^s}(A, B) = max\{0, \mu_R(A, B) - \mu_R(B, A)\}. \tag{10}$$

The value of the membership function $\mu_{ND}(x)$ stands for the degree to which the vector x is not dominated by any other vector from the set X (the degree of truth of the statement "x is not dominated by any other vector $y \in X$").

Definition 4. *A deterministic solution of problem (7) is the vector $x \in X$ for which function $\mu_{ND}(x)$ assumes the greatest value (a solution which,* *with respect to relation R, is non dominated by the other solutions to the maximal degree), i.e. a solution of the following mathematical programming problem:*

$$\begin{aligned} &\mu_{ND}(x) \longrightarrow max, \\ &x \in X. \end{aligned} \tag{11}$$

Problem (11), occurring in definition 4, is equivalent to the following mathematical programming problem:

$$\begin{aligned} &\sup_{y \in X} \mu_{R^s}(F(y), F(x)) \longrightarrow min, \\ &x \in X. \end{aligned} \tag{12}$$

It is worth noticing that in both problems ((11) and (12)) the set of feasible solutions is determined by linear conditions, but the objective functions are not linear. Both problems are thus non-linear programming problems with linear constraints.

The following definition of the unfuzzy non dominated solution is of vital importance for this paper:

Definition 5. *If for a maximally non dominated solution x (i.e. an optimal solution according to definition 4) the value of the membership function $\mu_{ND}(x)$ is equal to 1, $\mu_{ND}(x) = 1$, then x is called an unfuzzy non dominated solution. The set of unfuzzy non dominated solutions will be denoted by UND.*

The unfuzzy non dominated solutions - if such solutions exist, i.e. if $UND \neq \emptyset$ - seem to be of great importance from the practical point of view. They are solutions for which as if there were no doubts that they are the best ones - as the statement that they are not dominated by any other solution from the set of feasible solutions is true to the degree 1.

A direct consequence of formula (9) and definitions 4 and 5 is the following theorem, which we will use in section 5 of the paper.

Theorem 1. *A vector $x \in X$ is a unfuzzy non dominated solution with respect to a fuzzy relation R, i.e. $x \in UND$, if and only if for each $y \in X$ the following formula holds:*

$$\mu_{R^s}(F(y), F(x)) = 0. \tag{13}$$

4 Fuzzy preference relations

Let us quote from [1] the definitions of six fuzzy preference relations, defined on the set of fuzzy numbers, from among eight ones introduced there. We will not justify those definitions, referring the interested reader to the source paper [1]. However, we will prove some properties of theirs, which have not been included in [1] and which we will need in the following section.

We will consider six fuzzy preference relations R_i, $i = 1, 2, \ldots, 6$ defined on the set $FN(L, R)$. The membership functions of those relations, $\mu_i : FN(L,R)^2 \to [0,1]$, $i = 1, 2, \ldots, 6$, are given by the corresponding formulae (14)-(19):

$$\begin{aligned}
\mu_1(A,B) &= Prob\{\underline{a} - L^{-1}(Y)\alpha_A > \overline{b} + R^{-1}(Y)\beta_B\}, & (14)\\
\mu_2(A,B) &= Prob\{\underline{a} - L^{-1}(Y)\alpha_A > \overline{b} + R^{-1}(Z)\beta_B\}, & (15)\\
\mu_3(A,B) &= Prob\{\underline{a} - L^{-1}(Y)\alpha_A \geq \underline{b} - L^{-1}(Y)\alpha_B\}, & (16)\\
\mu_4(A,B) &= Prob\{\overline{a} + R^{-1}(Y)\beta_A > \overline{b} + R^{-1}(Y)\beta_B\}, & (17)\\
\mu_5(A,B) &= Prob\{\overline{a} + R^{-1}(Y)\beta_A \geq \underline{b} - L^{-1}(Y)\alpha_B\}, & (18)\\
\mu_6(A,B) &= Prob\{\overline{a} + R^{-1}(Z)\beta_A \geq \underline{b} - L^{-1}(Y)\alpha_B\}, & (19)
\end{aligned}$$

where $A, B \in FN(L, R)$ $(A = (\underline{a}, \overline{a}, \alpha_A, \beta_A)_{L-R},\ B = (\underline{b}, \overline{b}, \alpha_B, \beta_B)_{L-R})$ and the symbols Y and Z stand for independent random variables with the uniform distribution on the interval [0,1].

The membership functions of two from among the above six fuzzy preference relations, i.e. functions $\mu_1(A,B)$ and $\mu_5(A,B)$, are respectively equivalent to the indices of necessity and possibility introduced by Dubois and Prade [4].

The lemmas listed below contain the properties of relations (14)-(19) used in the following section, section 5, which constitutes the essential part of the paper.

Lemma 1. *Let $A, B \in FN(L, R)$ be two arbitrary fuzzy numbers of the type $L-R$. For both fuzzy relations R_i, $i = 1, 2$ the following condition is fulfilled:*

$$\mu_i(A,B) > 0 \Longrightarrow \underline{a} > \overline{b} \quad (\textit{or equivalently} \quad \underline{a} \leq \overline{b} \Longrightarrow \mu_i(A,B) = 0). \quad (20)$$

Proof. Obvious. Condition (20) follows directly from formulae (14), (15) and the properties of the shape function of a fuzzy number (it holds namely $L^{-1}(Y) \geq 0$ and $R^{-1}(Y) \geq 0$).

□

The following lemma is a consequence of Lemma 1:

Lemma 2. *Relation R_i, $i = 1, 2$ is antisymmetric, i.e. for any $A, B \in FN(L, R)$ the following condition is fulfilled:*

$$\mu_i(A, B) > 0 \Longrightarrow \mu_i(B, A) = 0. \tag{21}$$

Proof. Let $\mu_i(A, B) > 0$, $i = 1, 2$. From Lemma 1 we get $\underline{a} > \overline{b}$. From this inequality and from the properties of fuzzy numbers (it holds $\underline{a} \leq \overline{a}$ and $\underline{b} \leq \overline{b}$) we obtain the inequality $\underline{b} \leq \overline{a}$, from which, after applying Lemma 1, we come to the equality $\mu_i(B, A) = 0$.

□

Lemma 3. *For any two fuzzy numbers $A, B \in FN(L, R)$ the following conditions are fulfilled:*

$$\underline{a} - \underline{b} - L^{-1}(1/2)(\alpha_A - \alpha_B) \geq 0 \iff \mu_3(A, B) \geq 1/2, \tag{22}$$
$$\underline{a} - \underline{b} - L^{-1}(1/2)(\alpha_A - \alpha_B) > 0 \implies \mu_3(A, B) > 1/2. \tag{23}$$

Proof. Making use of formula (16) and the fact that L^{-1} is a strictly decreasing function on the interval [0,1], the following formula for calculating the value of the membership function $\mu_3(A, B)$ can be derived:

$$\mu_3(A, B) = \begin{cases} L\left(\frac{\underline{a}-\underline{b}}{\alpha_A - \alpha_B}\right) & for \quad \alpha_A < \alpha_B \quad and \quad \underline{a} < \underline{b}, \\ 1 & for \quad \alpha_A \leq \alpha_B \quad and \quad \underline{a} \geq \underline{b}, \\ 1 - L\left(\frac{\underline{a}-\underline{b}}{\alpha_A - \alpha_B}\right) & for \quad \alpha_A > \alpha_B \quad and \quad \underline{a} \geq \underline{b}, \\ 0 & for \quad \alpha_A \geq \alpha_B \quad and \quad \underline{a} < \underline{b}. \end{cases} \tag{24}$$

For its part, this formula allows to justify easily conditions (22) and (23).

□

Remark: Implication (23) is satisfied also in the other direction, but only for $A, B \in FN(L, R)$ fulfilling the condition $\underline{a} \neq \underline{b}$ or $\alpha_A \neq \alpha_B$. In the case $\underline{a} = \underline{b}$ and $\alpha_A = \alpha_B$ we have $\mu_3(A, B) = 1 > 1/2$ but $\underline{a} - \underline{b} - L^{-1}(1/2)(\alpha_A - \alpha_B) = 0$.

Making use also of formula (24), it is easy to prove the following lemma:

Lemma 4. *For any two fuzzy numbers $A, B \in FN(L,R)$ such that $\underline{a} \neq \underline{b}$ or $\alpha_A \neq \alpha_B$ the following condition is fulfilled:*

$$\mu_3(A,B) + \mu_3(B,A) = 1. \tag{25}$$

Two following lemmas concern relation R_4.

Lemma 5. *For any two fuzzy numbers $A, B \in FN(L,R)$ the following conditions are fulfilled:*

$$\overline{a} - \overline{b} + R^{-1}(1/2)(\beta_A - \beta_B) \geq 0 \iff \mu_4(A,B) \geq 1/2, \tag{26}$$

$$\overline{a} - \overline{b} + R^{-1}(1/2)(\beta_A - \beta_B) > 0 \iff \mu_4(A,B) > 1/2. \tag{27}$$

Proof. Making use of formula (17) and the properties of function R^{-1}, a formula similar to (24) for the values of the membership function μ_4 can be derived:

$$\mu_4(A,B) = \begin{cases} R\left(\frac{\overline{b}-\overline{a}}{\beta_A - \beta_B}\right) & for \quad \beta_A > \beta_B \quad and \quad \overline{a} \leq \overline{b}, \\ 1 & for \quad \beta_A \geq \beta_B \quad and \quad \overline{a} > \overline{b}, \\ 1 - R\left(\frac{\overline{b}-\overline{a}}{\beta_A - \beta_B}\right) & for \quad \beta_A < \beta_B \quad and \quad \overline{a} > \overline{b}, \\ 0 & for \quad \beta_A \leq \beta_B \quad and \quad \overline{a} \leq \overline{b}. \end{cases} \tag{28}$$

Making use of (28), it is easy to justify conditions (26) and (27).

□

Making use also of formula (28), the following lemma can be proved:

Lemma 6. *For any two fuzzy numbers $A, B \in FN(L,R)$ such that $\beta_A \neq \beta_B$ or $\overline{a} \neq \overline{b}$ the following condition is fulfilled:*

$$\mu_4(A,B) + \mu_4(B,A) = 1. \tag{29}$$

Lemma 7. *Let $A, B \in FN(L,R)$ be two arbitrary fuzzy numbers of the type $L - R$. For both fuzzy relations R_i, $i = 5, 6$, the following condition is fulfilled:*

$$\mu_i(A,B) < 1 \iff \overline{a} < \underline{b} \quad (\text{or equivalently} \quad \overline{a} \geq \underline{b} \iff \mu_i(A,B) = 1). \tag{30}$$

Proof. Obvious. Condition (30) follows directly from formulae (18), (19) and the properties of the shape function ($L^{-1}(Y) \geq 0$ and $R^{-1}(Y) \geq 0$).

□

Lemma 8. *Both relations R_i, $i = 5, 6$ fulfill the condition*

$$\mu_i(A, B) < 1 \Longrightarrow \mu_i(B, A) = 1. \tag{31}$$

Proof. Let us assume that $\mu_i(A, B) < 1$. From Lemma 7 we obtain condition $\overline{a} < \underline{b}$. From it and the properties of fuzzy numbers (it holds $\underline{a} \leq \overline{a}$ and $\underline{b} \leq \overline{b}$) we get the inequality $\overline{b} > \underline{a}$, from which, after applying Lemma 7 again, we obtain condition $\mu_i(B, A) = 1$.

□

5 A characterization of the set of unfuzzy non dominated solutions.

The first of the theorems of this section, theorem 2, is a very useful characterization, from the practical point of view, of the set of unfuzzy non dominated solutions of problem (7) with respect to relations R_1 and R_2. Before formulating this theorem we will present a simple lemma needed in its proof.

Lemma 9. *For any antisymmetric fuzzy relation R it holds $R^s = R$, i.e. $\mu_{R^s}(A, B) = \mu_R(A, B)$ for any A, B.*

Proof.

1. Let us assume that $\mu_R(A, B) > 0$. Then

$$\begin{aligned} \mu_{R^s}(A, B) &= max\{0, \mu_R(A, B) - \mu_R(B, A)\} \\ &= max\{0, \mu_R(A, B)\} \\ &= \mu_R(A, B), \end{aligned} \tag{32}$$

which follows from the antisymmetry of the relation R, as $\mu_R(B, A) = 0$.

2. Let us assume that $\mu_R(A,B)=0$. Then

$$\begin{aligned}\mu_{R^s}(A,B) &= max\{0,\mu_R(A,B)-\mu_R(B,A)\} \\ &= max\{0,-\mu_R(B,A)\} \\ &= 0=\mu_R(A,B),\end{aligned} \tag{33}$$

because it always holds $\mu_R(B,A) \geq 0$, from where it follows $-\mu_R(B,A)\leq 0$.

□

Theorem 2. *For the fuzzy relations R_i, $i=1,2$, the following statement is true:*

$$x \in UND \Longleftrightarrow \overline{f}(x)=\sum_{j=1}^{n}\overline{c}_j x_j \geq \underline{f}_{max}, \tag{34}$$

where

$$\underline{f}_{max}=\max_{y\in X}\underline{f}(y)=\sum_{j=1}^{n}\underline{c}_j y_j. \tag{35}$$

Proof.

1. $\Longrightarrow$ We will conduct an indirect proof. Let us assume that $x\in UND$, i.e. x is an unfuzzy non dominated solution of problem (7) with respect to relation R_i, $i=1,2$, and $\overline{f}(x)<\underline{f}_{max}$. Let $y\in X$ be a vector such that $\underline{f}(y)=\underline{f}_{max}$ (such a vector exists - it follows from the properties of the linear programming problem). From the definitions of relations R_1, R_2 (formulae (14), (15)) and lemmas 2, 9 it follows that

$$\begin{aligned}&\mu_{R_1^s}(F(y),F(x))=\mu_1(F(y),F(x))= \\ &=Prob\{\underline{f}(y)-\overline{f}(x)>L^{-1}(Y)\alpha_{F(y)}+R^{-1}(Y)\beta_{F(x)}\},\end{aligned} \tag{36}$$

$$\begin{aligned}&\mu_{R_2^s}(F(y),F(x))=\mu_2(F(y),F(x))= \\ &=Prob\{\underline{f}(y)-\overline{f}(x)>L^{-1}(Y)\alpha_{F(y)}+R^{-1}(Z)\beta_{F(x)}\},\end{aligned} \tag{37}$$

where Y and Z are independent random variables with the uniform distribution on the interval [0,1]. From the assumption of the indirect proof $\overline{f}(x)<\underline{f}_{max}$ and from the definition of vector y it follows that

$$\underline{f}(y)-\overline{f}(x)>0. \tag{38}$$

The following facts follow from the properties of the shape functions:

(a) function $L^{-1}(t)\alpha_{F(y)} + R^{-1}(t)\beta_{F(x)}$, $t \in [0,1]$, is continuous, strictly decreasing and positive, it attains 0 in $t = 1$;

(b) function $L^{-1}(t)\alpha_{F(y)} + R^{-1}(z)\beta_{F(x)}$, $t \in [0,1]^2$, is continuous, strictly decreasing with respect to each variable, it attains 0 in (1,1).

From statements (36), (37), inequality (38) and facts (a), (b) it follows that $\mu_1(F(y), F(x)) > 0$, $\mu_2(F(y), F(x)) > 0$ which, according to theorem 1, contradicts the fact $x \in UND$.

2. $\Longleftarrow$ Let us assume that $x \in X$ fulfills the condition $\overline{f}(x) \geq \underline{f}_{max}$. From the definition of the quantity $\underline{f}_{max}$ (see (35)) it follows that for each $y \in X$, $\underline{f}(y) \leq \underline{f}_{max}$. From it and from the assumption $\overline{f}(x) \geq \underline{f}_{max}$ it follows that

$$\underline{f}(y) \leq \overline{f}(x), \quad y \in X. \tag{39}$$

Subsequently, making use of the inequality (39) and lemmas 1, 2, 9 we obtain

$$\mu_i(F(y), F(x)) = \mu_{R_i^s}(F(y), F(x)) = 0, \quad y \in X. \tag{40}$$

Condition (40) and theorem 1 imply that $x \in UND$.

□

Now we will give a useful characterization of the set of unfuzzy non dominated solutions of problem (7) with respect to relation R_3. Indeed, the following theorem implies that the unfuzzy non dominated solutions of problem (7) with respect to the fuzzy relation R_3 are optimal solutions of a certain linear programming problem — function $f(y) - L^{-1}(1/2)\alpha_{F(y)}$ is linear with respect to vector $y \in X$ (see (8)).

Theorem 3. *For the fuzzy relation R_3 the following equivalence holds*

$$x \in UND \Longleftrightarrow \underline{f}(x) - L^{-1}(1/2)\alpha_{F(x)} = \max_{y \in X}(\underline{f}(y) - L^{-1}(1/2)\alpha_{F(y)}). \tag{41}$$

Proof.

1. $\Longrightarrow$ We will conduct an indirect proof. Let us assume that $x \in UND$ and at the same time x is not an optimal solution of the linear programming problem from condition (41). Hence, there exists a vector $y \in X$ such that

$$\underline{f}(y) - L^{-1}(1/2)\alpha_{F(y)} > \underline{f}(x) - L^{-1}(1/2)\alpha_{F(x)}. \tag{42}$$

From condition (42) and lemma 3 (condition (23)) it follows that:

$$\mu_3(F(y), F(x)) > 1/2. \tag{43}$$

(42) implies that the following condition is satisfied

$$\underline{f}(y) \neq \underline{f}(x) \quad or \quad \alpha_{F(y)} \neq \alpha_{F(x)}. \tag{44}$$

So, making use of lemma 4 and inequality (43) we obtain

$$\mu_3(F(x), F(y)) < 1/2. \tag{45}$$

From conditions (43), (45) we have

$$\mu_{R_3^s}(F(y), F(x)) = max\{0, \mu_3(F(y), F(x)) - \mu_3(F(x), F(y))\} > 0, \tag{46}$$

which contradicts the conclusion of theorem 1.

2. $\Longleftarrow$ Let us assume that $x \in X$ fulfills the condition

$$\underline{f}(x) - L^{-1}(1/2)\alpha_{F(x)} = \max_{y \in X}(\underline{f}(y) - L^{-1}(1/2)\alpha_{F(y)}). \tag{47}$$

We will show that $x \in UND$. In order to do so, it is enough to show, according to theorem 1, that for each $y \in X$ the following condition holds:

$$\mu_{R_3^s}(F(y), F(x)) = 0. \tag{48}$$

(47) implies that for each $y \in X$ the following inequality is fulfilled:

$$\underline{f}(x) - L^{-1}(1/2)\alpha_{F(x)} \geq \underline{f}(y) - L^{-1}(1/2)\alpha_{F(y)}. \tag{49}$$

Making use of (49) and lemma 3 (condition (22)) we obtain the following condition

$$\mu_3(F(x), F(y)) \geq 1/2, \quad y \in X. \tag{50}$$

Condition (50) and lemma 4 imply that for each $y \in X$ satisfying condition (44) the following inequality holds

$$\mu_3(F(y), F(x)) \leq 1/2. \tag{51}$$

From inequalities (50), (51) and the definition of the relation of strict preference (see (10)) it follows that for $y \in X$ satisfying condition (44) it holds:

$$\mu_{R_3^s}(F(y), F(x)) = max\{0, \mu_3(F(y), F(x)) - \mu_3(F(x), F(y))\} = 0. \tag{52}$$

Condition (52) holds true also for those $y \in X$ which do not satisfy condition (44). It is so, because then, since $\underline{f}(y) = \underline{f}(x)$ and $\alpha_{F(y)} = \alpha_{F(x)}$, the definition of relation R_3 (see formula (16)) implies $\mu_3(F(y), F(x)) = \mu_3(F(x), F(y)) = 1$.

□

We will show that also in the case of fuzzy preference relation R_4 the unfuzzy non dominated solutions are optimal solutions of a certain linear programming problem.

Theorem 4. *For fuzzy preference relation R_4 the following equivalence holds*

$$x \in UND \Longleftrightarrow \overline{f}(x) + R^{-1}(1/2)\beta_{F(x)} = \max_{y \in X}(\overline{f}(y) + R^{-1}(1/2)\beta_{F(y)}). \tag{53}$$

Proof.

1. $\Longrightarrow$ We will conduct an indirect proof. Let us assume that $x \in UND$ and at the same time there exists $y \in X$ such that:

$$\overline{f}(y) + R^{-1}(1/2)\beta_{F(y)} > \overline{f}(x) + R^{-1}(1/2)\beta_{F(x)}. \tag{54}$$

From inequality (54) and lemma 5 (condition (27)) it follows that

$$\mu_4(F(y), F(x)) > 1/2. \tag{55}$$

(54) implies that the following condition is satisfied

$$\overline{f}(y) \neq \overline{f}(x) \quad or \quad \beta_{F(y)} \neq \beta_{F(x)}. \tag{56}$$

Thus, making use of lemma 6 and inequality (55) we obtain

$$\mu_4(F(x), F(y)) < 1/2. \tag{57}$$

From conditions (55), (57) we get the following inequality

$$\mu_{R_4^s}(F(y), F(x)) = max\{0, \mu_4(F(y), F(x)) - \mu_4(F(x), F(y))\} > 0, \tag{58}$$

which contradicts theorem 1.

2. $\Longleftarrow$ Let us assume that $x \in X$ satisfies the condition

$$\overline{f}(x) + R^{-1}(1/2)\beta_{F(x)} = \max_{y \in X}(\overline{f}(y) + R^{-1}(1/2)\beta_{F(y)}). \tag{59}$$

(59) implies that for each $y \in X$ the following inequality is satisfied:

$$\overline{f}(x) + R^{-1}(1/2)\beta_{F(x)} \geq \overline{f}(y) + R^{-1}(1/2)\beta_{F(y)}. \tag{60}$$

Making use of (60) and lemma 5 (condition (26)) we come to the conclusion that for each $y \in X$ the following condition is fulfilled:

$$\mu_4(F(x), F(y)) \geq 1/2. \tag{61}$$

Lemma 6 and condition (61) imply that for each $y \in X$ satisfying condition (56) the following inequality holds true:

$$\mu_4(F(y), F(x)) \leq 1/2. \tag{62}$$

Both inequalities (61), (62) and the definition of relation R_4^s imply that for those $y \in X$ which satisfy condition (56) the following statement is true:

$$\mu_{R_4^s}(F(y), F(x)) = max\{0, \mu_4(F(y), F(x)) - \mu_4(F(x), F(y))\} = 0. \tag{63}$$

Condition (63) is fulfilled also for the other $y \in X$, i.e. those not satisfying condition (56). It is so, because in that case from $\overline{f}(y) = \overline{f}(x)$ and $\beta_{F(y)} = \beta_{F(x)}$, we obtain the following statement:

$$\mu_4(F(y), F(x)) = \mu_4(F(x), F(y)) = 0.$$

From condition (63) and theorem 1 we conclude that $x \in UND$.

$\square$

The following theorem gives a necessary and sufficient condition for an $x \in X$ to be an unfuzzy non dominated solution of problem (7) with respect to relations R_5, R_6. It is interesting to notice that this condition is identical to that for relations R_1, R_2. However, the proof of theorem 5 is different from that of theorem 2 and that is why the former is presented apart.

Theorem 5. *For fuzzy relations R_i, $i = 5, 6$, the following condition holds*

$$x \in UND \Longleftrightarrow \overline{f}(x) = \sum_{j=1}^{n} \overline{c}_j x_j \geq \underline{f}_{max}, \tag{64}$$

where

$$\underline{f}_{max} = \max_{y \in X} \underline{f}(y) = \sum_{j=1}^{n} \underline{c}_j y_j. \tag{65}$$

Proof.

1. $\Longrightarrow$ We will conduct an indirect proof. Let us assume that although $x \in UND$, the following condition is fulfilled

$$\overline{f}(x) < \underline{f}_{max}. \tag{66}$$

Let $y \in X$ be such a vector that

$$\underline{f}(y) = \underline{f}_{max}. \tag{67}$$

Such a vector exists, which follows from the properties of linear programming. Making use of (66), (67) we obtain the inequality $\overline{f}(x) < \underline{f}(y)$ from which, on the basis of lemma 7, it follows that

$$\mu_i(F(x), F(y)) < 1, \quad i = 5, 6. \tag{68}$$

Making use then of (68) and lemma 8 we obtain

$$\mu_i(F(y), F(x)) = 1, \quad i = 5, 6. \tag{69}$$

From inequalities (68), (69) we conclude that

$$\mu_{R_i^s}(F(y), F(x)) =$$
$$= max\{0, \mu_i(F(y), F(x)) - \mu_i(F(x), F(y))\} > 0, \ i = 5, 6, \tag{70}$$

which contradicts theorem 1.

2. $\Longleftarrow$ Let us assume that $x \in X$ satisfies the condition

$$\overline{f}(x) \geq \underline{f}_{max}. \tag{71}$$

From (71) and the fact that $\underline{f}(x) \leq \underline{f}_{max}$, $y \in X$, we obtain for each $y \in X$ the inequality $\underline{f}(y) \leq \overline{f}(x)$, which, in view of lemma 7, implies that for each $y \in X$ the following equality is satisfied

$$\mu_i(F(x), F(y)) = 1, \qquad i = 5, 6. \tag{72}$$

From (72) and the fact that for each $y \in X$, $\mu_i(F(y), F(x)) \leq 1$, $i = 5, 6$, we obtain

$$\mu_{R_i^s}(F(y), F(x)) =$$
$$= max\{0, \mu_i(F(y), F(x)) - \mu_i(F(x), F(y))\} = 0,\ i = 5, 6. \tag{73}$$

Making use of (73) and theorem 1 we conclude that $x \in UND$.

□

6 Summary

The essential theorems in the paper are theorems 2-5. They give a useful characterization of the set of unfuzzy non dominated solutions with respect to six fuzzy preference relations assumed in the paper. Let us illustrate the utility of those theorems with a simple numerical example.

Let us consider the following linear programming problem with fuzzy coefficients in the objective function:

$$\begin{array}{rrcrl} F(x) = & C_1x_1 & + & C_2x_2 & \longrightarrow \ max \\ & 2x_1 & + & 3x_2 & \leq 10 \\ & 3x_1 & + & 2x_2 & \leq 15 \\ & & & x_1, x_2 & \geq 0, \end{array} \tag{74}$$

where $C_1 = (5, 10, 8, 6)_{L-R}$ and $C_2 = (1, 2, 1, 10)_{L-R}$ are fuzzy numbers of the exponential type with the shape functions $L(y) = e^{-y}$ and $R(y) = e^{-2y}$.

Theorems 2 and 5 imply that the set of unfuzzy non dominated solutions of problem (74) with respect to each relation R_1, R_2, R_5 and R_6 coincides with the set of the solutions of the following system of inequalities:

$$\begin{array}{rcrl} 10x_1 & + & 2x_2 & \geq \underline{f}_{max} = 25 \\ 2x_1 & + & 3x_2 & \leq 10 \\ 3x_1 & + & 2x_2 & \leq 15 \\ & & x_1, x_2 & \geq 0, \end{array} \tag{75}$$

in which the number occurring on the right hand side of the first inequality, $\underline{f}_{max} = 25$, is the optimal value of the objective function in the following usual linear programming problem:

$$\begin{array}{rrrrl} \underline{f}(x) = & 5x_1 & + & x_2 & \longrightarrow \ max \\ & 2x_1 & + & 3x_2 & \leq 10 \\ & 3x_1 & + & 2x_2 & \leq 15 \\ & & & x_1, x_2 & \geq 0. \end{array} \tag{76}$$

Vector $x_{opt} = (5, 0)$ and $\underline{f}_{max} = \underline{f}(x_{opt}) = 25$ are the solution of this problem. Of course the set of solutions of the system of inequalities (75) may be infinite, and this will usually be the case. The decision maker can select any solution from this set, but he can also apply here an additional criterion. If the function characterizing this criterion is linear with respect to the decision variables, than the task of selecting the final decision will be reduced again to the solution of a corresponding linear programming problem.

As follows from theorems 3 and 4, in order to obtain an unfuzzy non dominated solution of the problem (74) with respect to each of relations R_3, R_4, it is necessary to solve a corresponding linear programming problem. In the case of relation R_3 it is

$$\begin{array}{rrrl} -0.545177x_1 & + & 0.306853x_2 & \longrightarrow \ max \\ 2x_1 & + & 3x_2 & \leq 10 \\ 3x_1 & + & 2x_2 & \leq 15 \\ & & x_1, x_2 & \geq 0 \end{array} \tag{77}$$

in the case of relation R_4

$$\begin{array}{rrrl} 12.0794x_1 & + & 5.46574x_2 & \longrightarrow \ max \\ 2x_1 & + & 3x_2 & \leq 10 \\ 3x_1 & + & 2x_2 & \leq 15 \\ & & x_1, x_2 & \geq 0. \end{array} \tag{78}$$

Vector $x_{opt} = (0, 3.3333)$ is the solution of the problem (77) and it is an unfuzzy non dominated solution of the initial problem (74) with respect to the fuzzy relation R_3. Vector $x_{opt} = (5, 0)$ is the solution of problem (78) and for its part it is an unfuzzy non dominated solution of problem (74) with respect to relation R_4.

To conclude with, it is worth emphasizing once again that the properties of relations R_1-R_6, included in lemmas 1-8, which in this paper were used only in the proofs of the essential theorems, are interesting per se. In this sense lemmas 1-8 constitute also an enlargement of the results included in [1].

References

[1] S. Chanas, M. Delgado, J.L. Verdegay, M.A. Vila, Ranking fuzzy interval numbers in the setting of random sets, *Information Sciences*, 69(1993), 201-217.

[2] S. Chanas, D. Kuchta, Linear programming with fuzzy coefficients in the objective function, in: M. Delgado, J. Kacprzyk, J.L. Verdegay, M.A. Vila (eds.), Fuzzy Optimisation, Recent Advances, Phisica-Verlag, A Springer-Verlag Company, 1994, 148-157.

[3] D. Dubois, H. Prade, Operations on fuzzy numbers, *Int. J. Systems Sci.* 30(1978), 613-626.

[4] D. Dubois, H. Prade, Ranking fuzzy numbers in the setting of possibility theory, *Information Sciences*, 30(1983), 183-224.

[5] I.R. Goodman, Fuzzy sets as equivalence classes of random sets, in: Recent Developments in Fuzzy Sets and Possibility Theory (R.R. Yager, Ed.), Pergamon Press, Oxford/New York, 1982, 327-432.

[6] S.A. Orlovsky, Decision-making with a fuzzy preference relation, *Fuzzy Sets and Systems*, 1(1978), 157-167.

[7] S.A. Orlovsky, Mathematical programming problems with fuzzy parameters, in: J. Kacprzyk, R.R. Yager (eds.), Management Decision Support Systems Using Fuzzy Sets and Possibility Theory, Verlag TÜV Rheinland, 1985, 136-145.

[8] H. Rommelfanger, R. Hanuscheck, J. Volf, Linear programming with fuzzy objectives, *Fuzzy Sets and Systems*, 29(1989), 31-48.