Criticality in the network with imprecise activity times

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Abstract

A natural generalization of the criticality notion in a project network model for the case of interval and fuzzy activities duration times is given. A problem of determining a critical path in a network with interval activities duration times is also considered. Two methods of calculation of the path degree of criticality in a network with fuzzy activities duration times are presented.

Keywords: Decision analysis, Fuzzy CPM, Fuzzy PERT, Critical path.

1 Introduction

The critical path method (CPM), worked out at the beginning of the sixties (see [10]), has become one of the tools that are most useful in practice and are applied in the planning and control of the realization of complex projects. It consists in the identification of the so called critical paths, critical activities, critical events in a network, which is the project model, under the assumption the earliest possible project completion time. By the way one calculates certain quantities that are useful for the decision maker such as: events and activities slacks, the earliest and the latest start and finish times of the activities. What is essential in the CPM method is the assumption that the activities duration times are deterministic and known. Of course in practice this assumption cannot always be fulfilled with a satisfying accuracy. That is why in the literature several methods have appeared that are destined for the analysis of projects with nondeterministic activities duration times, represented in the network with random variables or fuzzy numbers (see e.g. [1, 2, 6, 7, 8, 9, 11, 13, 15, 16]). However, the problems of determining exact characteristics in such networks (for example, the expected value of the project completion time) are not easy. The opposite is true only if far going simplifying assumptions are made in the method of network analysis, like it is done in the source PERT method (see [12]), but then only approximate estimates of the project characteristics are obtained.

In this paper we give a natural, unconsidered in the literature so far, generalization of the criticality notion for the case of a network with interval and fuzzy activities duration times. We also occupy ourselves with the problem of determining a critical path in a network with interval activities duration times and the problem of computing path degrees of criticality in a network with fuzzy activities duration times. Before we pass on to the essential considerations, let us remind the notions of criticality for a path, an activity and an event in a network with deterministic activities times.

2 CPM for network with crisp activity times

A network $S =< V, A, t >$ being a project arc model is given (activities are represented by arcs of a network). V is the set of nodes (events) and $A \subset V \times V$ is the set of arcs (activities). The network S is a directed, compact, acyclic graph. The set $V = \{1, 2, \dots , n\}$ is labelled in such a way that $i < j$ for each activity $(i, j) \in A$. Let t be a nonnegative real function defined on the set of the activities, $t: A \to \mathbb{R}^+$, where $t(i,j) \stackrel{\text{def}}{=} t_{ij}$ is duration time of activity $(i,j) \in A$. The essence of the CPM method (from numerical point of view) are two recurrence formulae which are used to determine the earliest and the latest moments of occurring the events $i \in V$. The earliest moment, T_i^e, of occurrence of the event $i \in V$ is determined according to the following recurrence formula:

$$T_i^e = \begin{cases} 0 & \text{for} \quad i = 1, \\ \max_{k \in P(i)}(T_k^e + t_{ki}) & \text{for} \quad i > 1, \end{cases} \tag{1}$$

where $P(i) = \{k \in V | (k,i) \in A\}$. And the latest moment, T_i^l, of occurrence of the event $i \in V$ can be found by use of the following formula:

$$T_i^l = \begin{cases} T_n^e & \text{for} \quad i = n, \\ \min_{k \in S(i)}(T_k^l - t_{ik}) & \text{for} \quad i < n, \end{cases} \tag{2}$$

where $S(i) = \{k \in V | (i,k) \in A\}$. The times obtained according to (1) and (2) are applied in the calculation of slack times, $L_i = T_i^l - T_i^e$, for events $i \in V$ and slack times, $Z(i,j) = T_j^l - T_i^e - t_{ij}$, for activities $(i,j) \in A$. By means of such determined quantities we can define notions of criticality of an event and an activity.

Definition 1. *An activity $(i,j) \in A$ is critical if and only if $Z(i,j) = 0$.*

Definition 2. *An event $i \in V$ is critical if and only if $L_i = 0$.*

Let us denote by P the set of all paths in S from node 1 to node n.

Definition 3. *A path $p \in P$ is critical if and only if the activities belonging to p are critical.*

3 CPM for network with interval activity times

A network $S = < V, A, T >$ is given. All elements of this network are the same as in deterministic case except for function T, which maps the set of activities A into the set $I(\mathbb{R}^+)$ (the set of intervals with nonnegative ends), $T : A \to I(\mathbb{R}^+)$. We denote $T(i,j) \stackrel{\text{def}}{=} T_{ij} = [\underline{t}_{ij}, \overline{t}_{ij}]$, $(i,j) \in A$.

Definition 4. *A path $p \in P$ is i-critical (interval critical) in S if and only if there exists a set of times t_{ij}, $t_{ij} \in [\underline{t}_{ij}, \overline{t}_{ij}]$, $(i,j) \in A$, such that p is a critical path in the sense of definition 3, after replacing the interval times T_{ij} with the exact values t_{ij}, $T_{ij} := t_{ij}$.*

Definition 5. *An activity $(k,l) \in A$ (an event $k \in V$) is i-critical (interval critical) in the network S if and only if there exists a set of times t_{ij}, $t_{ij} \in [\underline{t}_{ij}, \overline{t}_{ij}]$, $(i,j) \in A$, such that (k,l) (k) is a critical in the sense of definition 1 (definition 2) in the network S, after replacing the interval times T_{ij} with the exact values t_{ij}, $T_{ij} := t_{ij}$.*

The key theorem for further considerations is Theorem 1 which determines a necessary and sufficient condition of i-criticality of a given path $p \in P$.

Theorem 1. *A path $p \in P$ is i-critical in S if and only if it is critical in the sense of definition 3 in the network S, after replacing the interval times $T_{ij} = [\underline{t}_{ij}, \overline{t}_{ij}]$, $(i,j) \in A$, with the exact values t_{ij} determined by means of the following formula*

$$t_{ij} = \begin{cases} \overline{t}_{ij} & \text{if} \quad (i,j) \in p, \\ \underline{t}_{ij} & \text{if} \quad (i,j) \notin p. \end{cases}$$

Proof (see [4]). □

It follows from Definition 4 and Theorem 1 that the problem of determining an arbitrary i-critical path and that of the estimation of the i-criticality of a fixed path $p \in P$ are easy problems and they can be solved in time bounded by a polynomial in the size of the network.

In the first case it is enough to apply the classical CPM (section 2) to the network S with any fixed values t_{ij} of activity times chosen from the corresponding interval duration times, $t_{ij} \in [\underline{t}_{ij}, \overline{t}_{ij}]$. And in the second one it is sufficient to apply the classical CPM to the network S, after replacing the interval times $[\underline{t}_{ij}, \overline{t}_{ij}]$ with the exact values t_{ij} determined as in Theorem 1. However, not all the problems concerning the i-criticality are so easy. Some of them have turned out to be hard ones. In paper [3] we have proved that the problem of estimating the i-criticality of a fixed activity (a fixed event) is NP-complete in the strong sense. We have also shown that the problem of determining K i-critical paths is NP-hard.

4 CPM for network with fuzzy activity times

A network $S =< V, A, \tilde{T} >$ is given. All elements of this network are the same as in deterministic case except for function $\tilde{T}$ which is now defined in the following way: $\tilde{T} : A \to F(\mathbb{R}^+)$, where $F(\mathbb{R}^+)$ is the set of nonnegative fuzzy numbers. We introduce denotation $\tilde{T}(i,j) \stackrel{\text{def}}{=} \tilde{T}_{ij}$, $(i,j) \in A$.

Definition 6. *The fuzzy critical (f-critical) path in S is the fuzzy set $\tilde{P}$ in set P with the membership function $\mu_{\tilde{P}} : P \to [0,1]$ determined in the following way:*

$$\mu_{\tilde{P}}(p) = \sup_{\substack{t_{ij}\in\mathbb{R}^+,(i,j)\in A \\ \text{and } p \text{is critical with} \\ \text{activities duration times} \\ \text{equal to } t_{ij},(i,j)\in A}} \min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}(t_{ij}), p \in P$$

We say that a path p is f-critical with the degree $\mu_{\tilde{P}}(p)$. The value $\mu_{\tilde{P}}(p)$ stands for the path degree of criticality, possibility of the criticality of path p. To put it in another way, $\mu_{\tilde{P}}$ determines a possibility distribution of the criticality of the path in the set P which is generated by possibility distributions of activities duration times $\mu_{\tilde{T}_{ij}}$, $(i,j) \in A$ (generated according to extension principle of Zadeh – if the criticality, or lack of it, is treated as the activities duration times function in the network).

Definition 7. *The fuzzy critical (f-critical) activity (event) in S is the fuzzy set $\tilde{A}$ ($\tilde{E}$) in set A (V) with the membership function determined in the following way:*

$$\mu_{\tilde{A}}(k,l) = \sup_{\substack{t_{ij}\in\mathbb{R}^+,(i,j)\in A \\ \text{and } (k,l) \text{is critical with} \\ \text{activities duration times} \\ \text{equal to } t_{ij},(i,j)\in A}} \min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}(t_{ij}),$$

$$(k,l) \in A$$

$$(\mu_{\tilde{E}}(k) = \sup_{\substack{t_{ij}\in\mathbb{R}^+,(i,j)\in A \\ \text{and } k \text{is critical with} \\ \text{activities duration times} \\ \text{equal to } t_{ij},(i,j)\in A}} \min_{(i,j)\in A} \mu_{\tilde{T}_{ij}}(t_{ij}),$$

$$k \in V)$$

In [4], there have been shown relationships between the path degree of criticality and the degrees of the criticality of the activities and the events belonging to the path.

5 Determining the path degree of criticality

In this section, we present two effective method of determining the path degree of criticality. Before we pass on to the essentials we introduce a notion of *feasibility* of a value $\lambda \in [0,1]$ under a path $p \in P$ and then we formulate the theorem which we substantially use in the both methods.

Definition 8. *The value $\lambda \in [0,1]$ is called feasible under the path $p \in P$ if and only if p is i-critical in the network S with interval activities duration times $T_{ij} = \tilde{T}^{\lambda}_{ij} = [\underline{t}^{\lambda}_{ij}, \overline{t}^{\lambda}_{ij}]$, where $\underline{t}^{\lambda}_{ij} = \inf\{t_{ij} \in \mathbb{R}^+ | \mu_{\tilde{T}_{ij}}(t_{ij}) \geq \lambda\}$, $\overline{t}^{\lambda}_{ij} = \sup\{t_{ij} \in \mathbb{R}^+ | \mu_{\tilde{T}_{ij}}(t_{ij}) \geq \lambda\}$, $(i,j) \in A$.*

Theorem 2. *The following equality holds: $\mu_{\tilde{P}}(p) = \sup\{\lambda | \lambda$ is a feasible value under the path $p \in P\}$.*

Proof Obvious. It follows directly from the theorem which asserts that every fuzzy set can be decomposed according to its λ-cuts (see [14]). □

5.1 An algorithm for computing the path degree of criticality

Now we present an algorithm for computing the path degree of criticality. It is based on the idea of bisection of the unit interval of possible values of λ to compute the maximal feasible value λ under the path p. In this algorithm at each iteration k we test if the value $\lambda_k \in [0,1]$ is feasible under the path p. The testing can be reduced to applying the classical CPM method to the network S, after replacing the times $T_{ij} = \tilde{T}^{\lambda_k}_{ij} = [\underline{t}^{\lambda_k}_{ij}, \overline{t}^{\lambda_k}_{ij}]$, $(i,j) \in A$, with the exact values t_{ij} determined as in Theorem 1.

Algorithm 1.

Step 1 *Assign $k := 0$.*

Step 2 *Test if $\lambda = \epsilon$ is a feasible value under the path p. If it is not then assign $\lambda_{\max} := 0$ and go to step 6.*

Step 3 *Assign $\lambda_k := 1$ and test if λ_k is a feasible value under the path p. If it is then assign $\lambda_{\max} := 1$ and go to step 6.*

Step 4 *Assign* $k := k + 1$,

$$\lambda_k := \begin{cases} \lambda_{k-1} + 1/2^k & \text{if} \quad \lambda_{k-1} \text{ is a feasible value under the path } p, \\ \lambda_{k-1} - 1/2^k & \text{otherwise} \end{cases}$$

Test if λ_k *is a feasible value under the path p. If it is then assign* $\lambda_{\max} := \lambda_k$.

Step 5 *If* $k < K$ *then go to step 4.*

Step 6 *Assign* $\mu_{\tilde{P}}(p) := \lambda_{\max}$, *stop.*

The length K of the generated sequence in step 4 depends on the assumed accuracy of computation. If we want the absolute error of computation to be not greater than 10^{-N} then K has to fulfill the following condition

$$K \geq N / \log_{10} 2.$$

The value of ϵ used in step 2 should be positive and not grater than the assumed absolute error of computation.

5.2 A linear programming approach

Now we present another approach to the problem of determining the path degree of criticality showing that the problem can be reduced, under certain assumptions on membership functions of fuzzy activities duration times $\tilde{T}_{ij}$, to that of determining the optimal solutions of a classical linear programming problem.

Let us return to the interval case considered in Section 3. Testing if path $p \in P$ is i-critical in network S with activity times equal to $T_{ij} = [\underline{t}_{ij}, \overline{t}_{ij}]$ can be reduced to solution of the following system of linear equalities and inequalities:

$$\begin{array}{ll} t_j - t_i - \overline{t}_{ij} = 0, & (i,j) \in p, \\ t_j - t_i - \underline{t}_{ij} \geq 0, & (i,j) \notin p, \\ t_1 = 0, & \\ t_i \geq 0, & (i = 2, \dots, n), \end{array} \tag{3}$$

where unknowns t_i denote moments of occurring the events $i \in V$ in network S.

From Theorem 1 it follows that path p is i-critical if and only if the system (3) has a solution.

Thus, the statement of $\lambda \in (0, 1]$ feasibility under the path p (in the fuzzy case) can be reduced to testing if the following system of equalities and inequalities has a solution:

$$\begin{array}{ll} t_j - t_i - \overline{t}_{ij}^{\lambda} = 0, & (i,j) \in p, \\ t_j - t_i - \underline{t}_{ij}^{\lambda} \geq 0, & (i,j) \notin p, \\ t_1 = 0, & \\ t_i \geq 0, & (i = 2, \dots, n), \end{array} \tag{4}$$

where $\underline{t}_{ij}^{\lambda}$ and $\overline{t}_{ij}^{\lambda}$ are the ends of the interval $\tilde{T}_{ij}^{\lambda} = [\underline{t}_{ij}^{\lambda}, \overline{t}_{ij}^{\lambda}]$ (the λ-cut of $\tilde{T}_{ij}$).

Hence, determination of the criticality degree of path $p \in P$, $\mu_{\tilde{P}}(p)$, can be reduced, according to Theorem 2, to the following mathematical programming problem:

$$\lambda \to \max$$

$$\begin{array}{ll} t_j - t_i - \overline{t}_{ij}^{\lambda} = 0 & (i,j) \in p \\ t_j - t_i - \underline{t}_{ij}^{\lambda} \geq 0 & (i,j) \notin p \\ t_1 = 0 & \\ t_i \geq 0 & (i = 2, \dots, n), \end{array} \tag{5}$$

where $\lambda \in [0, 1]$. If $\lambda_{\max}$ is the optimal objective value of (5) then $\mu_{\tilde{P}}(p) = \lambda_{\max}$.

In the general case it may be difficult to find an optimal solution of the problem (5).

Let us assume that fuzzy activities duration times $\tilde{T}_{ij}$, $(i,j) \in A$, are given by means of fuzzy numbers of the same $L - L$ type (see [5]), i.e. $\tilde{T}_{ij} = (\underline{t}_{ij}, \overline{t}_{ij}, \alpha_{ij}, \beta_{ij})_{L-L}$. In this case λ-cuts of a fuzzy number $\tilde{T}_{ij}$ have the form

$$\tilde{T}_{ij}^{\lambda} = [\underline{t}_{ij} - L^{-1}(\lambda)\alpha_{ij}, \overline{t}_{ij} + L^{-1}(\lambda)\beta_{ij}].$$

and the problem (5) reduces itself to the following linear programming problem:

$$\Theta \to \min$$

$$\begin{array}{ll} t_j - t_i - \overline{t}_{ij} - \beta_{ij}\Theta = 0 & (i,j) \in p \\ t_j - t_i - \underline{t}_{ij} + \alpha_{ij}\Theta \geq 0 & (i,j) \notin p \\ t_1 = 0 & \\ t_i \geq 0 & (i = 2, \dots, n), \end{array} \tag{6}$$

where $\Theta \in [\underline{\Theta}, \overline{\Theta}]$, $\Theta = L^{-1}(\lambda)$, $\underline{\Theta} = L^{-1}(1)$, $\overline{\Theta} = L^{-1}(0)$. If $\Theta_{\min}$ is the optimal objective value of (6) then the path degree of criticality, $\mu_{\tilde{P}}(p)$, is equal to $L(\theta_{\min})$.

5.3 Examples

Let us illustrate the utility of the methods of computing the path degrees of criticality with simple numerical examples.

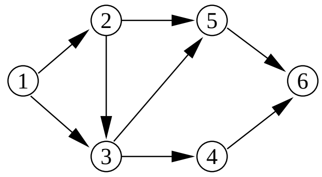

Figure 1: The structure of a project in the Examples 1, 2

Example 1. *The network representing a structure of a project is given in Figure 1. The activities duration times are fuzzy numbers of* $L_{ij} - L_{ij}$ *type,* $(i,j) \in A$. $\tilde{T}_{12} = (3,4,2,2)_{L_{12}-L_{12}}$, $\tilde{T}_{13} = (6,6,2,4)_{L_{13}-L_{13}}$, $\tilde{T}_{23} = (3.5,4,2,0)_{L_{23}-L_{23}}$, $\tilde{T}_{25} = (4,5,0,2)_{L_{25}-L_{25}}$, $\tilde{T}_{35} = (7,7,2,2)_{L_{35}-L_{35}}$, $\tilde{T}_{34} = (3,3,2,4)_{L_{34}-L_{34}}$, $\tilde{T}_{46} = (5,5,2,2)_{L_{46}-L_{46}}$, $\tilde{T}_{56} = (6,6,0,0)_{L_{56}-L_{56}}$, $L_{12}(x) = L_{34}(x) = L_{35}(x) = \max(1-x^2,0)$, $L_{13}(x) = \max(0,1-x^4)$, $L_{23}(x) = L_{46}(x) = \max(0,1-x)$, $L_{25}(x) = \mathrm{e}^{-\mathrm{x}^2}$ $(L_{25}^{-1}(y) = \sqrt{-\ln y}$, $y \in [0,1]$ *we assume that* $-\ln 0 = \infty)$, $L_{56}(x) = \mathrm{e}^{-\mathrm{x}}$.

Applying Algorithm 1 we have obtained the results which are listed in Table 1. The path degrees of criticality have been computed with accuracy 10^{-4}.

Table 1: The path degrees of criticality in Example 1.

$p \in P$	$\mu_{\tilde{P}}(p)$
1-2-5-6	0.2867
1-2-3-5-6	1
1-2-3-4-6	0.5870
1-3-5-6	0.9998
1-3-4-6	0.5870

Example 2. *The network representing a structure of a project is given in Figure. 1. The activities duration times are fuzzy numbers of the same* $L - L$ *type, where* $L(x) = \max(0, 1 - x^2)$: $\tilde{T}_{12} = (3,4,2,2)_{L-L}$, $\tilde{T}_{13} = (6,6,2,4)_{L-L}$, $\tilde{T}_{23} = (3.5,4,2,0)_{L-L}$, $\tilde{T}_{25} = (4,5,0,2)_{L-L}$, $\tilde{T}_{35} = (7,7,2,2)_{L-L}$, $\tilde{T}_{34} = (3,3,2,4)_{L-L}$, $\tilde{T}_{46} = (5,5,2,2)_{L-L}$, $\tilde{T}_{56} = (6,6,0,0)_{L-L}$.

In this case we have determined the degree of criticality solving for each paths $p \in P$ the linear programming problem (6). The results are listed in Table 2.

Table 2: The path degrees of criticality in Example 2.

$p \in P$	$\mu_{\tilde{P}}(p)$	$\Theta_{\min}$
1-2-5-6	0.1597	0.9167
1-2-3-5-6	1	0
1-2-3-4-6	0.6093	0.625
1-3-5-6	0.9961	0.0625
1-3-4-6	0.6093	0.625

6 Summary

In the paper we have given a natural generalization of the notion of criticality for the case of a network with interval duration times of the activities, by introducing the notion of i-critical path, activity and event. We have shown that both the problem of determining an arbitrary i-critical path and that of the estimation of the i-criticality of a fixed path are easy problems. Solving them reduces itself to applying the classical CPM method to a network with deterministic duration times of the activities, which are certain, suitably chosen values from the interval duration times.

We have also presented another natural approach to the generalization of the criticality notion for the case of the network with fuzzy activities duration times. This generalization is made directly without using generalized arithmetic operations on fuzzy numbers. Due to direct application of the extension principle of Zadeh to the classical criticality notion treated as a function of activity duration times in a network this approach to the criticality concept is devoid of faults which are characteristic for the definitions of fuzzy criticality proposed till now in the literature. Although Definition 6 of fuzzy critical path seems to be very complicated it is possible to calculate effectively the exact value of criticality degree for a given path. In the paper we have proposed two efficient methods of computing the path degree of criticality (according to the proposed concept of fuzzy criticality). The first one is adapted to fuzzy activity times given in general form and the second one, based on linear programming, is true only for fuzzy activity times determined in the network by means of fuzzy numbers of the same $L-L$ type. There is no a such method for calculating the criticality degree of an activity (an event). This problem is NP-hard since it includes, as a particular case, the problem of asserting the i-criticality of a fixed activity (a fixed event) in a network which is, as shown in [3], NP-complete in the strong sense.

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