Possibilistic Bottleneck Combinatorial Optimization Problems with Ill-known Weights[☆]

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Abstract

In this paper a general bottleneck combinatorial optimization problem with uncertain element weights modeled by fuzzy intervals is considered. A possibilistic formalization of the problem and solution concepts in this setting, which lead to compute robust solutions under fuzzy weights are given. Some algorithms for finding a solution according to the introduced concepts and evaluating optimality of solutions and elements are provided. These algorithms are polynomial for bottleneck combinatorial optimization problems with uncertain element weights, if their deterministic counterparts are polynomially solvable.

Keywords: bottleneck combinatorial optimization, interval, fuzzy interval, possibility theory, robust optimization

1. Introduction

A *combinatorial optimization problem* consists in finding an object composed of elements of a given ground set E. In a deterministic case, every element has a precise weight and in the class of *bottleneck combinatorial optimization problems*, we wish to find an object that minimizes the weight

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of its heaviest element. This object is called an optimal solution and all the elements, which belong to some optimal solution are also called optimal. Such formulation encompasses a large variety of classical combinatorial optimization problems, for instance the bottleneck path [1], the bottleneck assignment [2] and the bottleneck spanning tree [3] (or a more general the bottleneck matroid base problem [4]). All these problems are polynomially solvable when the weights of all elements (parameters) are precisely known. Unfortunately, in real world it is not easy to specify element weights precisely. In many cases, the exact values of weights are not known in advance and this uncertainty must be taken into account. One of the most popular settings of problems for hedging against uncertainty of parameters is *stochastic optimization*, in which uncertain parameters are modeled as random variables. (see, e.g., [5]). Usually, the goal is to optimize the expected value of a solution built. Some of the models of stochastic optimization consider other criteria of choosing a solution: *chance constrained model* (see, e.g., [6, 7]), *threshold model* [8, 9] in which we seek a solution maximizing the probability that its random value does not exceed a given threshold. This model is perhaps the closest one in spirit to the model assumed in this paper. Unfortunately, most of the stochastic optimization problems are inherently intractable (even if parameters are independent random variables). They are tractable only when some assumptions are imposed. Another difficulty, not always pointed out, is the possible lack of statistical data validating the choice of parameter distributions. In the overwhelming part of the stochastic optimization literature (see [10] for a bibliography), it is assumed that probability distributions describing uncertain parameters are known in advance. Usually, special classes of distributions such as: the normal, the Poisson, the exponential, the Bernoulli are applied to model the uncertainty of parameters. In fact, probability distributions permit to model the variability of repetitive parameters, but this approach becomes debatable when dealing with uncertainty caused by a lack of information [11, 12]. Even if statistical data are available, they may be partially inadequate because each problem may take place in a specific environment, and is not the exact replica of the past ones.

A simple approach for handling uncertain parameters is modeling the uncertainty in the form of intervals. It is natural in practice - a decision maker just needs to provide a minimal value of a parameter and a maximal one. Assigning some interval to a parameter means that it will take some value within the interval, but it is not possible to predict at present which one. There is no probability distribution in the interval. The interval uncertainty representation may also be considered as poorly expressive. So, we do not propose the use of intervals as the definite answer to modeling uncertain parameters. A more elaborate approach could be to collect both intervals and plausible values from decision makers and, in this case, fuzzy intervals may be useful. Resorting to fuzzy sets and *possibility theory* [13] for modelling ill-known parameters, the model considered in this paper may help building a trade-off between the lack of expressive power of mere intervals and the computational difficulties of stochastic optimization techniques.

In this paper, we use fuzzy intervals to model uncertain element weights. Namely, the membership function $\mu_{\widetilde{W}}$ of a fuzzy interval $\widetilde{W}$ is a possibility distribution describing, for each value w of the element weight, the extent to which it is a possible value. Equivalently, it means that the value of this weight belongs to a λ-cut interval $\widetilde{W}^{\lambda} = \{t : \mu_{\widetilde{W}}(t) \geq \lambda\}$ with confidence (or degree of necessity) $1 - \lambda$. Now to each solution or element a *degree of possible optimality* and a *degree of necessary optimality* can be assigned. The notion of the necessary optimality of a solution may be weaken by assigning a *degree of necessary soft optimality*. Moreover, all the degrees of optimality of a solution (an element) can be derived from a *fuzzy deviation*, that is a possibility distribution representing the set of plausible values of deviations of a solution (an element) from optimum. In order to choose a "robust solution" under fuzzy weights, we adopt two criteria. The first one consists in choosing a solution of the maximum degree of necessary optimality, called a *best necessarily optimal solution*. The second criterion is weaker than the first one and consists in choosing a solution of the maximum degree of necessary soft optimality, called a *best necessarily soft optimal solution*. This criterion has been originally proposed in [14, 15] for the linear programming problem with a fuzzy objective function. For a review of various concepts of the robustness of solutions in optimization and a bibliography we refer the reader to [16].

In this paper, we provide some methods for the optimality evaluation and for choosing a solution under fuzzy weights. In Section 3, we investigate the interval case, that is the class of problems where the element weights are specified as closed intervals. A closed interval can be viewed as a fuzzy interval with sharp bounds. We show that it is possible to construct polynomial algorithms for such problems if only their deterministic counterparts are polynomially solvable. In consequence, the interval bottleneck problems are easier to solve than the interval problems with a linear sum objective discussed in [17, 18] (see also [18, 19, 20]). In particular, we obtain polynomial algorithms for such classical problems as the bottleneck path, the bottleneck assignment and the bottleneck matroid base under interval weights. In Section 4, we show that the optimality evaluation and the problem of choosing a solution under fuzzy weights can be reduced to examining a small number of interval problems. In particular, we prove that a best necessarily soft optimal solution can be computed in polynomial time for a wide class of problems. In order to make the presentation more clear we place all technical proofs in the appendix.

2. Preliminaries

Let $E = \{e_1, \dots, e_n\}$ be a finite ground set and let $\Phi \subseteq 2^E$ be a set of subsets of E called the set of the *feasible solutions*. A nonnegative weight w_e is given for every element $e \in E$. A *bottleneck combinatorial optimization problem* $\mathcal{BP}$ consists in finding a feasible solution X that minimizes the weight of its heaviest element, namely:

$$\mathcal{BP}: \quad \min_{X \in \Phi} F(X) = \min_{X \in \Phi} \max_{e \in X} w_e. \tag{1}$$

We call $F(X) = \max_{e \in X} w_e$ a *bottleneck objective function*, in contrast to the more popular in literature linear sum objective, which is of the form $F(X) = \sum_{e \in X} w_e$. A solution to (1) is called an *optimal solution*. An element $e \in E$ is said to be *optimal* if it is a part of an optimal solution.

The formulation (1) encompasses a large variety of problems. In *network problems*, E is a set of edges of a given graph $G = (V, E)$ and Φ consists of all subsets of the edges that form some objects in G such as paths, spanning trees, matchings, cuts etc. (see e.g. [21, 22, 23]). In general, (1) includes the problems, which can be stated as 0-1 programming ones. To see this, we need to associate a binary variable $x_i \in \{0, 1\}$ with every element $e_i \in E$ and describe Φ using a system of constraints involving the binary variables. Notice that $\mathcal{BP}$ may be polynomially solvable or NP-hard and in this paper, we assume that it is polynomially solvable. In particular, some polynomial algorithms for the bottleneck path, the bottleneck assignment, the bottleneck spanning tree and the bottleneck matroid base problems can be found for instance in [1, 2, 3, 4].

In theory and practice the class of *matroidal problems* is of great importance. Recall that a *matroid* is a pair $(E, \mathcal{I})$, where E is a nonempty element set and $\mathcal{I}$ is a set of subsets of E such that $\mathcal{I}$ is closed under inclusion (if

$A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$) and fulfills the so-called *growth property* (if $A, B \in \mathcal{I}$ and $|A| < |B|$, then there is $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$) (see e.g. [22]). The maximal (under inclusion) sets in $\mathcal{I}$ are called *bases* and the minimal (under inclusion) sets not in $\mathcal{I}$ are called *circuits*. We will denote the set of all bases by $\mathcal{B}$. In a matroidal problem the set of feasible solutions Φ consists of all bases of a given matroid, that is $\Phi = \mathcal{B}$. Perhaps, the best known example of a bottleneck matroidal problem is the bottleneck spanning tree, where E is a set of edges of a given undirected graph and $\mathcal{I}$ consists of all subsets of the edges that form acyclic subgraphs of G. Then $(E, \mathcal{I})$ is called a *graphic matroid* and its base is a spanning tree of G. Another example is the bottleneck selecting items problem. In this problem, E is a set of items and $\mathcal{I}$ consists of all subsets of E, whose cardinalities are less than or equal to a given number p. The system $(E, \mathcal{I})$ is called an *uniform matroid* and $X \subseteq E$ is a base if and only if $|X| = p$. We will see later in this paper that a particular structure of matroidal problems allows us to design efficient algorithms under uncertainty.

In the approach presented in this paper a crucial role will be played by the concept of a deviation. A *deviation of solution* $X \in \Phi$ and a *deviation of element* $f \in E$ are defined in the following way:

$$\delta_X = F(X) - \min_{Y \in \Phi} F(Y), \quad \delta_f = \min_{Y \in \Phi_f} F(Y) - \min_{Y \in \Phi} F(Y),$$

where Φ_f is the set of all feasible solutions that contain element f. The deviations express a "distance" of a solution or element from optimum and it is clear that solution X (element f) is optimal if and only if $\delta_X = 0$ ($\delta_f = 0$). Thus, the solution (element) deviation gives us an information how far from optimality a solution (element) is.

3. Bottleneck combinatorial optimization problems with interval-valued weights

In practice, the precise values of element weights in problem $\mathcal{BP}$ may be not well known. To take this uncertainty into account we first apply one of the simplest uncertainty representation, where each uncertain weight is modeled by a closed interval. This representation is based on the fact that it is often possible to give a minimal and a maximal expected value of an element weight. In consequence, we know that the value of the weight of $e \in E$ will fall within a closed interval $W_e = [\underline{w}_e, \overline{w}_e]$ and W_e contains all possible values of the weight of e. We assume that there is no probability distribution in W_e, $e \in E$, and all weights are unrelated, that is the value of every weight does not depend on the values of the remaining weights.

A vector $S = (s_e)_{e \in E}$ such that $s_e \in W_e$ for all $e \in E$ is called a *scenario* and it represents the state of the world in which $w_e = s_e$ for all $e \in E$. Thus, every scenario is a precise instantiation of the element weights, which may occur. We denote by Γ the set of all scenarios, i.e. $\Gamma = \times_{e \in E} [\underline{w}_e, \overline{w}_e]$ and we use $w_e(S)$ to denote the weight of element $e \in E$ in a fixed scenario $S \in \Gamma$, $w_e(S) \in W_e$. Among the scenarios of Γ, we distinguish the *extreme* ones, which belong to $\times_{e \in E} \{\underline{w}_e, \overline{w}_e\}$. Let $A \subseteq E$ be a fixed subset of elements. In scenario S_A^+ all elements $e \in A$ have weights $\overline{w}_e$ and all the remaining elements have weights $\underline{w}_e$. Similarly, in scenario S_A^- all elements $e \in A$ have weights $\underline{w}_e$ and all the remaining elements have weights $\overline{w}_e$. For a given solution $X \in \Phi$, we define its weight under a fixed scenario $S \in \Gamma$ as $F(X, S) = \max_{e \in X} w_e(S)$. We will denote by $F^*(S)$ the value of the weight of an optimal solution under scenario $S \in \Gamma$, that is

$$F^*(S) = \min_{X \in \Phi} F(X, S) = \min_{X \in \Phi} \max_{e \in X} w_e(S).$$

Therefore, in order to obtain $F^*(S)$, we have to solve the deterministic problem $\mathcal{BP}$ under the weight realization specified by scenario S. Now solution and element deviations also depend on scenario S and we will denote them as $\delta_X(S)$ and $\delta_f(S)$, respectively. Hence $\delta_X(S) = F(X, S) - F^*(S)$ and $\delta_f(S) = \min_{Y \in \Phi_f} F(Y, S) - F^*(S)$.

3.1. Optimality evaluation

Similarly to the deterministic case, where the deviations of a solution and an element give a full characterization of their optimality, in the interval case we can give a full characterization of optimality of a solution and an element in terms of the so-called *deviation intervals*. Consider the following optimization problems:

$$\underline{\delta}_X = \min_{S \in \Gamma} \delta_X(S), \quad \overline{\delta}_X = \max_{S \in \Gamma} \delta_X(S) \tag{2}$$

$$\underline{\delta}_f = \min_{S \in \Gamma} \delta_f(S), \quad \overline{\delta}_f = \max_{S \in \Gamma} \delta_f(S). \tag{3}$$

The solutions to (2) determine a *deviation interval* $\Delta_X = [\underline{\delta}_X, \overline{\delta}_X]$ containing all possible values of deviation for solution X. Similarly $\Delta_f = [\underline{\delta}_f, \overline{\delta}_f]$ is a deviation interval for element f. It is worth pointing out that in literature (see e.g. [24]) the quantity $\overline{\delta}_X$ is called the *maximal regret* or *robust deviation* of X and it expresses the maximal possible deviation of X from optimum (the largest "distance of X from optimality").

Since intervals Δ_X and Δ_f contain all values of solution and element deviations which may occur, they allow us to give the following optimality characterization in problem $\mathcal{BP}$ with interval weights: a solution X (element $f \in E$) is *possibly optimal* if $\underline{\delta}_X = 0$ ($\underline{\delta}_f = 0$) and solution X (element $f \in E$) is *necessarily optimal* if $\overline{\delta}_X = 0$ ($\overline{\delta}_f = 0$). Clearly, a solution $X \in \Phi$ (element $f \in E$) is possibly optimal if and only if it is optimal in some scenario $S \in \Gamma$ and it is necessarily optimal if and only if it is optimal in all scenarios $S \in \Gamma$. It is easily seen that every possibly (necessarily) optimal solution is composed of possibly (necessarily) optimal elements.

We now show how to solve the optimization problems (2) and (3) and, consequently, how to determine the deviation intervals.

Proposition 1. *Let X be a given feasible solution. Then*

$$\underline{\delta}_X = \max\{0, \max_{e \in X} \underline{w}_e - F^*(S_E^+)\}, \tag{4}$$

$$\overline{\delta}_X = \max_{e \in X} \max\{0, \overline{w}_e - F^*(S_{\{e\}}^+)\}. \tag{5}$$

PROOF. Equality (5) has been proved in [25]. The proof of equality (4) can be found in Appendix A. □

Making use of Proposition 1, we can determine $\Delta_X = [\underline{\delta}_X, \overline{\delta}_X]$ of a given solution X in polynomial time if only the underlying bottleneck deterministic problem $\mathcal{BP}$ is polynomially solvable. In order to compute the lower bound $\underline{\delta}_X$, it suffices to compute the value of $F^*(S_E^+)$, using an algorithm for the deterministic problem $\mathcal{BP}$, and the value of $F(X, S_E^-)$. Therefore, this can be done in $O(|X| + f(|E|))$ time, where $f(|E|)$ is the running time of an algorithm for the deterministic problem $\mathcal{BP}$. Notice that computing the lower bound of a next solution, say $X' \in \Phi$, requires only $O(|X'|)$ time because $F^*(S_E^+)$ does not depend on X'. Determining the upper bound $\overline{\delta}_X$ is a little more complex, since it requires computing the difference $\overline{w}_e - F^*(S_{\{e\}}^+)$ for each $e \in X$ and, consequently, the overall running time of determining $\overline{\delta}_X$ is $O(|X| f(|E|))$.

The following two corollaries are direct consequences of Proposition 1. They establish sufficient and necessary conditions for possible and necessary optimality of a given solution.

Corollary 1. *A solution $X \in \Phi$ is possibly optimal if and only if* $\max_{e\in X} \underline{w}_e \leq F^*(S_E^+)$.

Corollary 2. *A solution $X \in \Phi$ is necessarily optimal if and only if $\overline{w}_e \leq F^*(S_{\{e\}}^+)$ for all $e \in X$.*

Let us consider the problem of computing the element deviation interval $\Delta_f = [\underline{\delta}_f, \overline{\delta}_f]$ of a specified element $f \in E$. The following proposition gives a formula for computing the lower bound of Δ_f:

Proposition 2. *Let $f \in E$ be a specified element. Then*

$$\underline{\delta}_f = \max\{0, \min_{X\in\Phi_f} F(X, S_X^-) - F^*(S_E^+)\}. \tag{6}$$

Proof. See Appendix A. □

From Proposition 2 we immediately get the following corollary:

Corollary 3. *An element $f \in E$ is possibly optimal if and only if the inequality* $\min_{X\in\Phi_f} F(X, S_X^-) \leq F^*(S_E^+)$ *holds.*

Proposition 2 and Corollary 3 show a significant difference between the problems with bottleneck objective, studied here, and the problems with linear sum objective discussed for instance in [17, 18]. For the latter problems, deciding whether $\underline{\delta}_f = 0$ may be NP-hard even if a deterministic counterpart is polynomially solvable [18]. For the bottleneck problems the situation is much better. Also, for the problems with linear sum objective, there are nonpossibly optimal solutions entirely composed of possibly optimal elements [26]. The following proposition shows that this does not hold true for the problems with bottleneck objective.

Proposition 3. *A solution X is possibly optimal if and only if it is composed of possibly optimal elements.*

Proof. See Appendix A. □

From Proposition 2, it follows that if problem $\mathcal{BP}$ is solvable in $f(|E|)$ time, then the bound $\underline{\delta}_f$ for a given element f can be determined in $O(f(|E|))$ time. Namely, we need to compute the value of $F^*(S_E^+)$ by an algorithm for the deterministic problem $\mathcal{BP}$ and the value of $\min_{X \in \Phi_f} F(X, S_X^-)$ by a slight modification of the algorithm for problem $\mathcal{BP}$.

We are unable here to provide a general formula for computing the upper bound of an element deviation $\overline{\delta}_f$. Also, the complexity status of the problem of checking whether a specified element f is necessarily optimal is unknown. This is an interesting subject of further research.

We now show how to compute efficiently the quantities $\underline{\delta}_f$ and $\overline{\delta}_f$ when $\mathcal{BP}$ has matroidal structure.

Proposition 4. *Let f be a specified element. If $\mathcal{BP}$ is a matroidal problem, then*

$$\underline{\delta}_f = \max\{0, \underline{w}_f - F^*(S_E^+)\}, \tag{7}$$

$$\overline{\delta}_f = \max\{0, \overline{w}_f - F^*(S_{\{f\}}^+)\}. \tag{8}$$

PROOF. See Appendix A. □

Proposition 4 leads to the following two corollaries:

Corollary 4. *Suppose that $\mathcal{BP}$ is a matroidal problem. Then element $f \in E$ is possibly optimal if and only if $\underline{w}_f \leq F^*(S_E^+)$.*

Corollary 5. *Suppose that $\mathcal{BP}$ is a matroidal problem. Then element $f \in E$ is necessarily optimal if and only if $\overline{w}_f \leq F^*(S_{\{f\}}^+)$.*

Proposition 4 allows us to determine efficiently the bounds $\underline{\delta}_f$ and $\overline{\delta}_f$ of a specified element $f \in E$ in all matroidal problems. Computing the values of $F^*(S_E^+)$ and $F^*(S_{\{f\}}^+)$ in (7) and (8) can be done in $O(|E| \log^*(|E|))$ time [4], where $\log^* |E|$ is the iterated logarithm of $|E|$. Note also that using formula (7), we can compute the lower bounds of all elements of E in $O(|E| \log^*(|E|))$ because we need to execute an algorithm for the deterministic problem only once. On the other hand, formula (8) does not allow us to determine the upper bounds of all elements without extra effort. Evaluating the possible and necessary optimality of f costs the same time as computing the deviation interval Δ_f (see Corollaries 4 and 5).

Consider an example of an interval-valued bottleneck spanning tree shown in Figure 1. Using Propositions 1 and 4, we get deviation intervals of spanning trees: $\Delta_{\{a,c\}} = [0, 1]$, $\Delta_{\{b,c\}} = [0, 1]$ and edges: $\Delta_a = [0, 1]$, $\Delta_b = [0, 1]$, $\Delta_c = [0, 0]$. Hence all the spanning trees and the edges are possibly optimal. There is no necessary optimal spanning tree, although there is one isolated necessarily optimal edge c.

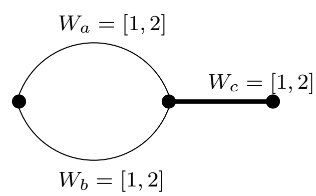

Figure 1: An example of an interval-valued bottleneck spanning tree with isolated necessarily optimal edge c (in bold).

3.2. Choosing a robust solution

An important task in the interval-valued problem is to choose a *robust solution*, that is the one which performs reasonably well under any possible scenario. It is clear that if one finds a necessarily optimal solution, then it is the ideal choice because it is optimal regardless of the scenario that will occur. Unfortunately, such a solution rarely exists. On the other hand, a possibly optimal solution always exists. It is enough to choose any scenario and compute an optimal solution under this scenario. But such a possibly optimal solution will be chosen by an optimistic decision maker, because it may be poor if some bad scenario will occur. Hence the possible optimality is too weak criterion while the necessary optimality seems to be too strong and a compromise between the possible and necessary optimality is required. A solution that minimizes the maximal regret $\overline{\delta}_X$ seems to be a compromise choice. In literature [24] it is called an *optimal minmax regret solution.* So, under the interval uncertainty representation we focus on the following optimization problem:

$$\min_{X \in \Phi} \overline{\delta}_X. \tag{9}$$

Notice that every necessarily optimal solution is an optimal min-max regret one with zero maximal regret. Moreover, due to the following proposition, every optimal minmax regret solution is possibly optimal:

Proposition 5. *Every optimal minmax regret solution X is possibly optimal and it is composed of possibly optimal elements.*

PROOF. See Appendix A. □

So, the deviation interval of an optimal minmax regret solution is of the form $\Delta_X = [0, \overline{\delta}_X]$ where $\overline{\delta}_X$ is the smallest among all $X \in \Phi$. Consequently, X minimizes the distance to the necessary optimality.

The problem (9) has been studied in [25]. It turns out that if the deterministic $\mathcal{BP}$ problem is polynomially solvable, then its minmax regret version is polynomially solvable as well. The crucial fact is equality (5) proved in [25]. Indeed, combining (5) with (9) we obtain

$$\min_{X\in\Phi} \overline{\delta}_X = \min_{X\in\Phi} \max_{e\in X} \hat{w}_e, \tag{10}$$

where weights $\hat{w}_e = \max\{0, \overline{w}_e - F^*(S^+_{\{e\}})\}$, $e \in E$, are deterministic. So, the minmax regret problem (9) can be reduced to problem $\mathcal{BP}$ with deterministic weights $\hat{w}_e$, $e \in E$. It can be shown [25] that the minmax regret problem (9) can be solved in $O(|E|+|X^*|f(|E|))$ time, where X^* is such that $F(X^*, S^-_E) = F^*(S^-_E)$ and $f(|E|)$ is the running time of an algorithm for problem $\mathcal{BP}$.

It is worth pointing out that replacing the bottleneck objective function with the linear sum one radically changes the complexity of the minmax regret problem (9). In this case the minmax regret problem turns out to be NP-hard for such classical combinatorial optimization problems as: the shortest path [19], the minimum spanning tree [19] and the minimum assignment [20], which are polynomially solvable with deterministic weights (see also [27] for a survey). A weaker criterion for choosing a robust solution in combinatorial optimization problems with interval weights has been proposed in [28]. In this approach, the robust counterparts of polynomially solvable (α-approximable) combinatorial optimization problems remain polynomially solvable (α-approximable).

4. Bottleneck combinatorial optimization problems with fuzzy-valued weights

In this section, we study problem $\mathcal{BP}$ with uncertain element weights, where the uncertainty is modeled by fuzzy intervals. We first recall some notions from possibility theory, which we will use later in this paper (a detailed description of this theory can be found for instance in [13]). We then give a rigorous possibilistic interpretation of the fuzzy problem and provide some solution concepts and algorithms in this setting.

4.1. Selected notions of possibility theory

A *fuzzy set* allows us to express the uncertainty connected with an ill-known quantity in a more sophisticated manner than a closed interval. A *fuzzy set* $\widetilde{A}$ is defined by means of a reference set $\mathcal{V}$ together with a mapping

$\mu_{\widetilde{A}}$ from $\mathcal{V}$ into $[0,1]$, called a *membership function*. The value of $\mu_{\widetilde{A}}(v)$, $v \in \mathcal{V}$, is a degree of membership of v in the fuzzy set $\widetilde{A}$. The λ-*cut*, $\lambda \in (0,1]$, of $\widetilde{A}$ is a classical set defined as $\widetilde{A}^{\lambda} = \{v \in \mathcal{V} : \ \mu_{\widetilde{A}}(v) \geq \lambda\}$. The λ-cuts of $\widetilde{A}$ form a family of nested sets, i.e if $\lambda_1 \geq \lambda_2$, then $\widetilde{A}^{\lambda_1} \subseteq \widetilde{A}^{\lambda_2}$. The *support* of $\widetilde{A}$ is the set $\{v : \mu_{\widetilde{A}}(v) > 0\}$.

A fuzzy set $\widetilde{A}$ in $\mathbb{R}$ with a compact support, whose membership function $\mu_{\widetilde{A}} : \mathbb{R} \to [0,1]$ is normal, quasi-concave and upper semi-continuous is called a *fuzzy interval*. We will denote by $\widetilde{A}^0$ the smallest closed set containing the support of $\widetilde{A}$. Now, it can be easily verified (see e.g. [13]) that $\widetilde{A}^{\lambda}$ is a closed interval for every $\lambda \in [0,1]$. So, we can represent a fuzzy interval $\widetilde{A}$ as a nested family of closed intervals $\widetilde{A}^{\lambda} = [\underline{a}^{\lambda}, \overline{a}^{\lambda}]$ parametrized by values of $\lambda \in [0,1]$. Functions $\underline{a}^{\lambda}$ and $\overline{a}^{\lambda}$ of λ are called *left* and *right profiles* of $\widetilde{A}$ (see [29]). The membership function $\mu_{\widetilde{A}}$ can be retrieved from the family of λ-cuts in the following way:

$$\mu_{\widetilde{A}}(v) = \sup\{\lambda \in [0,1] : v \in \widetilde{A}^{\lambda} = [\underline{a}^{\lambda}, \overline{a}^{\lambda}]\}, \tag{11}$$

and $\mu_{\widetilde{A}}(v) = 0$ if $v \notin \widetilde{A}^0$; $\underline{a}^{\lambda} = \inf\{v \in \mathbb{R} : \mu_{\widetilde{A}}(v) \geq \lambda\}$, $\overline{a}^{\lambda} = \sup\{v \in \mathbb{R} : \mu_{\widetilde{A}}(v) \geq \lambda\}$. A classical closed interval $A = [\underline{a}, \overline{a}]$ is a special case of a fuzzy one with membership function $\mu_A(v) = 1$ if $v \in A$ and $\mu_A(v) = 0$ otherwise. In this case we have $A^{\lambda} = [\underline{a}, \overline{a}]$ for all $\lambda \in [0,1]$. A *fuzzy interval of the L-R type*, denoted as $(\underline{a}, \overline{a}, \alpha, \beta)_{L-R}$ is very popular and convenient in applications. Its membership function is of the following form:

$$\mu_{\widetilde{A}}(v) = \begin{cases} 1 & \text{for} \quad v \in [\underline{a}, \overline{a}], \\ L\left(\frac{\underline{a}-v}{\alpha}\right) & \text{for} \quad v \leq \underline{a}, \\ R\left(\frac{v-\overline{a}}{\beta}\right) & \text{for} \quad v \geq \overline{a}, \end{cases}$$

where L and R are continuous and nonincreasing functions, defined on $[0, +\infty)$, called *shape functions*. The parameters α and β are nonnegative real numbers. Every fuzzy interval of the L-R type $(\underline{a}, \overline{a}, \alpha, \beta)_{L-R}$ with a compact support can be described by the following family of λ-cuts, $\lambda \in [0,1]$:

$$A^{\lambda} = [\underline{a}^{\lambda}, \overline{a}^{\lambda}] = [\underline{a} - L^{-1}(\lambda)\alpha, \overline{a} + R^{-1}(\lambda)\beta]. \tag{12}$$

If $L(v) = R(v) = \max\{0, 1-v\}$, then we obtain *a trapezoidal fuzzy interval*, which is shortly denoted by quadruple $(\underline{a}, \overline{a}, \alpha, \beta)$. If additionally $\underline{a} = \overline{a}$, then we get *a triangular fuzzy interval* denoted by triple (a, α, β).

Let us now recall the possibilistic interpretation of a fuzzy set. *Possibility theory* [13] is an approach to handle incomplete information and it relies on two dual measures: *possibility* and *necessity*, which express plausibility and certainty of events. Both measures are built from a *possibility distribution*. Let a fuzzy set $\widetilde{A}$ be attached with a single-valued variable a. The membership function $\mu_{\widetilde{A}}$ is understood as a possibility distribution, $\pi_a = \mu_{\widetilde{A}}$, which describes the set of more or less plausible, mutually exclusive values of the variable a. It plays a role similar to a probability density, while it can encode a family of probability functions [30]. In particular, a degree of possibility can be viewed as the upper bound of a degree of probability [30]. The value of $\pi_a(v)$ represents the possibility degree of the assignment $a = v$, i.e.

$$\Pi(a = v) = \pi_a(v) = \mu_{\widetilde{A}}(v),$$

where $\Pi(a = v)$ is the possibility of the event that a will take the value of v. In particular, $\pi_a(v) = 0$ means that $a = v$ is impossible and $\pi_a(v) > 0$ means that $a = v$ is plausible, that is, not surprising. Equivalently, it means that the value of a belongs to a λ-cut $\widetilde{A}^\lambda$ with confidence (or degree of necessity) $1-\lambda$. A detailed interpretation of the possibility distribution and some methods of obtaining it from the possessed knowledge are described in [13, 31].

Let $\widetilde{G}$ be a fuzzy set in $\mathbb{R}$. Then "$a \in \widetilde{G}$" is a *fuzzy event*. The *possibility* of "$a \in \widetilde{G}$", denoted by $\Pi(a \in \widetilde{G})$, is as follows [32]:

$$\Pi(a \in \widetilde{G}) = \sup_{v\in\mathbb{R}} \min\{\pi_a(v), \mu_{\widetilde{G}}(v)\}. \tag{13}$$

$\Pi(a \in \widetilde{G})$ evaluates the extent to which "$a \in \widetilde{G}$" is possibly true. The *necessity* of event "$a \in \widetilde{G}$", denoted by $\mathrm{N}(a \in \widetilde{G})$, is as follows:

$$\begin{aligned} \mathrm{N}(a \in \widetilde{G}) &= 1 - \Pi(a \notin \widetilde{G}) = 1 - \sup_{v\in\mathbb{R}} \min\{\pi_a(v), 1 - \mu_{\widetilde{G}}(v)\} \qquad (14) \\ &= \inf_{v\in\mathbb{R}} \max\{1 - \pi_a(v), \mu_{\widetilde{G}}(v)\}, \end{aligned}$$

where $1 - \mu_{\widetilde{G}}$ is the membership function of the complement of the fuzzy set $\widetilde{G}$. $\mathrm{N}(a \in \widetilde{G})$ evaluates the extent to which "$a \in \widetilde{G}$" is certainly true. Observe that if G is a classical set, then $\Pi(a \in G) = \sup_{v\in G} \pi_a(v)$ and $\mathrm{N}(a \in G) = 1 - \sup_{v\notin G} \pi_a(v)$.

4.2. A possibilistic formalization of the problem

We now give a possibilistic formalization of problem $\mathcal{BP}$, in which the weights of elements are modeled by fuzzy intervals $\widetilde{W}_e$, $e \in E$. Here, a membership function of $\widetilde{W}_e$ is regarded as a possibility distribution for the values of the unknown weight w_e (see the previous section). The possibility degree of the assignment $w_e = s$ is $\Pi(w_e = s) = \pi_{w_e}(s) = \mu_{\widetilde{W}_e}(s)$. Let $S = (s_e)_{e\in E}$ be a scenario that represents a state of the world where $w_e = s_e$, for all $e \in E$. It is assumed that the weights are unrelated one to each other. This assumption makes the fuzzy valued problem $\mathcal{BP}$ very tractable (a review of some attempts to handle related fuzzy parameters can be found in [33]). Hence, the possibility distributions associated with the weights induce the following possibility distribution over all scenarios in $S \in \mathbb{R}^n$ (see [34]):

$$\pi(S) = \Pi\left(\bigwedge_{e\in E}[w_e = s_e]\right) = \min_{e\in E} \Pi(w_e = s_e) = \min_{e\in E} \mu_{\widetilde{W}_e}(s_e). \tag{15}$$

The value of $\pi(S)$ stands for the possibility of the event that scenario $S \in \mathbb{R}^n$ will occur. We have thus generalized scenario set Γ, defined in Section 3 and now $\widetilde{\Gamma}$ is a fuzzy set of scenarios with membership function $\mu_{\widetilde{\Gamma}}(S) = \pi(S)$, $S \in \mathbb{R}^n$. Making use of (15) and the definition of λ-cut it is easy to check that the λ-cuts of $\widetilde{\Gamma}$ for every $\lambda \in (0,1]$ fulfill the following equality:

$$\widetilde{\Gamma}^{\lambda} = \{S:\ \pi(S) \geq \lambda\} = \times_{e\in E}[\underline{w}_e^{\lambda}, \overline{w}_e^{\lambda}]. \tag{16}$$

We also define $\widetilde{\Gamma}^0 = \times_{e\in E}[\underline{w}_e^0, \overline{w}_e^0]$. Notice that $\widetilde{\Gamma}^{\lambda}$, $\lambda \in [0,1]$, is the classical scenario set containing all scenarios whose possibility of occurrence is not less than λ. This property allows us to decompose the fuzzy problem into a family of interval problems. We will make use of this fact later in this paper.

As in the deterministic and interval cases (see Sections 2 and 3.1) we can characterize the optimality of a solution X and an element f using the concept of deviation. In the fuzzy problem the solution and element deviations are unknown quantities that fall within fuzzy intervals $\widetilde{\Delta}_X$ and $\widetilde{\Delta}_f$, *fuzzy deviations*, whose membership functions $\mu_{\widetilde{\Delta}_X}$ and $\mu_{\widetilde{\Delta}_f}$ are possibility distributions for values of δ_X and δ_f, respectively, defined as follows:

$$\mu_{\widetilde{\Delta}_X}(v) = \Pi(\delta_X = v) = \sup_{\{S:\, \delta_X(S)=v\}} \pi(S), \tag{17}$$

$$\mu_{\widetilde{\Delta}_f}(v) = \Pi(\delta_f = v) = \sup_{\{S:\, \delta_f(S)=v\}} \pi(S). \tag{18}$$

Fuzzy deviations (17) and (18) allow us to characterize possible and necessary optimality of solutions and elements. Recall that the statement "X is optimal" is equivalent to the condition $\delta_X = 0$, so we can define the *degrees of possible and necessary optimality of solution* X, as the possibility and necessity of the event "X is optimal" (see (13) and (14)):

$$\begin{aligned} \Pi(X \text{ is optimal}) &= \Pi(\delta_X = 0) = \mu_{\widetilde{\Delta}_X}(0), &(19)\\ \mathrm{N}(X \text{ is optimal}) &= \mathrm{N}(\delta_X = 0) = 1 - \Pi(\delta_X > 0) &(20)\\ &= 1 - \sup_{v>0} \mu_{\widetilde{\Delta}_X}(v). \end{aligned}$$

In the same manner, we can define the *degrees of possible* and *necessary optimality of element* f as the possibility and necessity of the event "f is optimal". It suffices to replace X with f in (19) and (20).

4.3. Computing the optimality degrees and fuzzy deviations

Let us first consider the problem of computing the degrees of possible and necessary optimality of a given solution X. Denote by $\widetilde{\Delta}^{\lambda}_X = [\underline{\delta}^{\lambda}_X, \overline{\delta}^{\lambda}_X]$, a λ-cut of the fuzzy deviation of X. From (19) and (11), it follows easily that

$$\begin{aligned} \Pi(X \text{ is optimal}) &= \sup\{\lambda \in [0,1] : 0 \in \widetilde{\Delta}^{\lambda}_X = [\underline{\delta}^{\lambda}_X, \overline{\delta}^{\lambda}_X]\} \\ &= \sup\{\lambda \in [0,1] : \underline{\delta}^{\lambda}_X = 0\} &(21) \end{aligned}$$

and $\Pi(X$ is optimal$) = 0$ if $\underline{\delta}^{0}_X > 0$. A similar reasoning leads to the following equality:

$$\mathrm{N}(X \text{ is optimal}) = 1 - \inf\{\lambda \in [0,1] : \overline{\delta}^{\lambda}_X = 0\} \qquad (22)$$

and $\mathrm{N}(X$ is optimal$) = 0$ if $\overline{\delta}^{1}_X > 0$. Note that the interval $\widetilde{\Delta}^{\lambda}_X = [\underline{\delta}^{\lambda}_X, \overline{\delta}^{\lambda}_X]$ is the deviation interval of X in problem $\mathcal{BP}$ under scenario set $\widetilde{\Gamma}^{\lambda}$, that is the one with interval weights $\widetilde{W}^{\lambda}_e = [\underline{w}^{\lambda}_e, \overline{w}^{\lambda}_e]$ for all $e \in E$.

Equations (21) and (22) form a theoretical basis for calculating the values of the optimality degrees. They are links between the interval and fuzzy problems. So, in order to compute the degree of possible optimality of X, we need to find the largest value of λ such that X is possibly optimal under scenario set $\widetilde{\Gamma}^{\lambda}$ (which is equivalent to the condition $\underline{\delta}^{\lambda}_X = 0$). Since $\underline{\delta}^{\lambda}_X$ is nondecreasing function of λ, we can apply a binary search technique on $\lambda \in [0,1]$. At each iteration the possible optimality of X under scenario set $\widetilde{\Gamma}^{\lambda}$ is checked, which can be done by using Proposition 1 or Corollary 1.

Similarly, in order to compute the degree of necessary optimality of X, we need to find the smallest value of λ such that X is necessarily optimal under scenario set $\widetilde{\Gamma}^{\lambda}$ (which is equivalent to the condition $\overline{\delta}_X^{\lambda} = 0$). Function $\overline{\delta}_X^{\lambda}$ is nonincreasing of λ and hence the degree of necessary optimality can also be computed by using a binary search. At each iteration the necessary optimality of X under scenario set $\widetilde{\Gamma}^{\lambda}$ is tested using Proposition 1 or Corollary 2. If $I(|E|)$ is the time required to assert whether a given solution is possibly (necessarily) optimal in the corresponding interval problem, then its degree of possible (necessary) optimality can be calculated in $\mathcal{O}(I(|E|)\log \epsilon^{-1})$ time, where $\epsilon > 0$ is a given error tolerance.

The possibility distribution for solution deviation can be determined approximately via the use of λ-cuts. That is, we compute intervals $\widetilde{\Delta}_X^{\lambda} = [\underline{\delta}_X^{\lambda}, \overline{\delta}_X^{\lambda}]$ for suitably chosen λ-cuts using Proposition 1. Then the fuzzy interval $\widetilde{\Delta}_X$ can be reconstructed from the obtained λ-cuts by using equality (11). This method gives an approximation of $\widetilde{\Delta}_X$. Its overall running time is $O(rI(|E|))$, where r is the number of computed λ-cuts and $I(|E|)$ is time required to determine $\widetilde{\Delta}_X^{\lambda}$.

In order to obtain an analytical and exact representation of possibility distribution $\widetilde{\Delta}_X$ a parametric technique can be applied. Proposition 1 yields:

$$\underline{\delta}_X^{\lambda} = \max\{0, \max_{e\in X} \underline{w}_e^{\lambda} - F^*(S_E^{+\lambda})\}, \quad \lambda \in [0,1], \tag{23}$$

$$\overline{\delta}_X^{\lambda} = \max_{e\in X} \max\{0, \overline{w}_e^{\lambda} - F^*(S_{\{e\}}^{+\lambda})\}, \quad \lambda \in [0,1], \tag{24}$$

where $\underline{w}_e^{\lambda}$ and $\overline{w}_e^{\lambda}$ $e \in E$, are parametric weights (functions of the parameter λ), $S_{\{e\}}^{+\lambda}$ is the scenario in which we fix the weight of element e to $\overline{w}_e^{\lambda}$ and the weights of elements $g \in E \setminus \{e\}$ to $\underline{w}_g^{\lambda}$ and $S_E^{+\lambda}$ is the scenario in which all the weights are at $\overline{w}_e^{\lambda}$. The main difficulty in determining bounds $\underline{\delta}_X^{\lambda}$ and $\overline{\delta}_X^{\lambda}$ is the computation of $F^*(S_E^{+\lambda})$ and $F^*(S_{\{e\}}^{+\lambda})$, which are functions of parameter λ, $\lambda \in [0,1]$. Note that for any fixed $\lambda' \in [0,1]$, $F^*(S_E^{+\lambda'})$ (resp. $F^*(S_{\{e\}}^{+\lambda'})$) is the value of the weight of an optimal solution under scenario $S_E^{+\lambda'}$ (resp. $S_{\{e\}}^{+\lambda'}$). Hence, in order to describe analytically function $F^*(S_E^{+\lambda})$ (resp. $F^*(S_{\{e\}}^{+\lambda})$) for $\lambda \in [0,1]$, we have to determine sequences $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k = 1$ and $X_0, \ldots, X_{k-1}$ such that X_i is an optimal solution under $S_E^{+\lambda}$ (resp. $S_{\{e\}}^{+\lambda}$) for $\lambda \in [\lambda_i, \lambda_{i+1}]$, λ_i and λ_{i+1} are two adjacent points. At points $\lambda_1 < \lambda_2 < \cdots < \lambda_{k-1}$ optimal solutions change. Therefore,

X_i is optimal over the entire closed interval $[\lambda_i, \lambda_{i+1}]$ and nowhere else with the parametric weight $F(X_i, S_E^{+\lambda}) = F^*(S_E^{+\lambda})$ (resp. $F(X_i, S_{\{e\}}^{+\lambda}) = F^*(S_{\{e\}}^{+\lambda})$). Having computed $\lambda_1 < \lambda_2 < \cdots < \lambda_{k-1}$ and parametric weights $F(X_i, S_E^{+\lambda})$ (resp. $F(X_i, S_{\{e\}}^{+\lambda})$ for every $e \in X$), $\lambda \in [\lambda_i, \lambda_{i+1}]$, $i = 0, \ldots, k-1$, it is easy to describe analytically $F^*(S_E^{+\lambda})$ (resp. $F^*(S_{\{e\}}^{+\lambda})$ for every $e \in X$) for $\lambda \in [0, 1]$ and, in consequence, $\underline{\delta}_X^\lambda$ (resp. $\overline{\delta}_X^\lambda$) in (23) (resp. (24)). It turns out that if $\underline{w}_e^\lambda$ and $\overline{w}_e^\lambda$ are linear functions of λ for each $e \in E$, then for some particular bottleneck problems their parametric counterparts can be solved by algorithms proposed in [35, 36] and by algorithms being some adaptations to the bottleneck case of the ones given in [37, 38, 39].

The functions $F(X_i, S_E^{+\lambda})$ (resp. $F(X_i, S_{\{e\}}^{+\lambda})$), $\lambda \in [\lambda_i, \lambda_{i+1}]$, $i = 0, \ldots, k-1$, are piecewise linear and thus $F^*(S_E^{+\lambda})$ (resp. $F^*(S_{\{e\}}^{+\lambda})$) are piecewise linear for $\lambda \in [0, 1]$. In consequence, $F^*(S_E^{+\lambda})$, $F^*(S_{\{e\}}^{+\lambda})$ and also the family of intervals $\widetilde{\Delta}_X^\lambda$, $\lambda \in [0, 1]$, can be computed if the uncertain weights are modeled by trapezoidal or triangular fuzzy intervals or by more general fuzzy intervals of the L-L type (see (12), where $R = L$) since the bounds can be then easily linearized. Namely, it is sufficient to substitute $L^{-1}(\lambda)$ with parameter Θ in (12). The parametric approach requires performing operations on linear functions. Some methods for handling piecewise linear qualities and some operations that preserve piecewise linearity (maximum, minimum, addition and subtraction) can be found in [40].

The same reasoning applies to elements because it is enough to replace X with f in formulae (21) and (22). However, the computational complexity for an element depends now on the structure of problem $\mathcal{BP}$. For instance, if $\mathcal{BP}$ has matroidal structure, then the reasoning is exactly the same. We only need to use Corollaries 4, 5 and Proposition 4. Otherwise (if problem $\mathcal{BP}$ is not matroidal one) we can only compute the degree of possible optimality of an element f (Corollary 3) or determine the left profile $\underline{\delta}_f^\lambda$, $\lambda \in [0, 1]$, of the possibility distribution $\widetilde{\Delta}_f$ (Proposition 2).

Let us illustrate the above approach using a simple example of the bottleneck path problem with fuzzy weights shown in Figure 2. The arc weights are specified as triangular fuzzy intervals $\widetilde{W}_f = (w_f, \alpha_f, \beta_f)$, $f \in A = \{a, b, c, d, e\}$, so the bounds $\underline{w}_f^\lambda = w_f - \alpha_f(1-\lambda)$ and $\overline{w}_f^\lambda = w_f + \beta_f(1-\lambda)$ are linear functions of $\lambda \in [0, 1]$. Let us examine path $X = \{b, e\}$. We wish to determine its fuzzy deviation $\Delta_{\{b,e\}}$, i.e. functions $\underline{\delta}_{\{b,e\}}^\lambda$ and $\overline{\delta}_{\{b,e\}}^\lambda$, $\lambda \in [0, 1]$. In order to determine analytical representation of $F^*(S_A^{+\lambda})$, we

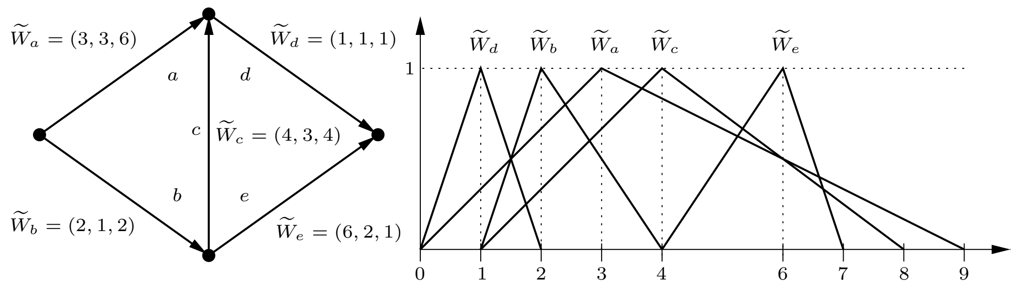

Figure 2: An example of a fuzzy-valued bottleneck path problem – a network with fuzzy weights (triangular fuzzy intervals).

need to compute a sequence of λ's and corresponding optimal paths under scenario $S_A^{+\lambda}$. The resulting sequence is $0 < \frac{1}{3} < \frac{1}{2} < 1$, path $\{b, e\}$ is optimal with weight $F(\{b,e\}, S_A^{+\lambda}) = 6 + (1-\lambda)$ for $\lambda \in [0, \frac{1}{3}]$, path $\{b, c, d\}$ is optimal with weight $F(\{b,c,d\}, S_A^{+\lambda}) = 4 + 4(1-\lambda)$ for $\lambda \in [\frac{1}{3}, \frac{1}{2}]$ and path $\{a, d\}$ is optimal with weight $F(\{a,d\}, S_A^{+\lambda}) = 3 + 6(1-\lambda)$ for $\lambda \in [\frac{1}{2}, 1]$. Accordingly, $F^*(S_A^{+\lambda})$ is a piecewise linear function whose value is $6+(1-\lambda)$ for $\lambda \in [0, \frac{1}{3}]$; $4 + 4(1-\lambda)$ for $\lambda \in [\frac{1}{3}, \frac{1}{2}]$; and $3 + 6(1-\lambda)$ for $\lambda \in [\frac{1}{2}, 1]$. Moreover, $\max_{f \in \{b,e\}} \underline{w}_f^\lambda = 6 - 2(1-\lambda)$, $\lambda \in [0,1]$. Applying (23) yields $\underline{\delta}_{\{b,e\}}^\lambda$ whose value is 0 for $\lambda \in [0, \frac{5}{8}]$; $3 - 8(1-\lambda)$ for $\lambda \in [\frac{5}{8}, 1]$ (see Figure 3).

Let us pass on to determining $\overline{\delta}_{\{b,e\}}^\lambda$. Path $\{a, d\}$ is optimal under scenarios $S_{\{b\}}^{+\lambda}$ and $S_{\{e\}}^{+\lambda}$ for every $\lambda \in [0,1]$. Thus $F^*(S_{\{b\}}^{+\lambda}) = F^*(S_{\{e\}}^{+\lambda}) = 3 - 3(1-\lambda)$ for $\lambda \in [0,1]$. Function $\max\{0, \overline{w}_b^\lambda - F^*(S_{\{b\}}^{+\lambda})\}$ is piecewise linear and its value is $-1 + 5(1-\lambda)$ for $\lambda \in [0, \frac{4}{5}]$; 0 for $\lambda \in [\frac{4}{5}, 1]$. It holds $\max\{0, \overline{w}_e^\lambda - F^*(S_{\{e\}}^{+\lambda})\} = 3 + 4(1-\lambda)$ for $\lambda \in [0,1]$. Applying (24) gives $\overline{\delta}_{\{b,e\}}^\lambda = 3 + 4(1-\lambda)$ for $\lambda \in [0,1]$. Having $\underline{\delta}_{\{b,e\}}^\lambda$ and $\overline{\delta}_{\{b,e\}}^\lambda$ for $\lambda \in [0,1]$, we can form possibility distribution $\mu_{\widetilde{\Delta}_{\{b,e\}}}$ for the deviations of $\{b, e\}$ (see Figure 3).

Having the possibility distribution we can get some information about path $\{b, e\}$. For example, we can compute $\Pi(\{b,e\}$ is optimal$) = \Pi(\delta_{\{b,e\}} = 0) = 5/8$ and $\mathrm{N}(\{b,e\}$ is optimal$) = 0$. In fact, we can compute the possibility and necessity of any event of the form $\delta_{\{b,e\}} \in [u, v]$. In particular, the interval $[0, 7]$ contains all possible values of $\delta_{\{b,e\}}$, so $\mathrm{N}(\delta_{\{b,e\}} \in [0,7]) = 1 - \Pi(\delta_{\{b,e\}} \notin [0,7]) = 1$

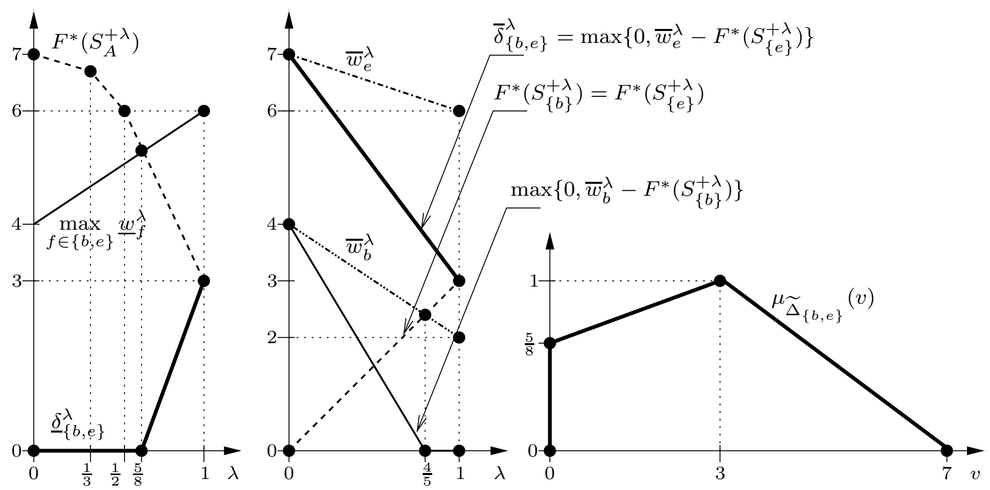

Figure 3: Bounds $\underline{\delta}_{\{b,e\}}^\lambda$ and $\overline{\delta}_{\{b,e\}}^\lambda$ for path $\{b,e\}$ and possibility distribution $\mu_{\widetilde{\Delta}_{\{b,e\}}}(v) = \Pi(\delta_{\{b,e\}} = v)$ for values of $\delta_{\{b,e\}}$.

4.4. Choosing a robust solution

We now propose some concepts of choosing a robust solution in the fuzzy-valued problem $\mathcal{BP}$. The first idea is to choose a solution with the highest degree of certainty that it will be optimal, i.e. an optimal solution to the following problem:

$$\max_{X\in\Phi} \mathrm{N}(X \text{ is optimal}) = \max_{X\in\Phi} \mathrm{N}(\delta_X = 0). \tag{25}$$

A solution to (25) is called a *best necessarily optimal solution*. Unfortunately, this concept has a drawback, since the criterion used in (25) is very strong. Namely, a solution X such that $\mathrm{N}(X \text{ is optimal}) > 0$ may not exist or even if it exists, its necessary optimality degree may be very small. On the other hand, maximizing the degree of possible optimality is trivial, since there is always at least one solution $X \in \Phi$ for which the degree of possible optimality attains its maximal value equal to 1. We thus meet the same problem as in the interval uncertainty representation - the possible optimality is too weak criterion of choosing a solution and the necessary optimality is too strong (see Section 3.2). To overcome this drawback, we replace the strong optimality requirement with a weaker one.

Suppose that a decision maker knows her/his preferences about δ_X and expresses it by a *fuzzy goal* $\widetilde{G}$, a fuzzy set in $\mathbb{R}$ with a compact support.

The membership function of the fuzzy goal $\mu_{\widetilde{G}}$ is a nonincreasing mapping from $[0, \infty)$ into $[0, 1]$ such that $\mu_{\widetilde{G}}(0) = 1$. The value of $\mu_{\widetilde{G}}(\delta_X)$ expresses the degree to which deviation δ_X satisfies the decision maker. We can now replace the strong requirement "X is optimal" ($\delta_X = 0$) with the weaker "$\delta_X \in \widetilde{G}$". Recall that δ_X is an unknown quantity characterized by possibility distribution $\pi_{\delta_X} = \mu_{\widetilde{\Delta}_X}$ (see (17)). So, "$\delta_X \in \widetilde{G}$" is a fuzzy event and we can compute the necessity that this event holds using (14):

$$\mathrm{N}(\delta_X \in \widetilde{G}) = 1 - \Pi(\delta_X \notin \widetilde{G}) = 1 - \sup_{v \in \mathbb{R}} \min\{\pi_{\delta_X}(v), 1 - \mu_{\widetilde{G}}(v)\}.$$

Using (17) we get an equivalent formula:

$$\mathrm{N}(\delta_X \in \widetilde{G}) = \inf_{S} \max\{1 - \pi(S), \mu_{\widetilde{G}}(\delta_X(S))\}.$$

One can check that $\mathrm{N}(\delta_X \in \widetilde{G}) = 1$ means that for all scenarios S such that $\pi(S) > 0$ the deviation of X in scenario S, $\delta_X(S)$, is totally accepted or equivalently the degree of possibility of event "$\delta_X \notin \widetilde{G}$" equals zero. If $\mathrm{N}(\delta_X \in \widetilde{G}) = 0$, then $\Pi(\delta_X \notin \widetilde{G}) = 1$ and, with possibility equal to 1 a scenario may occur, in which the deviation of X is not at all accepted. More generally, $\mathrm{N}(\delta_X \in \widetilde{G}) = 1-\lambda$, means that for all scenarios S such that $\pi(S) > \lambda$, the degree of satisfaction is not less than $1 - \lambda$, i.e. $\mu_{\widetilde{G}}(\delta_X(S)) \geq 1 - \lambda$, or equivalently, by (11), $\delta_X(S) \in \widetilde{G}^{1-\lambda} = [0, \overline{g}^{1-\lambda}]$. The above three cases are illustrated in Figure 4. We use the possibility distribution from Figure 3 and three different goals. The right profile of $\widetilde{\Delta}_{\{b,e\}}$, representing the largest deviation of $\{b, e\}$, is shown in bold.

There is an obvious connection between $\mathrm{N}(X \text{ is optimal})$ and $\mathrm{N}(\delta_X \in \widetilde{G})$, that is

$$\mathrm{N}(X \text{ is optimal}) \leq \mathrm{N}(\delta_X \in \widetilde{G}). \tag{26}$$

Hence, we have generalized and weakened the notion of the necessary optimality.

Accordingly, it is reasonable to choose a solution whose deviation belongs to $\widetilde{G}$ with the highest confidence. This leads to the following optimization problem:

$$\max_{X \in \Phi} \mathrm{N}(\delta_X \in \widetilde{G}). \tag{27}$$

An optimal solution to (27) is called a *best necessarily soft optimal solution*. We recall that the criterion of choosing a solution in (27) has been originally

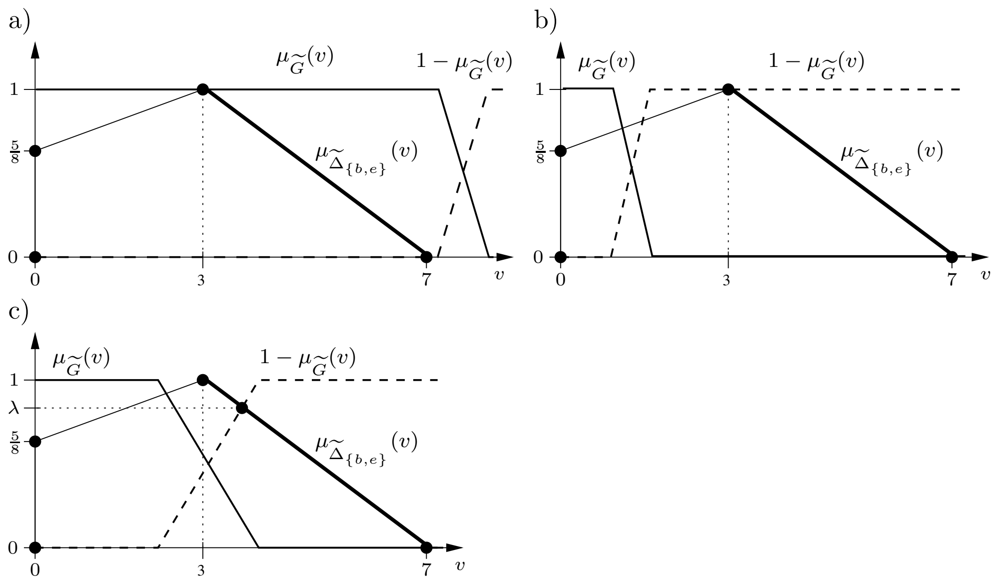

Figure 4: Three different cases depending on the choice of fuzzy goal $\widetilde{G}$: a) $\mathrm{N}(\delta_{\{b,e\}} \in \widetilde{G}) = 1$, b) $\mathrm{N}(\delta_{\{b,e\}} \in \widetilde{G}) = 0$, c) $\mathrm{N}(\delta_{\{b,e\}} \in \widetilde{G}) = 1 - \lambda$.

proposed in [14, 15] and applied to determining a robust solution for fuzzy linear programming problems. Obviously, if $\widetilde{G}$ is zero fuzzy interval, then we arrive to the problem of finding a best necessarily optimal solution (25).

It is not difficult to show that problem (27) is equivalent to the following one:

$$\begin{array}{ll} \min & \lambda \\ \text{s.t.} & \overline{\delta}_X^{\lambda} \leq \overline{g}^{1-\lambda}, \\ & \lambda \in [0,1], \\ & X \in \Phi. \end{array} \tag{28}$$

If λ^* is the optimal objective value of (28) and X^* is a best necessarily soft optimal solution, then $\mathrm{N}(X^* \text{ is soft optimal}) = 1 - \lambda^*$. If (28) is infeasible then $\mathrm{N}(\delta_X \in \widetilde{G}) = 0$ for all solutions $X \in \Phi$. In the next two sections we show two methods of solving (27).

4.4.1. Binary search algorithm

Since $\overline{\delta}_X^{\lambda}$ is nonincreasing and $\overline{g}^{1-\lambda}$ is nondecreasing function of λ, problem (28) can also be solved by binary search technique on $\lambda \in [0, 1]$ (see

Algorithm 1). In order to find the optimal value of λ^* in $[0,1]$, we seek at each iteration, for a fixed λ, a solution $X \in \Phi$ that satisfies inequality $\overline{\delta}_X^{\lambda} \leq \overline{g}^{1-\lambda}$. Observe that $\overline{\delta}_X^{\lambda}$ is the maximal regret of X under scenario set $\widetilde{\Gamma}^{\lambda}$ (see Section 3.2). Thus, inequality $\overline{\delta}_X^{\lambda} \leq \overline{g}^{1-\lambda}$ is satisfied for some $X \in \Phi$ if and only if it is satisfied by an optimal min-max regret solution under $\widetilde{\Gamma}^{\lambda}$.

Algorithm 1: Finding a best necessarily soft optimal solution

Input: Problem $\mathcal{BP}$ with fuzzy weights $\widetilde{W}_e$, $e \in E$, error tolerance ϵ, fuzzy goal $\widetilde{G}$.
Output: A best necessarily soft optimal solution.

Find an optimal minmax regret solution X under scenario set $\widetilde{\Gamma}^1$
if $\overline{\delta}_X^1 > \overline{g}^0$ **then**
  return X /* $\mathrm{N}(\delta_X \in \widetilde{G}) = 0$ for all $X \in \Phi$ */
$\lambda_1 \leftarrow \frac{1}{2}$; $\lambda_2 \leftarrow 0$; $k \leftarrow 1$
while $|\lambda_1 - \lambda_2| \geq \epsilon$ **do**
  $\lambda_2 \leftarrow \lambda_1$
  Find an optimal minmax regret solution Y under scenario set $\widetilde{\Gamma}^{\lambda_1}$
  if $\overline{\delta}_Y(\lambda_1) \leq \overline{g}^{1-\lambda_1}$ **then** $X \leftarrow Y$, $\lambda_1 \leftarrow \lambda_1 - \frac{1}{2^{k+1}}$
  else $\lambda_1 \leftarrow \lambda_1 + \frac{1}{2^{k+1}}$
  $k \leftarrow k+1$
return X /* $\mathrm{N}(\delta_X \in \widetilde{G}) = 1 - \lambda_1$ */

The running time of Algorithm 1 is $O(I(|E|)\log \epsilon^{-1})$ time, where $\epsilon > 0$ is a given error tolerance and $I(|E|)$ is the time required for seeking an optimal minmax regret solution under scenario set $\widetilde{\Gamma}^{\lambda}$. Note that $I(|E|) = O(|E| + |E|f(|E|))$ (see Section 3.2), where $f(|E|)$ is the running time of an algorithm for problem $\mathcal{BP}$ with deterministic weights. Thus, the overall running time is $O((|E| + |E|f(|E|))\log \epsilon^{-1})$. Consequently, Algorithm 1 is polynomial if the running time $f(|E|)$ is polynomial. In [1, 2, 3, 4] some polynomial algorithms for the bottleneck path, the bottleneck assignment, the bottleneck spanning tree, and the bottleneck matroid base problems with deterministic weights can be found. Therefore, Algorithm 1 is polynomial for a wide class of bottleneck combinatorial optimization problems.

4.4.2. Parametric technique

We now present a parametric approach to finding a best necessary optimal solution. The problem (28) can be rewritten as follows (see also (10)):

$$\min\{\lambda \in [0,1] : \min_{X\in\Phi}\max_{e\in X} \hat{w}_e^{\lambda} \le \overline{g}^{1-\lambda}\}, \tag{29}$$

where $\hat{w}_e^{\lambda} = \max\{0, \overline{w}_e^{\lambda} - F^*(S_{\{e\}}^{+\lambda})\}$, $e \in E$, are parametric weights. Determining these weights requires computing functions $F^*(S_{\{e\}}^{+\lambda})$ for each $e \in E$. This can be done by applying the parametric approach for finding the possibility distribution of a solution (see Section 4.3 for details). As the result we obtain a parametric bottleneck problem with weights $\hat{w}_e^{\lambda}$, $e \in E$:

$$\overline{\delta}^{\lambda} = \min_{X\in\Phi}\max_{e\in X} \hat{w}_e^{\lambda}, \quad \lambda \in [0,1], \tag{30}$$

Solving (30) we get sequence $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k = 1$ and the optimal minmax regret solutions $X_0, \ldots, X_{k-1}$ with maximal regrets $\overline{\delta}_{X_i}^{\lambda}$, $\lambda \in [\lambda_i, \lambda_{i+1}]$, $i = 0, \ldots, k-1$, which provide an analytical description of the function $\overline{\delta}^{\lambda}$ for $\lambda \in [0,1]$ (see Section 4.3 for references to algorithms for some particular parametric bottleneck problem). The function $\overline{\delta}^{\lambda}$ is nonincreasing and, by (29), we conclude that in order to obtain a best necessarily soft optimal solution we have to find the intersection point λ^* of $\overline{\delta}^{\lambda}$ with $\overline{g}^{1-\lambda}$. If $\lambda^* \in [\lambda_i, \lambda_{i+1}]$, then X_i is a best necessarily soft optimal solution. If such an intersection point does not exist, then two cases are possible: either $\overline{\delta}^{1} > \overline{g}^{0}$ or $\overline{\delta}^{0} < \overline{g}^{1}$. In the former case $\mathrm{N}(\delta_X \in \widetilde{G}) = 0$ for all feasible solutions $X \in \Phi$ and, in the latter one, $\mathrm{N}(\delta_{X_0} \in \widetilde{G}) = 1$ and X_0 is a best necessarily soft optimal solution.

The above solution procedure is more complex than binary search. It has, however, two advantages. It gives an exact best necessarily soft optimal solution. Furthermore, it provides some additional information in the fuzzy problem. Observe that, regardless of fuzzy goal, a best necessarily soft optimal solution is always among $X_0, \ldots, X_{k-1}$. One can easily check how the solution changes when the fuzzy goal $\widetilde{G}$ is changed. So, we can perform a sensitivity analysis of the obtained solution.

In the absence of fuzzy goal, we can treat the set of solutions $\boldsymbol{X} = \{X_0, \ldots, X_{k-1}\}$ as a solution of the fuzzy problem with the following interpretation. The first solution X_0 is the most conservative one. It minimizes the maximal regret over all possible scenarios S such that $\pi(S) > 0$. It should be chosen by very pessimistic or very risk-averse decision maker. On the other hand, the last solution X_{k-1} minimizes the maximal regret only over the most plausible scenarios S such that $\pi(S) = 1$ and it may be chosen by an optimistic decision maker, who considers only the most possible states of the world. So, $\boldsymbol{X}$ contains solutions of different degree of risk or conservatism. After introducing a fuzzy goal, which expresses the decision maker's preferences (or averse to risk), exactly one of the sequences of $\boldsymbol{X}$ will be chosen.

5. Conclusions

In this paper, we have studied a general bottleneck combinatorial optimization problem with uncertain element weights modeled by fuzzy intervals. The membership functions of theses fuzzy intervals are regarded as possibility distributions for the values of the unknown weights. We have described, in this setting, the notions of possible and necessary optimality of a solution and an element and the necessary soft optimality of a solution. These notions are natural generalizations of the ones introduced in the interval-valued case. In order to choose a robust solution, we have determined a best necessary soft optimal solution. This concept of choosing a solution is also a generalization of the minmax regret criterion to the fuzzy case. We have thus shown that there exists a link between interval and possibilistic uncertainty representation. Hence, we have discussed first the interval-valued case and then we have extended the notions and the methods introduced for the interval-valued problem to the fuzzy-valued one. Indeed, the optimality evaluation and choosing a robust solution in the fuzzy problem boil down to solving a number of problems $\mathcal{BP}$ with interval weights. Both problems can be solved in polynomial time if the corresponding deterministic counterparts (problems $\mathcal{BP}$ with precise weights) are polynomially solvable. This holds true for a wide class of classical bottleneck combinatorial problems. This is in contrast to the problems with a linear sum objective, where the optimality evaluation and computing a robust solution is mostly NP-hard.

Appendix A.

Proof (Proposition 1). Let $S \in \Gamma$ be a scenario that minimizes the deviation, that is $\underline{\delta}_X = \delta_X(S) = F(X, S) - F^*(S)$ (see (2)). Since $\max_{e\in X} \underline{w}_e \leq$

$F(X,S)$, $F^*(S_E^+) \geq F^*(S)$ and $\underline{\delta}_X \geq 0$, we conclude that

$$\underline{\delta}_X \geq \max\{0, \max_{e\in X} \underline{w}_e - F^*(S_E^+)\}. \tag{A.1}$$

It remains to show that the inequality $\leq$ also holds in (A.1). Let Y be an optimal solution under S_E^+ and let $g = \arg\max_{e\in Y} \overline{w}_e$. We consider two cases. (i) $\max_{e\in X} \underline{w}_e > \overline{w}_g$. Denote $h = \arg\max_{e\in X} \underline{w}_e$. Consider scenario S such that $w_e(S) = \min\{\underline{w}_h, \overline{w}_e\}$ for all $e \in X$ and $w_e(S) = \overline{w}_e$ for all $e \in E \setminus X$. Since $\underline{w}_h \geq \underline{w}_e$ for all $e \in X$, $S \in \Gamma$. It is easy to check that $F(X,S) = \underline{w}_h$ and $F^*(S) = F^*(S_E^+)$. Hence $\underline{\delta}_X \leq \delta_X(S) = \max_{e\in X} \underline{w}_e - F^*(S_E^+) \leq \max\{0, \max_{e\in X} \underline{w}_e - F^*(S_E^+)\}$, which together with (A.1) yield (4). (ii) $\max_{e\in X} \underline{w}_e \leq \overline{w}_g$. Consider scenario S such that under this scenario all elements $e \in E \setminus X$ have weights $\overline{w}_e$ and all the elements $e \in X$ have weights $\min\{\overline{w}_e, \overline{w}_g\}$. Since $\underline{w}_e \leq \overline{w}_g$ for all $e \in X$, $S \in \Gamma$. One can easily verify that X is optimal under S, which means that $\underline{\delta}_X = 0 \leq \max\{0, \max_{e\in X} \underline{w}_e - F^*(S_E^+)\}$. This, together with (A.1), give (4). □

Proof (Proposition 2). It is easy to see that $\underline{\delta}_f = \min_{X\in\Phi_f} \underline{\delta}_X$. So, by Proposition 1, $\underline{\delta}_f = \min_{X\in\Phi_f} \max\{0, F(X, S_E^-) - F^*(S_E^+)\}$, which immediately leads to (6). □

Proof (Proposition 3). ($\Rightarrow$) Obvious. ($\Leftarrow$) Suppose, by contradiction, that X is composed of possibly optimal elements and X is not possibly optimal. According to Corollary 1, we get $\underline{w}_f = \max_{e\in X} \underline{w}_e > F^*(S_E^+)$. But $\min_{X\in\Phi_f} F(X, S_E^-) \geq \underline{w}_f$ and $\min_{X\in\Phi_f} F(X, S_E^-) > F^*(S_E^+)$, which contradicts the assumption that $f \in X$ is possibly optimal (see Corollary 3). □

Proof (Proposition 4). We will use the following well known property of matroids: if $B \in \mathcal{B}$ is a base and f is an element such that $f \notin B$, then $B \cup \{f\}$ contains the unique circuit C. Furthermore, for every $e \in C$, set $(B \cup \{f\}) \setminus \{e\}$ is a base. We first prove equality (7). Let $S \in \Gamma$ be a scenario that minimizes $\delta_f(S)$ (see (3)), $\underline{\delta}_f = \delta_f(S) = \min_{B\in\mathcal{B}_f} F(B,S) - F^*(S)$. Therefore, we have the following inequality

$$\underline{\delta}_f \geq \max\{0, \underline{w}_f - F^*(S_E^+)\}, \tag{A.2}$$

because $\underline{w}_f \leq \min_{B\in\mathcal{B}_f} F(B,S)$, $F^*(S) \leq F^*(S_E^+)$ and $\underline{\delta}_f \geq 0$. We now need to show that inequality $\leq$ holds in (A.2). Let B^* be an optimal base under S_E^+, $F(B^*, S_E^+) = F^*(S_E^+)$, and set $g = \arg\max_{e\in B^*} \overline{w}_e$. (i) Assume that $\underline{w}_f >$

$\overline{w}_g$. Consider scenario $S \in \Gamma$ such that $w_f(S) = \underline{w}_f$ and $w_e(S) = \overline{w}_e$ for all $E \setminus \{f\}$. Since $\underline{w}_f > F^*(S^+_E)$, $F(B^*, S) = F^*(S) = F^*(S^+_E)$. Observe, $B^* \cup \{f\}$ contains an unique circuit C. Set $B^{'} = (B^* \setminus \{e\}) \cup \{f\}$, $e \in C \setminus \{f\}$, is a base and $B^{'} \in \mathcal{B}_f$, where $\mathcal{B}_f$ stands for the set all bases that contain element f. From the above and $\underline{w}_f > \overline{w}_g$, we obtain $\min_{B \in \mathcal{B}_f} F(B, S) \leq F(B^{'}, S) = \underline{w}_f$. Therefore, $\underline{\delta}_f \leq \delta_f(S) = \min_{B \in \mathcal{B}_f} F(B, S) - F^*(S) \leq \max\{0, \underline{w}_f - F^*(S^+_E)\}$ which together with (A.2) imply equality (7). (ii) Assume that $\underline{w}_f \leq \overline{w}_g$. We will show that in this case $\underline{\delta}_f = 0$, which together with (A.2) yields (7). Consider scenario $S \in \Gamma$ such that $w_f(S) = \min\{\overline{w}_f, \overline{w}_g\}$ and $w_e(S) = \overline{w}_e$ for all $E \setminus \{f\}$. It is easily seen that $F(B^*, S) = F^*(S) = F^*(S^+_E)$. If $f \in B^*$, then $\underline{\delta}_f = 0$ and we are done. Otherwise, $B^* \cup \{f\}$ contains an unique circuit C. Set $B^{'} = (B^* \setminus \{e\}) \cup \{f\}$, $e \in C \setminus \{f\}$, is a base and $F(B^{'}, S) \leq F(B^*, S)$. In consequence $B^{'}$ is optimal under S and $\underline{\delta}_f = 0$.

We now show equality (8). Let $S \in \Gamma$ be a scenario that maximizes $\delta(S)$, that is $\overline{\delta}_f = \delta_f(S)$. Since $\delta_f(S) \geq \delta_f(S^+_{\{f\}}) = \min_{B \in \mathcal{B}_f} F(B, S^+_{\{f\}}) - F^*(S^+_{\{f\}})$, $\min_{B \in \mathcal{B}_f} F(B, S^+_{\{f\}}) \geq \overline{w}_f$ and $\overline{\delta}_f \geq 0$, we see that

$$\overline{\delta}_f \geq \max\{0, \overline{w}_f - F^*(S^+_{\{f\}})\}. \tag{A.3}$$

It remains to show that inequality $\leq$ holds in (A.3). It is obviously true if $\overline{\delta}_f = 0$. So, suppose that $\overline{\delta}_f = \delta_f(S) > 0$. Thus, $\min_{B \in \mathcal{B}_f} F(B, S) > F^*(S) = F(B^*, S)$. Obviously, $f \notin B^*$ and $w_f(S) > F^*(S)$. Moreover, $w_f(S) \geq \min_{B \in \mathcal{B}_f} F(B, S)$. Otherwise, base $B^{'} = (B^* \setminus \{e\}) \cup \{f\}$, $e \in C \setminus \{f\}$, where C is an unique circuit in $B^* \cup \{f\}$, is such that $F(B^{'}, S) < \min_{B \in \mathcal{B}_f} F(B, S)$, $B^{'} \in \mathcal{B}_f$, a contradiction. Therefore, $\overline{w}_f \geq w_f(S) \geq \min_{B \in \mathcal{B}_f} F(B, S) > F^*(S)$. Since $f \notin B^*$, $F^*(S) \geq F^*(S^+_{\{f\}})$. Hence, $\overline{\delta}_f = \delta_f(S) = \min_{B \in \mathcal{B}_f} F(B, S) - F^*(S) \leq \max\{0, \overline{w}_f - F^*(S^+_{\{f\}})\}$, which together with (A.3) give equality (8). □

Proof (Proposition 5). We use a proof by contraposition. Assume that X is not possibly optimal. From Corollary 1, we have $\max_{e \in X} \underline{w}_e > F^*(S^+_E)$. Let Y^* be an optimal solution in scenario S^+_E. Define $g = \arg\max_{e \in Y^*} \overline{w}_e$ and $h = \arg\max_{e \in X} \underline{w}_e$. Thus $w_h(S) > \overline{w}_g$ for all scenarios $S \in \Gamma$. But $\overline{w}_g \geq w_e(S)$ for all $e \in Y^*$ in every scenario S, which yields $w_h(S) > w_e(S)$ for $e \in Y^*$. From this we conclude that $F(X, S) > F(Y^*, S)$ for all $S \in \Gamma$, which implies $\overline{\delta}_X > \overline{\delta}_Y$. In consequence, X cannot be an optimal minmax regret solution. It is obvious that every possibly optimal solution is composed of possibly optimal elements. □

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