On the Sure Criticality of Tasks in Activity Networks With Imprecise Durations

Stefan Chanas, Didier Dubois, and Paweł Zieliński

Abstract—**The notion of the necessary criticality (both with respect to path and to activity) of a network with imprecisely defined (by means of intervals or fuzzy intervals) activity duration times is introduced and analyzed. It is shown, in the interval case, that both the problem of asserting whether a given path is necessarily critical and the problem of determining an arbitrary necessarily critical path (more exactly, a subnetwork covering all the necessarily critical paths) are easy. The corresponding solution algorithms are proposed. However, the problem of evaluating whether a given activity is necessarily critical does not seem to be such. Certain conditions are formulated which in some situations (but not in all possible) allow evaluating the necessary criticality of activities.**

The results obtained for networks with interval activity duration times are generalized to the case of networks with fuzzy activity duration times. Two effective algorithms of calculating the degree of necessary criticality of a fixed path, as well as an algorithm of determining the paths that are necessarily critical to the maximum degree, are proposed.

Index Terms—**Fuzzy CPM, possibility and necessity, project management and scheduling.**

I. Introduction

THE BASIC problem in scheduling is that of finding the critical activities and determining optimal starting times of activities in an activity network representing a project, that is, a partially ordered set of activities of prescribed durations, forming a directed acyclic graph. Of major concern as well is to determine the earliest ending time of the project. This problem was posed in the 1950s, in the framework of project management, by Malcolm *et al.* [29] who proposed the basic underlying graph-theoretic approach, called Project Evaluation and Review Technique (PERT). The determination of critical activities is carried out via the so-called critical path method [24]. The usual assumption in scheduling is that the duration of tasks is precisely known, so that solving the PERT problem is rather simple. However, in project management, the duration of tasks is seldom precisely known in advance, at the time when the plan of the project is designed. Detailed specification of the methods and resources involved for the realization of activities are often not available when the tentative plan is made up. This difficulty has been noticed very early by the authors that introduced the PERT approach, since they proposed to model the duration of tasks by probability distributions and tried to evaluate the mean value and standard deviation of earliest starting times of activities. Since then, there is an extensive literature on probabilistic PERT (see [27] for a bibliography). Even if the task durations are independent random variables, it is admitted that the problem of finding the distribution of the ending time of a project is intractable, due to the dependencies induced by the topology of the network. Another difficulty, not always pointed out, is the possible lack of statistical data validating the choice of activity duration distributions. Even if statistical data are available, they may be partially inadequate because each project takes place in a specific environment and is not the exact replica of past projects.

The simplest form of noncommittal uncertainty representation for activity duration is the interval. Assigning some time interval I to an activity duration means that the actual duration of this activity will take some value within I, but it is not possible at present to predict which one. Strangely enough, the PERT analysis with ill-known processing times modeled by simple intervals has not received much attention in the literature. Yet, the predictive computation of the minimal completion time of a project, the determination of critical paths and activities, and the determination of activity floats have been considered as important problems and have been widely acknowledged to be pervaded with uncertainty. However, the overwhelming part of the literature devoted to this topic adopts an orthodox stochastic approach, thus leading to a very complex problem that is still partially unsolved to-date. Until recently, and to the best of the authors' knowledge, interval-valued PERT analysis seems to have existed only as a special case of fuzzy PERT studies that appeared in the late seventies.

Resorting to fuzzy set and possibility theory for the modeling of ill-known task durations may help building a tradeoff between the expressive power and the computational difficulties of stochastic scheduling techniques while tackling uncertainty and possibly accounting for local specifications of preferences. This kind of methodology is not yet so common in operational research, even if quite a few works in fuzzy PERT-CPM and other types of fuzzy scheduling methods have been around for more than two decades [10], [36], [3], [4], that is, quite early in the development of fuzzy set theory. Apart from the book by Loostma [27], overviews on various aspects of fuzzy scheduling can be found in a recent edited volume [40] and in papers by Chanas and Kuchta [5] on graph-theoretic aspects, Werners [43] on fuzzy project management, and Turksen [41] on fuzzy rule-based production management. An abundant bibliography

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on fuzzy set applications in production management is supplied by Guiffrida and Nagi [20].

This paper is concerned with the critical path analysis of activity networks when the lack of knowledge about activity durations are modeled by intervals or fuzzy intervals. The membership function of a fuzzy interval M is a possibility distribution describing, for each value t of the activity duration, the extent to which it is a possible value. Equivalently, it means that the duration belongs to the level-cut interval $M^\alpha = \{t : \mu_M(t) \geq \alpha\}$ with confidence (or degree of necessity) $1 - \alpha$.

A basic result in standard (deterministic) activity networks analysis states that a task is critical if and only if its earliest and latest starting times are equal, and that critical tasks form critical paths, that is, longest paths from the initial node (event) to the final one. So finding the critical paths yields the critical tasks. When the durations of tasks are ill-known and modeled by intervals (and *a fortiori*, fuzzy intervals), it is not longer true that these results are valid. Namely, floats can no longer be recovered from the intervals containing earliest and latest starting times, and critical paths may no longer exist.

This paper is devoted to the determination of surely critical paths and activities in networks with imprecise (and fuzzy) durations. In the interval-valued case, while a task will eventually turn out to be critical or not, three situations may be observed at the time when the project is designed: a task will be either surely not critical, or surely critical, or possibly critical. A task is necessarily critical if it is critical whatever the actual values of task durations turn out to be. A task is possibly critical only when there are values of durations leading to a configuration of the network where the task is critical. In the fuzzy case, each task can be assigned both a degree of possible criticality and a degree of necessary criticality, since durations are modeled by possibility distributions.

Degrees of possible criticality have been studied by Dubois and Prade [13] and Chanas and Zieliński [6], [7]. The latter authors have carried out a full-fledged analysis of this notion, including computational methods and complexity estimation. This paper focuses on the dual notion of necessary criticality. The next section proposes a brief survey of fuzzy methods in scheduling problems and discusses their limitations for the PERT analysis under incomplete data. Its aim is to better situate the present work in a large body of existing literature applying fuzzy sets to scheduling, and highlight our motivation. Section III presents the criticality analysis of interval-valued activity networks. It proves that, in the case where all the durations are imprecise, the necessarily critical path is unique when it exists. However, the new features of the possibility-theoretical criticality analysis is that there may be no surely critical tasks, and, if some exist, they may be isolated and may not form a critical path. While the computation of the necessarily critical path when it exists is easy, the detection of isolated necessarily critical activities turns out be much harder. This state of facts seems not to have been known before. Algorithms for criticality analysis are provided that partly solve the problem. Section IV extends these results to the fuzzy case. Rigorous definitions of various criticality indices are provided. Their actual computation heavily relies on the previous section because it comes down to criticality analysis on level-cuts. Algorithms for the computation of paths with maximal degrees of necessary criticality are given. Only preliminary results for the graded criticality analysis of isolated activities have been obtained.

II. Fuzzy Sets in Scheduling: A Critical Review

One difficulty with fuzzy scheduling is to figure out what problem is really addressed in the various works found in the literature. If we set aside the use of fuzzy sets in the modeling of priority rules applied to deterministic formulations, fuzzy scheduling addresses two very distinct issues: scheduling under flexible constraints and scheduling under incomplete or imprecise information. Indeed, in the scope of decision theory, fuzzy sets can be used either as substitute of utility functions or as the substitute of probability functions. It reflects the ambiguity of membership functions that can be used both for preference modeling and for uncertainty handling. In the first group of papers, fuzzy sets are used to model local or global requirements in the form of flexible constraints (see [45] and [15] for the general setting for fuzzy constraint propagation). Flexible requirements include due-dates, release times of activities, and durations (see [14]). In this case, durations are controllable via decisions on the amount of resources, or the tuning of these resources. The problem is then to find the best schedule that achieves a compromise between these requirements. This methodology is akin to constraint-directed methods and includes the optimization of a single criterion as a particular case. It is similar to Artificial Intelligence methods, like the work of Sadeh *et al.* [38], except that the latter maximizes the sum of the local degrees of satisfaction of valued constraints, while in the fuzzy methods, the maximin approach tends to balance the local degrees of satisfaction. Such fuzzy constraint-based scheduling methods are applied to job-shop problems rather than project scheduling, and so fall out of the scope of this paper. However, project scheduling under fuzzy constraints is considered by Wang and Fu [42]. Their aim is to minimize costs under flexible constraints on activity times or available budget.

In the second group of papers, the aim is to analyze the main characteristics of a scheduling problem (including minimal makespan, critical paths, earliest and latest starting times of tasks) when data, especially noncontrollable task durations, are ill-known and modeled by fuzzy intervals, in the setting of possibility theory [13]. Possibility theory proposes a natural framework, simpler and less data-demanding than probability theory, for handling incomplete knowledge about scheduling data. Some scheduling problems involve both flexible constraints and uncertain data. Then, instead of optimizing average behaviors like in stochastic scheduling, fuzzy techniques rather aim at finding robust fault-tolerant schedules where all constraints are satisfied to some extent, with a sufficient level of confidence [14].

The past fuzzy PERT literature has sometimes relied on the assumption that, since in deterministic PERT, most parameters of interest are obtained by means of simple algorithms involving addition, subtraction, minimum and maximum, the same algorithms, once fuzzified, can be straightforwardly used: the same calculations can be carried out, changing numbers into fuzzy numbers, exploiting results in fuzzy arithmetics (on this topic, see [17] for an extensive survey).

The first interesting problem is that of computing the fuzzy completion time of the project. It is the possibility distribution of the minimal completion time. There is no constraint on the release date nor the ending date. This question, and the related one of finding the shortest or the longest distance between nodes in a graph with fuzzy-valued arcs, is easy using straightforward extensions of deterministic algorithms (contrary to the same question in stochastic PERT). It has been solved for a long time (Chanas and Kamburowski [4], Dubois and Prade [10],[11], Gazdik [19], Mareš [30], Mačák [28]). Lootsma [26] compares the fuzzy and the stochastic approaches on this question.

The critical path analysis in deterministic PERT is based on the computation of latest starting times of activities from the knowledge of the earliest ending time of the project. Then paths containing critical activities, that is, activities with zero floats, are identified. When durations are ill-know, it is tempting to compute the fuzzy latest starting times using the standard backward local propagation method, using the fuzzy ending time of the project for initializing the process and using fuzzy subtraction [36], [32]. However, as pointed out by several authors [8], [35], [37], [21], this method does not work for reasoning under uncertainty.

McCahon and Lee [32], Mon *et al.* [34], and Yao and Lin [44] propose to go back to standard critical path methods via defuzzification of the fuzzy processing times. McCahon [33] proposes to compute fuzzy slack times of activities from the fuzzy starting times obtained by the forward and backward recursions, but these fuzzy variables are interactive so that what is obtained is only a rough imprecise approximation of the fuzzy range of the actual float of the activity. Such a computation makes sense only if the fuzzy due-date and the fuzzy release date of the projects are prescribed independently of each other [13]. Kaufmann and Gupta [23], Hapke *et al.* [21] and Rommelfanger [37] suggest substitutes to the fuzzy subtraction, so as to improve the situation, but these techniques remain *ad hoc*. Nasution [35] resorts to symbolic computations on the variable processing times. However this technique is unwieldy and highly combinatorial. The computation of distributions of latest starting times of activities cannot be achieved using elementary techniques of fuzzy arithmetics, not even of interval arithmetics in the interval case. It is clear that the difficulty stems for the presence of intervals, be they fuzzy or not.

Chanas [2] proposes to substitute each fuzzy processing time with a random variable whose distribution faithfully reflects the membership function of the fuzzy processing time. The activity network is then analyzed by means of probabilistic methods.

Another way of approaching the criticality analysis of activities is to directly check if a path or an activity is critical, which in the fuzzy case is a matter of degree. Kamburowski [22] tries and computes a criticality index for path and activities directly. The criticality of an activity is obtained from two fuzzy evaluations of starting times of an activity: one using activities that precede it, the other from the activities that take place after, then comparing the sum of these fuzzy evaluations to the maximal fuzzy length of paths. For checking the extent to which a path is critical, one computes the height of the intersection of the fuzzy length of this path and the fuzzy completion time. It is clear that if this intersection is empty, then the path is not critical. However, the height of this intersection can be maximal for paths that are surely not critical. Slyeptsov and Tyshchuk [39] offer indices of criticality for path and activity which are also, as in Kamburowski's approach [22], a generalization of the criticality in the deterministic case to the fuzzy case. The criticality of a path is based on comparing the fuzzy earliest finish time of each activity lying on this path to the earliest start time of the activity that takes place after and lying on this path. The criticality degree of an activity is equal to the maximal criticality degree of paths crossing this activity. They aggregate such defined indices of criticality with the ones proposed by Kamburowski [22], thus providing generalized indices of criticality for path and activity. Mareš's evaluation [31] of the criticality of a path consists in comparing the fuzzy length of this path to fuzzy lengths of all paths in the network. Another view of criticality of activities is based on the notion of "most vital arcs" in fuzzy graph problems [25]: the idea is to delete each activity in the network and see how it affects the fuzzy duration of the project. The most critical task is then the one that maximally decreases the project length (using a fuzzy number ranking method).

Actually, a correct solution to the whole problem of critical path analysis under fuzzy uncertainty cannot be reached by mending existing algorithms. It requires a mathematically clean statement of the problem in the setting of possibility theory. This step was taken by Buckley [1]. However, as seen below, the main difficulty of the criticality analysis in fuzzy PERT, when fuzzy intervals represent ill-known processing times, does not lie in the introduction of fuzzy sets. It is already present when only usual intervals are involved. Solving the interval valued case is the main difficulty. The fuzzy case can then be rather easily solved, via the use of level-cuts. While Buckley [1] states the problem of computing fuzzy latest starting times of tasks and their floats in a correct way, he points out its difficulty without proposing a solving method. Such a method has been provided by Fargier *et al.* [18] for activity networks having a special topology, namely the one of series-parallel graphs, for which the proposed algorithms are polynomial. In the general case, the problem seems to be of exponential complexity. The possibilistic criticality analysis is carried out by Chanas and Zieliński [6] for interval-valued durations, and Chanas and Zieliński [7] for fuzzy durations. They provide algorithms that decide if a path is possibly or not possibly critical. Already in the interval-valued case, deciding if an activity is possibly critical is proved to be of exponential complexity. However, the question of finding paths or activities in a project that are surely critical despite the uncertainty about durations was not addressed in these works. So, the criticality analysis carried out on the basis of possibly and nonpossibly critical tasks was incomplete. In the following, the full picture of the criticality analysis of activity networks in an imprecise environment is provided, along with results for determining surely critical paths and activities, if any.

III. Criticality in Networks With Interval Activity Times

A network $S = \langle V, A, T \rangle (|V| = n, |A| = m)$ being a project activity-on-arc model is given (activities are represented by arcs of a network). V is the set of nodes (events) and $A \subseteq V \times V$ is the set of arcs (activities). The network S is a directed, connected, acyclic graph with one initial node and one end node. The set $V = \{1, 2, \ldots, n\}$ is labeled in such a way that $i < j$ for each activity $(i,j) \in A$. Thus, the initial and end nodes have labels 1 and n, respectively. By means of function $\mathcal{T}, \mathcal{T}: A \to I(\mathbb{R}_+)$, interval activity times in the network are assigned, $\mathcal{T}(i,j) \stackrel{\mathrm{def}}{=} T_{ij} = [\underline{t}_{ij}, \bar{t}_{ij}]$, where $I(\mathbb{R}_+)$ is the set of nonnegative intervals, i.e., $\underline{t}_{ij} \geq 0$ for each $(i,j) \in A$. The interval T_{ij} contains possible duration times of $(i,j) \in A$.

A. Basic Notions of Criticality Under Interval Activity Times

Before we pass on to the essential considerations, let us recall the notions of criticality, in the usual sense, of a path and of an activity in a network S with precise activity times.

Let us denote by P the set of all paths in S from the initial event 1 to the final event n.

Definition 1: A path $p \in P$ is critical if and only if it is a longest path in S (assuming that weights of the arcs are activities duration times).

The length of a critical path p is the minimum time required for completion of the whole project.

Definition 2: An activity $(i,j) \in A$ is critical if and only if it belongs to a critical path $p \in P$.

The definitions presented above are equivalent to those known from the literature, where the criticality is defined by zero floats of activities and events.

The following statement is obvious.

Statement 1: A path $p \in P$ is critical if and only if all activities belonging to p are critical.

Now we introduce the criticality notions for a network with interval activity duration times.

Definition 3: A path $p \in P$ is possibly critical in S if and only if there exists a set of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$, such that p is critical in the usual sense in S, after replacing the interval times T_{ij} with the exact values $t_{ij}, (i,j) \in A$.

Definition 4: An activity $(k,l) \in A$ is possibly critical in the network S if and only if there exists a set of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$, such that (k,l) is critical in the usual sense in the network S, after replacing the interval times T_{ij} with the exact values $t_{ij}, (i,j) \in A$.

The above two notions have been proposed in [6], [7], where instead of the term "possibly critical" the term "i-critical" has been used.

The following two notions are complementary to these given in Definitions 3 and 4.

Definition 5: A path $p \in P$ is necessarily noncritical in S if and only if for each set of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A, p$ is not critical in the usual sense in S, after replacing the interval times T_{ij} with the exact values $t_{ij}, (i,j) \in A$.

Definition 6: An activity $(k,l) \in A$ is necessarily noncritical in the network S if and only if for each set of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A, (k,l)$ is not critical in the usual sense in S, after replacing the interval times T_{ij} with the exact values $t_{ij}, (i,j) \in A$.

The next notion of necessary criticality has been proposed in [16], [18].

Definition 7: A path $p \in P$ is necessarily critical in S if and only if for each set of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A, p$ is critical in the usual sense in S, after replacing the interval times T_{ij} with the exact values $t_{ij}, (i,j) \in A$.

Definition 8: An activity $(k,l) \in A$ is necessarily critical in the network S if and only if for each set of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A, (k,l)$ is critical in the usual sense in S, after replacing the interval times T_{ij} with the exact values $t_{ij}, (i,j) \in A$.

And the last two notions are complementary to these given in Definition 7 and 8.

Definition 9: A path $p \in P$ is possibly noncritical in S if and only if there exists a set of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$, such that p is noncritical in the usual sense in S, after replacing the interval times T_{ij} with the exact values $t_{ij}, (i,j) \in A$.

Definition 10: An activity $(k,l) \in A$ is possibly noncritical in the network S if and only if there exists a set of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$, such that (k,l) is noncritical in the usual sense in the network S, after replacing the interval times T_{ij} with the exact values $t_{ij}, (i,j) \in A$.

The following statements are obvious. They result from the previously given definitions.

Statement 2: A path $p \in P$ (resp. an activity $(k,l) \in A$) is necessarily noncritical in S if and only if it is not possibly critical.

Statement 3: A path $p \in P$ (resp. an activity $(k,l) \in A$) is possibly noncritical in S if and only if it is not necessarily critical.

Statement 4: If a path $p \in P$ (resp. an activity $(k,l) \in A$) is necessarily critical in S, then it is possibly critical.

Note that from Statement 3 it results that on the basis of necessary criticality of paths and activities we also have an information about their possible noncriticality.

The notion of possible criticality is thoroughly investigated in [6], [7]. It has been shown there that the problem of determining an arbitrary possibly critical path and that of asserting if a fixed path $p \in P$ is possibly critical in a network S are easy and they can be solved in time bounded by a polynomial in the size of the network. However, not all the problems concerning the possible criticality are easy. Some of them have turned out to be hard ones. In [6] it has been proven that the problem of asserting the possible criticality of a fixed activity is NP-complete in the strong sense. It is also shown there that the problem of determining K possibly critical paths is NP-hard.

In the following points of this section the notion of necessary criticality (with regard to a path as well as to an activity) is analyzed.

B. Necessarily Critical Paths

Lemma 1 gives necessary and sufficient conditions for establishing the necessary criticality of a given path $p \in P$. It is a direct consequence of Definition 1 and Definition 7.

Lemma 1: A path $p \in P$ is necessarily critical in S if and only if it is critical in the usual sense in the network S, in which the interval activity times $T_{ij} = [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$, have been replaced with the exact values t_{ij} determined by means of the following formula:

$$t_{ij} = \begin{cases} \underline{t}_{ij} & \text{if } (i,j) \in p, \\ \bar{t}_{ij} & \text{if } (i,j) \notin p. \end{cases} \quad (1)$$

Similarly, Lemma 2 provides necessary and sufficient conditions for possible noncriticality (complementary to the notion necessary criticality) of a given path $p \in P$. It follows directly from Definition 9 and Definition 1.

Lemma 2: A path $p \in P$ is possibly noncritical in S if and only if it is not critical in the usual sense in the network S, in which the interval activity times $T_{ij} = [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$, have been replaced with the exact values t_{ij} determined by means of the following formula:

$$t_{ij} = \begin{cases} \underline{t}_{ij} & \text{if } (i,j) \in p, \\ \bar{t}_{ij} & \text{if } (i,j) \notin p. \end{cases}$$

From Lemma 1 it follows that the problem of asserting if a fixed path $p \in P$ is necessarily critical is easy. It reduces itself to applying the classical CPM method to the network S, after replacing the interval times $[\underline{t}_{ij}, \bar{t}_{ij}]$ with the exact values t_{ij} determined with formula (1).

The problem of determining an arbitrary necessarily critical path is also easy. Algorithm 1, proposed by us, determines the connected subnetwork $S_c = \langle V_c, A_c, \mathcal{T}\rangle (V_c \subseteq V, A_c \subseteq A)$ of the network S composed of all necessarily critical paths in S.

Algorithm 1 Determining a subnetwork $S_c = \langle V_c, A_c, \mathcal{T}\rangle$ composed of all necessarily critical paths

Require: $S = \langle V, A, \mathcal{T}\rangle$ with the interval activity times $T_{ij} = [\underline{t}_{ij}, \bar{t}_{ij}]$, $(i,j) \in A$.

Ensure: $S_c = \langle V_c, A_c, \mathcal{T}\rangle$—if such a subnetwork exists.

Step 1 Compute $\underline{A}_c$ (critical activities with durations equal to $\underline{t}_{ij}$).

Step 2 Compute $\bar{A}_c$ (critical activities with durations equal to $\bar{t}_{ij}$).

Step 3 Assign $A' := \underline{A}_c \cap \bar{A}_c$.

Step 4 Determine any path $p \in P$ composed of activities from A'.
If such a path does not exist then stop (there are no necessarily critical paths in S).
Otherwise go to Step 5.

Step 5 Check, according to Lemma 1, if the path p is necessarily critical in S.
If it is not then stop (there are no necessarily critical paths in S).
Otherwise go to Step 6.

Step 6 Remove from A' all activities (i,j) such that $i \neq 1$ and there is no $(k,l) \in A'$ with $i = l$ and all activities (i,j) such that $j \neq n$ and there is no $(k,l) \in A'$ with $j = k$.
Assign $A_c := A'$, $V_c := \{i \mid \exists(i,j) \in A' \text{ or } \exists(j,i) \in A'\}$, stop (the subnetwork $S_c = \langle V_c, A_c, \mathcal{T}\rangle$ covers exactly all necessarily critical paths in S).

The subnetwork S_c does not always exist, i.e., it may happen that there are no necessarily critical paths in S (see Step 4 and 5). Naturally, the number of paths in S_c may be very great as it may increase exponentially together with the network size extension. However, the subnetwork S_c is available in a time bounded by a polynomial in the size of the network S.

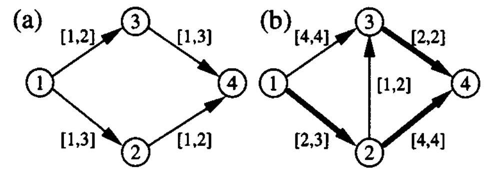

Fig. 1. (a) A network with A' containing paths but none are necessarily critical. (b) A network with A' (bold arcs) containing a necessarily critical path and a "protruding" activity.

The label $\underline{A}_c$ (resp. $\bar{A}_c$) denotes the set of all critical activities in S in which the interval activity times $T_{ij} = [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$, have been replaced with the values $\underline{t}_{ij}$ (resp. $\bar{t}_{ij}$).

Since it may happen that there exist in S paths belonging to P composed of activities from $A' = \underline{A}_c \cap \bar{A}_c$ but none of them are necessarily critical, Step 5 is essential in Algorithm 1. An example of such a network is given in Fig. 1(a). All the activities of the network belong to A'. There are two paths $1 \rightarrow 3 \rightarrow 4$ and $1 \rightarrow 2 \rightarrow 4$ and none is necessarily critical.

Step 6 consists in removing all "protruding" activities from the set A', i.e., activities which do not lay on paths belonging to P and composed of activities from $A^{'}$. In this way $A^{'}$ after rejecting the "protruding" activities contains exactly one initial node 1 and one final node n and covers all necessarily critical paths in S. An example of a network in which $A^{'}$ contains "protruding" activities is given in Fig. 1(b). $A^{'}$ is composed of the activities (1, 2), (2, 4) and (3, 4). (1, 2) and (2, 4) form a necessarily critical path and (3, 4) is a "protruding" activity.

Now we give the lemma (Lemma 3), which justifies Algorithm 1.

Lemma 3: Any path $p \in P$ is necessarily critical in S if and only if it belongs to the subnetwork $S_c = \langle V_c, A_c, \mathcal{T}\rangle$ constructed by Algorithm 1.

Proof: The *only if* direction: Let $p \in P$ be any necessarily critical path in S. From necessary criticality of p (see Definition 7) it follows that each activity forming p belongs to the set $\underline{A}_c$ as well as to the set $\bar{A}_c$. Hence, and from the fact that $p \in P$ we get that p is contained in S_c.

The *if* direction: We must show that any path $p^{'} \in P$ formed by activities from S_c is necessarily critical in S.

Let $p \in P$ be the path in S, whose necessary criticality has been checked in Step 5 of Algorithm 1.

Assume that $p^{'} \neq p$ (if $p^{'} = p$ then we immediately obtain that $p^{'}$ is necessarily critical).

We claim that interval duration times T_{ij} of activities, which distinguish $p^{'}$ from p, i.e., activities belonging to set $(p^{'} \setminus p) \cup (p \setminus p^{'})$, have to be precise, i.e., $\underline{t}_{ij} = \bar{t}_{ij}$.

Indeed, assume on the contrary that there exists an activity $(i^*, j^*) \in p^{'} \setminus p$ such that $\underline{t}_{i^*j^*} < \bar{t}_{i^*j^*}$. Since the paths $p^{'}$ and p belong to S_c the following equation holds $\sum_{(i,j)\in p'} \underline{t}_{ij} = \sum_{(i,j)\in p} \underline{t}_{ij}$. If we increase duration time of $(i^*, j^*) \in p^{'}$ to $\bar{t}_{i^*j^*}$ then we get a configuration of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$, for which $p^{'}$ is longer than p. This contradicts that p is necessarily critical.

Similarly, assume that there exists an activity $(i^*, j^*) \in p \setminus p'$ such that $\underline{t}_{i^*j^*} < \bar{t}_{i^*j^*}$. Since the paths $p^{'}$ and p belong to S_c the following equation holds $\sum_{(i,j)\in p'} \bar{t}_{ij} = \sum_{(i,j)\in p} \bar{t}_{ij}$. If we decrease duration time of $(i^*, j^*) \in p$ to $\underline{t}_{i^*j^*}$ then we get a configuration of times for which p is shorter than p'. This fact contradicts that p is necessarily critical.

Since interval duration times T_{ij} of activities belonging to $(p' \setminus p) \cup (p \setminus p')$ are such that $\underline{t}_{ij} = \bar{t}_{ij}$, the lengths of paths p and p' are the same for each configuration of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$. Thus, from necessary criticality of p it follows that p' is necessarily critical. ■

An interesting property of necessarily critical paths has been obtained in the proof of Lemma 3.

Lemma 4: For any necessarily critical paths $p, p' \in P, p \neq p'$, the following equality holds:

$$\{(i,j) \,|\, \underline{t}_{ij} < \bar{t}_{ij}, (i,j) \in p\} = \{(i,j) \,|\, \underline{t}_{ij} < \bar{t}_{ij}, \, (i,j) \in p'\}.$$

Corollary 1: Let $p \in P$ be a necessarily critical path in S. If duration times of activities $(i,j) \in p$ are such that $\underline{t}_{ij} < \bar{t}_{ij}$ then p is the unique necessarily critical path in S.

It is clear at this point that the critical path analysis of activity networks when durations of activities are imprecisely known is very different from the deterministic case: a necessarily critical path may not exist, and if it does, and contains only imprecise duration activities then it is unique.

C. Necessarily Critical Activities

Now we focus on the problems of determining necessarily critical (or equivalently possibly noncritical) activities. We deal with two problems. The first one concerns determining all necessarily critical activities and the second one consists in asserting the necessary criticality of a fixed activity. Of course, these problems are strictly connected with each other. If the first problem may be solved easily then this is true also for the second one, and the other way round. We distinguish two situations. Both will be analyzed in this section. The first one concerns the case, when there exist necessarily critical paths in the network. The second situation concerns the case, when there are no such paths in the network. As we have mentioned in Section III.B such case may occur and it does not seem to be rare. It turns out that in the first case the problem of determining all necessarily critical activities is computationally solvable by means of the polynomial algorithm (Algorithm 1) presented in Section III.B, that determines all necessarily critical paths in the network. It is so because all the activities forming necessarily critical paths are the only necessarily critical activities in the network. This property will be shown formally in Lemmas 5 and 6.

The second situation mentioned above, i.e., the case when there are no necessarily critical paths in the network and only isolated necessarily critical activities may exist (activities which belong to no necessarily critical path), seems to be hard to solve.

Before we pass on to the basic consideration let us define some sets of paths and activities, which will be helpful in formulating and proving the lemmas and statements in this section.

- The set of all paths from P, which contain an activity $(k,l) \in A$:

$$P^+(k,l) = \{p \,|\, p \in P, (k,l) \in p\}.$$

- The set of all paths from P, which do not contain an activity $(k,l) \in A$:

$$P^-(k,l) = \{p \,|\, p \in P, (k,l) \notin p\}.$$

- The set of all activities, which belong to paths from the set $P^+(k,l)$ and do not belong to paths from $P^-(k,l)$:

$$A^+(k,l) = \{(i,j) \,|\, (i,j) \in p, p \in P^+(k,l)\} \setminus \{(i,j) \,|\, (i,j) \in p, p \in P^-(k,l)\}.$$

- The set of all activities, which belong to paths from the set $P^-(k,l)$ and do not belong to paths from $P^+(k,l)$:

$$A^-(k,l) = \{(i,j) \,|\, (i,j) \in p, p \in P^-(k,l)\} \setminus \{(i,j) \,|\, (i,j) \in p, p \in P^+(k,l)\}.$$

- The configuration of activity duration times in S consisting of the left (resp. right) ends of intervals $T_{ij}, (i,j) \in A$:

$$\bar{T} = (\bar{t}_{ij})_{(i,j) \in A} \left(\text{resp.} \, \underline{T} = (\underline{t}_{ij})_{(i,j) \in A}\right).$$

- The set of all paths in $P^+(k,l)$ being critical, in the usual sense, with $\bar{T}$ (resp. $\underline{T}$):

$$\bar{P}_c^+(k,l) = \{p \,|\, p \text{ is critical with } \bar{T}\} \subseteq P^+(k,l)$$
$$(\underline{P}_c^+(k,l) = \{p \,|\, p \text{ is critical with } \underline{T}\} \subseteq P^+(k,l)).$$

$A^-(k,l)$ is composed of tasks which are parallel to (k,l), which are performed concurrently with (k,l). $A^+(k,l)$ is composed of tasks which, if different from (k,l), either precede or succeed to (k,l) only. Of course, the following relationship holds

$$A^+(k,l) \cap A^-(k,l) = \emptyset.$$

All these sets are easy to determine, i.e., they can be calculated in a time bounded by a polynomial in the size of the network S. To be precise, we should say in the case of sets $P^+(k,l)$ and $P^-(k,l)$ that only subnetworks covering these sets can be easily determined.

Lemma 5: If there exists a necessarily critical path $p \in P$ in S, then all necessarily critical activities belong to necessarily critical paths in S (equivalently there does not exist an isolated critical activity i.e., an activity which belongs to no necessarily critical path).

Proof: We conduct an indirect proof. Let us assume that there exists an isolated necessarily critical activity $(k,l) \in A$, i.e., an activity which belongs to no necessarily critical path in S.

We claim that there exists in set $P^+(k,l)$ a path p^* being critical with $\bar{T}$ (i.e., $p^* \in \bar{P}_c^+(k,l)$), and all its activities (i,j) such that $(i,j) \in A^+(k,l)$ are assigned precise times, i.e., $\underline{t}_{ij} = \bar{t}_{ij}$. Assume on the contrary that such path p^* does not exist. Thus, each path from $\bar{P}_c^+(k,l)$ contains at least one activity (i', j') such that $(i', j') \in A^+(k,l)$ and $\underline{t}_{i'j'} < \bar{t}_{i'j'}$. From necessary criticality of the path p it follows that p is critical with activity times $\bar{T}$. Indeed, the lengths of all paths from $\bar{P}_c^+(k,l)$, with activity times $\bar{T}$, are equal to the length of the path p (the remaining paths from $P^+(k,l)$ are shorter than p).

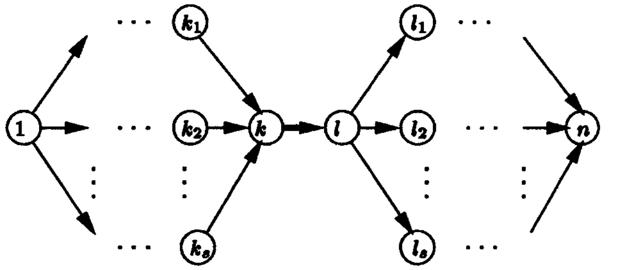

Fig. 2. A network with the necessarily critical activity (k, l) without parallel activities.

If we simultaneously decrease duration times $\bar{t}_{i'j'}$ of the activities (i', j') to $\underline{t}_{i'j'}$ we get a new configuration of activity times for which all the paths from $\bar{P}_c^+(k,l)$ are shorter than p. Therefore, (k,l) is not critical with this configuration of activity times and from Definition 8 we obtain that (k,l) is not necessarily critical. This fact contradicts necessary criticality of (k,l). Thus, we have shown that there exists a path $p^*, p^* \in \bar{P}_c^+(k,l)$, and all its activities (i,j) such that $(i,j) \in A^+(k,l)$ are assigned precise times, i.e., $\underline{t}_{ij} = \bar{t}_{ij}$.

We claim that all activities (i,j) such that $(i,j) \in p$ and $(i,j) \notin p^*$ are assigned precise times. Indeed, assume on the contrary that there exists an activity (i', j') such that $(i', j') \in p, (i', j') \notin p^*$ and $\underline{t}_{i'j'} < \bar{t}_{i'j'}$. Naturally, the length of p^*, with activity times $\bar{T}$, is equal to the length of the path p.

If we decrease duration time $\bar{t}_{i'j'}$ of the activity (i', j') to $\underline{t}_{i'j'}$ we get a new configuration of activity times for which the path p^* is longer than p. Therefore, p is not critical with this configuration of activity times and from Definition 7 we get that p is not necessarily critical. This fact contradicts necessary criticality of p. Thus, we have shown that all activities (i,j) such that $(i,j) \in p$ and $(i,j) \notin p^*$ have precise duration times.

Since the interval duration times T_{ij} of activities (i,j), such that $(i,j) \in p^*$ and $(i,j) \notin p$ or $(i,j) \in p$ and $(i,j) \notin p^*$ are precise, the lengths of the paths p and p^* are the same for each configuration of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$. From that and the necessary criticality of p it follows that the path p^* is necessarily critical. So, (k,l) belongs to a necessarily critical path. This fact contradicts that (k,l) is the isolated necessarily critical activity. ■

Lemma 6: A path $p \in P$ is necessarily critical in S if and only if all activities belonging to p are necessarily critical.

Proof: It results directly from Definitions 7, 8 and Statement 1. ■

From Lemmas 5 and 6 it follows that if there exist necessarily critical activities in the network S, then either they all belong to necessarily critical paths or they are isolated. Therefore, in the first case, as we have mentioned before, it is enough to apply Algorithm 1 for determining all necessarily critical activities in S. In the second one, problems arise. Further on, in this section, we investigate the problem of finding necessarily critical activities in the case when necessarily critical paths do not exist in S.

The simplest example of an activity which is certainly necessarily critical, though a necessarily critical path may not exist in a network, is the activity (k,l) in the network given in Fig. 2.

In this network there are no parallel activities to (k,l) (i.e, $A^-(k,l) = \emptyset$), so each path leading from the initial to the final node uses (k,l) and this is the real reason of its necessary criticality. According to that one may formulate the following statement.

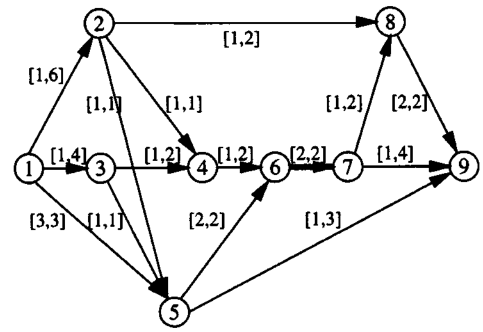

Fig. 3. A network with only one isolated necessarily critical activity (6, 7).

Statement 5: If a set $A^-(k,l)$ is empty, then activity (k,l) is necessarily critical.

Let us consider a more complicated case occurring in the network presented in Fig. 3. There is only one isolated necessarily critical activity (6, 7) but proving this claim is not so obvious as in the previous example.

This last example indicates that the problem of evaluation of necessary criticality (or possible noncriticality) of an activity in a general case may be not easy. On the contrary, it seems to be a hard one. However, the question of complexity of the problem is not resolved yet. At this moment we can only formulate the following conjectures.

Conjecture 1: The problem of asserting if a fixed activity $(k,l) \in A$ is necessarily critical in S is NP-Hard.

Conjecture 2: The problem of asserting if a fixed activity $(k,l) \in A$ is possibly noncritical in S is NP-Complete.

The following lemma (Lemma 7), which confirms the combinatorial nature of the investigated problem, supports the above conjectures. Before formulating the lemma let us introduce the notion of an extreme configuration.

Definition 11: Let $B \subseteq A$ be a fixed subset of activities. The assignment of t_{ij} to activities $(i,j) \in A$ such that

$$t_{ij} = \begin{cases} \underline{t}_{ij}, & \text{for } (i,j) \in B, \\ \bar{t}_{ij}, & \text{for } (i,j) \notin B. \end{cases}$$

will be called an extreme configuration generated by B and denoted with T_B.

Let us denote by $A_c(B)$ the set of all critical activities in configuration T_B. For special cases we have $T_A = \underline{T}, T_\emptyset = \bar{T}, A_c(A) = \underline{A}_c, A_c(\emptyset) = \bar{A}_c$, where $\underline{T}, \bar{T}, \underline{A}_c, \bar{A}_c$ are the notations defined earlier.

The following lemma holds.

Lemma 7: The set of all necessarily critical activities, denoted A_{nc}, is exactly equal to the intersection of all sets $A_c(B), B \subseteq A$, i.e.,

$$A_{nc} = \bigcap_{B \subseteq A} A_c(B).$$

Proof: We only have to prove that $\bigcap_{B \subseteq A} A_c(B) \subseteq A_{nc}$—the reverse inclusion is obvious.

For suppose not. It means that there exists an activity $(i,j) \in \bigcap_{B\subseteq A} A_c(B)$ which is not necessary critical, i.e, $(i,j) \notin A_{nc}$. Thus there is a configuration T, not extreme, where (i,j) is not critical. It means each path from $P^+(i,j)$, i.e, a path belonging to P and including the activity (i,j), is not critical in T. Let $p \in P$ be any critical path in T. Of course, $p \in P^-(i,j)$ and it is longer than any path from $P^+(i,j)$.

Let us take now the extreme configuration $T_{A\setminus p}$ (see Definition 11). We claim that also in the configuration $T_{A\setminus p}$ all the paths are shorter than the path p. Really, let p' be any path belonging to $P^+(i,j)$. The length difference between p and p' may only occur in arcs which are not common, i.e., in arcs $(p' \setminus p) \cup (p \setminus p')$. But replacing T with $T_{A\setminus p}$ one only may increase the length of p comparing with the length of p' since the durations t_{kl} at the most are increased for $(i,j) \in p \setminus p'$ and decreased for $(i,j) \in p' \setminus p$. Hence we have shown that no path $p' \in P^+(i,j)$ is critical in the extreme configuration $T_{A\setminus p}$ and therefore (i,j) is not critical in $T_{A\setminus p}$, i.e., $(i,j) \notin A_c(B)$ for $B = A \setminus p$. So, we obtain $(i,j) \notin \bigcap_{B\subseteq A} A_c(B)$, contradiction. ■

Lemma 7 suggests an algorithm, unfortunately ineffective (with exponential complexity), for determining all necessarily critical activities. Thus, according to this lemma it is enough to determine the sets of critical activities $A_c(B)$ for all extreme configurations T_B and then take the intersection of them. But the number of all possible configurations is equal to 2^m.

Now we formulate, in the form of statements, conditions allowing to reject some activities, which are not necessarily critical.

The first of the statements (Statement 6) is a direct consequence of Lemma 7. It seems to narrow down the set of candidates for necessarily critical activities drastically, though in some cases other two extreme configurations T_B and $T_D, B, D \subseteq A$ (instead of T_A and $T_\emptyset$) would provide, may be, a better preliminary approximation of the set of all necessarily critical activities. Of course, if $\underline{A}_c \cap \bar{A}_c = \emptyset$, then there are no necessarily critical activities in the network.

Statement 6: If an activity $(k,l) \notin \underline{A}_c \cap \bar{A}_c$, then (k,l) is not necessarily critical (equivalently it is possibly noncritical).

In particular, the case of Fig. 3 $\underline{A}_c \cap \bar{A}_c$ reduces to tasks (5, 6) and (6, 7). In order to prove (6, 7) is the only necessarily critical task one must use a configuration $t_{12} = 1, t_{13} = 4, t_{34} = 2, t_{46} = 2$, so that path $1 \to 3 \to 4 \to 6$ has length 8 while path $1 \to 5 \to 6$ has length 5 and path $1 \to 3 \to 5 \to 6$ has length 7. Such configurations are hard to guess in a systematic way.

The following statements (Statements 7, 8) hold for the hybrid network when all activities have precise or imprecise durations.

Statement 7: If an activity $(k,l) \in \underline{A}_c \cap \bar{A}_c, A^-(k,l) \cap \bar{A}_c \neq \emptyset$ and there does not exist in set $P^+(k,l)$ a path being critical with $\bar{T}$, all activities $(i,j) \in A^+(k,l)$ of which have precise durations, i.e., $\underline{t}_{ij} = \bar{t}_{ij}$, then (k,l) is not necessarily critical (equivalently it is possibly noncritical).

Proof: Let us consider the configuration of activity times $\bar{T}$. From the second assumption it follows that there exists an activity $(r,s) \in A^-(k,l)$ (i.e., an activity parallel to the activity (k,l)) being critical with the activity times equal to $\bar{T}$. Thus, there exists a path $p \in \bar{P}_c^+(r,s)$. Since $(k,l) \in \bar{A}_c$ (i.e., (k,l) is also critical with $\bar{T}$), there exists a nonempty subset $\bar{P}_c^+(k,l) \subseteq P^+(k,l)$. Naturally, the lengths of all paths from $\bar{P}_c^+(k,l)$, under activity times $\bar{T}$, are equal to the length of the path p (the remaining paths from $P^+(k,l)$ are shorter than p).

Let us decrease in $\bar{T}$ the activity times $\bar{t}_{ij}$ to $\underline{t}_{ij}$ of all the activities from $\bar{P}_c^+(k,l)$, which simultaneously belong to the set $A^+(k,l)$. From the last assumption of the statement it follows that all paths from $\bar{P}_c^+(k,l)$ are shorter than p for the new configuration of activity times. Thus, (k,l) is not critical with this configuration and from Definition 8 we obtain that (k,l) is not necessarily critical (equivalently it is possibly noncritical). ■

Statement 8: If an activity $(k,l) \in \underline{A}_c \cap \bar{A}_c$ and there exists an activity (r,s), such that $(r,s) \in A^-(k,l), (r,s) \in \underline{A}_c$ and $\underline{t}_{rs} < \bar{t}_{rs}$, then (k,l) is not necessarily critical (equivalently it is possibly noncritical).

Proof: Let us consider the configuration of activity times $\underline{T}$. From the second assumption it follows that there exists an activity $(r,s) \in A^-(k,l)$ being critical with $\underline{T}$. So, there exists a path $p \in \underline{P}_c^+(r,s)$. Since $(k,l) \in \underline{A}_c$, the subset $\bar{P}_c^+(k,l)$ is nonempty. Naturally, the lengths of all paths from $\underline{P}_c^+(k,l)$, for activity times $\underline{T}$, are equal to the length of the path p (the remaining paths from $P^+(k,l)$ are shorter than p).

Let us increase in $\underline{T}$ the activity time $\underline{t}_{rs}$ to $\bar{t}_{rs}$. We get a new configuration of activity times for which all the paths from $\underline{P}_c^+(k,l)$ are shorter than p. Thus, (k,l) is not critical with this configuration of activity times and from Definition 8 we obtain that (k,l) is not necessarily critical (equivalently it is possibly noncritical). ■

The Statements 6–8 are useful from a practical point of view, as the conditions contained in their assumptions are easy to check. It may be done in a time bounded by a polynomial in the size of S. Therefore, Statements 6–8 may be applied to the effective evaluation of necessary criticality of some (but not all) activities. Let us pay attention to the example in Fig. 3 again, it is worth noticing that (5, 6) is a candidate for being a necessarily critical activity, i.e., $(5,6) \in \underline{A}_c \cap \bar{A}_c$. However, by means of presented conditions (Statements 6, 7 and 8), one can not reject it.

Algorithm 2, proposed by us, determining an upper bound for the set of all necessarily critical activities, is a kind of summary of the results presented in this and the previous section.

Algorithm 2 Determining an approximation set of all necessarily critical activities

Require: $S = \langle V, A, \mathcal{T} \rangle$ with the interval activity times $T_{ij} = [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$, a number of generated extreme configuration ng.

Ensure: A_{pnc}—a set of potentially necessarily critical activities.

Step 1 Execute Algorithm 1.
If there is in S a necessarily critical path then assign $A_{pnc} := A_c$ and stop—A_{pnc} is the exact set of all necessarily activities in S.
Otherwise, go to Step 2.

Step 2 Assign $A_{pnc} := \underline{A}_c \cap \bar{A}_c$. If $A_{pnc} = \emptyset$ then stop—there are no necessary critical activities in S.
Otherwise, go to Step 3.

Step 3 For $k = 1$ to ng
generate a random extreme configuration T_{B_k}, compute $A_c(B_k)$ and assign $A_{pnc} := A_{pnc} \cap A_c(B_k)$.

Step 4 If $A_{pnc} = \emptyset$ then stop—there are no necessary critical activities in S.

Otherwise, use Statements 7 and 8 to remove, if possible, from set A_{pnc} surely not necessarily critical activities and stop—A_{pnc} contains potentially necessarily critical activities in S.

Step 3 of Algorithm 2 is based on Lemma 7. A number of generated extreme configuration, ng, should be given by a decision maker. The greater number ng the better approximation of the set A_{nc} may be probably obtained.

The following three Statements 9–11 are different from the previous statements and they are rather of theoretical importance. They allow to assert necessary criticality of certain activity only in some cases, in which one has information about other necessarily critical activities. They provide knowledge about the mutual location of necessarily critical activities in the network.

Statement 9: If an activity $(k,l) \in A$ is necessarily critical and $\underline{t}_{kl} < \bar{t}_{kl}$, then no activity from $A^-(k,l)$ is necessarily critical.

Proof: Let us consider any activity (i,j) from the set $A^-(k,l)$.

Case 1: $(i,j) \in \underline{A}_c \cap \bar{A}_c$. From necessary criticality of (k,l) we get that $(k,l) \in \underline{A}_c$. Since $(i,j) \in A^-(k,l)$, we have $(k,l) \in A^-(i,j)$. Consequently, as also $\underline{t}_{kl} < \bar{t}_{kl}$, from Statement 8 we obtain that (i,j) is not necessarily critical.

Case 2: $(i,j) \notin \underline{A}_c \cap \bar{A}_c$. It follows directly from Statement 6 that (i,j) is not necessarily critical.

Statement 10: If any two activities $(k,l) \in A, (r,s) \in A$ are such that $A^-(k,l) = A^-(r,s)$ and (k,l) is necessarily critical, then (r,s) is also necessarily critical.

Proof: Since the sets of activities $A^-(k,l)$ and $A^-(r,s)$ are equal, the sets of paths $P^+(k,l)$ and $P^+(r,s)$ are also equal. From that and necessary criticality of (k,l), we get that (r,s) is also necessarily critical (see Definitions 2 and 8). ■

Statement 11: Let A^* be sequence of successive necessarily critical activities forming a subpath in S, starting with event i^* and ending with event v^*. Let $S^{''} = \langle V^{''}, A^{''}, \mathcal{T}\rangle$ be subnetwork of S formed by paths starting in i^* and ending in v^*, made of activities $\underline{A}_c \cap \bar{A}_c$. If $(i,j) \in A^{''}$, then (i,j) is a necessarily critical activity in S.

Proof: Let us consider the subnetwork $S^{''}$ determined as in the assumptions. We prove that all activities of $S^{''}$ are necessarily critical. □

Case 1: $i^* = 1$ and $v^* = n$. $(i^*,j^*),(j^*,k^*),\ldots,(u^*,v^*)$ is a path in P containing necessarily critical activities, hence a necessarily critical path, from Lemma 6. $S^{''}$, from its definition, is thus the subnetwork determined in Step 6 of Algorithm 1 (see Section III-B). After applying Lemma 3, we have that $S^{''}$ covers all necessarily critical paths in S. From Lemma 5 we get that $S^{''}$ is composed of all necessarily critical activities in S.

Case 2: $i^* \neq 1$ or $v^* \neq n$. In this case $(i^*,j^*),(j^*,k^*),\ldots,(u^*,v^*)$ is an isolated sequence of necessarily critical activities in S. These activities are necessarily critical in $S^{''}$ as well. They form a path leading from i^* to v^* in $S^{''}$. From this fact and necessary criticality of activities $(i^*,j^*),(j^*,k^*),\ldots,(u^*,v^*)$ and after applying Lemma 6, we have that $(i^*,j^*),(j^*,k^*),\ldots,(u^*,v^*)$ is a necessarily critical path in $S^{''}$. Lemma 5 implies that there are no isolated necessarily critical activities in $S^{''}$.

Let us treat the subnetwork $S^{''}$ as a counterpart of the subnetwork determined in Step 6 of Algorithm 1 (i^* is the single initial node and v^* is the single final node instead of 1 and n). Making use of Lemma 3, we obtain that all paths in $S^{''}$ leading from i^* to v^* are necessarily critical in $S^{''}$. Lemma 4 implies that these paths differ from each other only with activities, which have precise duration times. Hence, the lengths of the paths are the same for each configuration of times $t_{ij}, t_{ij} \in [\underline{t}_{ij}, \bar{t}_{ij}], (i,j) \in A$. On the basis of above facts and necessary criticality of the sequence $(i^*,j^*),(j^*,k^*),\ldots,(u^*,v^*)$ we conclude that all activities from $A^{''}$ are necessarily critical in S. ■

The above results show that the traditional criticality analysis of activity networks is not robust and ceases to be valid when the activity durations are ill known. The fact that under imprecise information, surely critical activities can be isolated ones had never been pointed out in the literature previously. Also new is the claim that modeling durations by intervals is enough to move the criticality analysis from the class of easy problems to the class of hard ones. The above results leave the open problem of finding an efficient complete algorithm for detecting isolated surely critical activities, and of determining its computational complexity. The next section shows that going from intervals to fuzzy intervals provides a more expressive representation framework without significantly increasing the difficulty of the problem.

IV. Criticality in Networks With Fuzzy Activity Times

Before going to the fundamental consideration regarding necessary criticality in fuzzy networks let us recall the notion of a fuzzy number and give some remarks on its interpretation.

A. A Fuzzy Number Notion and Its Interpretation

The most general definition of a fuzzy number is the following one.

Definition 12: A fuzzy number $\tilde{A}$ is a normal convex fuzzy set in the space of real numbers $\mathbb{R}$ with an upper semi-continuous membership function $\mu_{\tilde{A}}$.

Let us remind that a fuzzy set $\tilde{A}$ in $\mathbb{R}$ is convex if and only if its membership function is quasiconcave, i.e., it fulfills the condition: $\mu_{\tilde{A}}(z) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$ for each x,y,z such that $z \in [x,y]$. A fuzzy set $\tilde{A}$ is normal if and only if there exists $x \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x) = 1$.

Let us notice that an interval $[\underline{a}, \bar{a}]$ is a special case of a fuzzy number with the membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1, & \text{for } x \in [\underline{a}, \bar{a}], \\ 0, & \text{for } x \notin [\underline{a}, \bar{a}]. \end{cases}$$

Thus, a real number y is also a fuzzy number since it may be treated as a one-point interval $[y,y]$.

A fuzzy number $\tilde{A}$, which we could call a *partially precise fuzzy number*, determined by a membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} 1, & \text{for } x = a, \\ f_{\tilde{A}}(x), & \text{for } \underline{a} \leq x < a, \\ g_{\tilde{A}}(x), & \text{for } a < x \leq \bar{a}, \\ 0, & \text{for } x \notin [\underline{a}, \bar{a}], \end{cases} \tag{2}$$

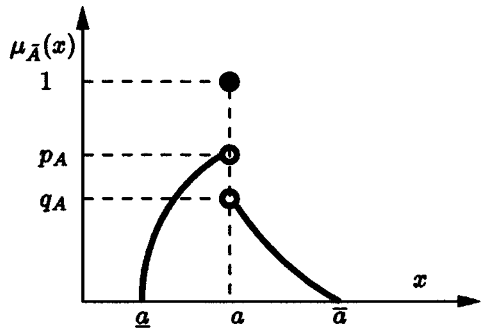

Fig. 4. The partially precise fuzzy number.

is another interesting special case which may be useful for application (see Fig. 4). Functions $f_{\tilde{A}}(x)$ and $g_{\tilde{A}}(x)$ are continuous, $f_{\tilde{A}}(x)$ is increasing (from 0 to p_A), $g_{\tilde{A}}(x)$ is decreasing (from q_A to 0) and p_A, q_A are any numbers from the unit interval $[0,1]$.

Now we recall the definition of the special case of a fuzzy number, introduced by Dubois and Prade [9]:

Definition 13: A fuzzy number $\tilde{A}$ is called a fuzzy number of the $L-R$ type if its membership function $\mu_{\tilde{A}}$ has the following form:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1, & \text{for } x \in [\underline{a}, \bar{a}], \\ L\left(\frac{\underline{a}-x}{\alpha_A}\right), & \text{for } x < \underline{a}, \\ R\left(\frac{x-\bar{a}}{\beta_A}\right), & \text{for } x > \bar{a}, \end{cases} \tag{3}$$

where L and R are continuous nonincreasing functions, defined on $[0, +\infty)$, strictly decreasing to zero in those subintervals of the interval $[0, +\infty)$ in which they are positive, and fulfilling the conditions $L(0) = R(0) = 1$. The parameters α_A and β_A are nonnegative real numbers.

We will use the following notation, introduced in [9], for a fuzzy number $\tilde{A}$ of $L-R$ type.

$$A = (\underline{a}, \bar{a}, \alpha_A, \beta_A)_{L-R}.$$

In Definition 13 the cases $\alpha = 0$ and/or $\beta = 0$ are admissible. Then, it is assumed that $L((\underline{a} - x)/0) = 0$ and/or $R((x - \bar{a})/0) = 0$. Thus, any interval $[\underline{a}, \bar{a}]$ and any precise number y are also fuzzy numbers of $L-R$ type

$$[\underline{a}, \bar{a}] = (\underline{a}, \bar{a}, 0, 0)_{L-R},$$
$$[y, y] = (y, y, 0, 0)_{L-R}.$$

Let us recall the notion of λ-cut of a fuzzy number.

Definition 14: Let $\tilde{A}$ be a fuzzy number. The interval

$$\tilde{A}^\lambda = [\underline{a}^\lambda, \bar{a}^\lambda] = \{x \in \mathbb{R} \,|\, \mu_{\tilde{A}}(x) \geq \lambda\} \quad \text{for } \lambda \in (0,1]$$

is called λ-cut of the fuzzy number $\tilde{A}$.

Let us give in this place some comments on the interpretation of a fuzzy number assumed in this paper. There are two possible interpretations of a fuzzy number applied to the description of an unknown parameter in a model. In the first case a fuzzy number expresses uncertainty connected with the ill known parameter modeled by this number. It generates possibility and necessity functions for sets of values likely to contain the unknown parameter. More formally, we say that the assertion of the form "V is $\tilde{A}$", where V is a variable and $\tilde{A}$ is a fuzzy number, generates the possibility distribution of V with respect to the following formula (see [46]):

$$\text{Poss}(V = x) = \mu_{\tilde{A}}(x), \quad x \in \mathbb{R}. \tag{4}$$

The same assertion induces the possibility and necessity measures for V in the following way:

$$\text{Poss}(V \in [a,b]) = \sup\{\lambda \,|\, \tilde{A}^\lambda \cap [a,b] \neq \emptyset\}, \tag{5}$$
$$\text{Nec}(V \in [a,b]) = 1 - \inf\{\lambda \,|\, \tilde{A}^\lambda \subseteq [a,b]\}, \tag{6}$$

where $[a,b] \in I(\mathbb{R})$ and $I(\mathbb{R})$ is the set of all intervals. Thus, for any continuous fuzzy number $\tilde{A}$ of the $L-R$ type, excluding the case when $\tilde{A}$ is a crisp interval, the following formula holds;

$$\text{Nec}(V \in \tilde{A}^\lambda) = 1 - \lambda, \quad \lambda \in (0,1]. \tag{7}$$

For a partially precise fuzzy number with a membership function (2) we have the following relations: $\text{Poss}(\tilde{A} \neq a) = \max\{p_A, q_A\}, \text{Poss}(\tilde{A} < a) = p_A, \text{Poss}(\tilde{A} > a) = q_A, \text{Nec}(\tilde{A} \geq a) = 1 - p_A, \text{Nec}(\tilde{A} \leq a) = 1 - q_A, \text{Nec}(\tilde{A} = a) = \min\{\text{Nec}(\tilde{A} \geq a), \text{Nec}(\tilde{A} \leq a)\} = \min\{1-p_A, 1-q_A\}$.

In the second interpretation it is assumed that a parameter modeled by a fuzzy number is controllable and the membership function describes a preference distribution for values assigned to the parameter (see e.g., [14]).

Further on we use the fuzzy numbers in the first role, i.e., we assume that all fuzzy parameters (fuzzy activity times) are ill known and their membership functions generate corresponding possibility and necessity functions in the sense of the formulae (4), (5) and (6).

B. Basic Notions of Fuzzy Criticality

A network $S = \langle V, A, \tilde{T} \rangle$ is given. All elements of this network are the same as in the interval case except for function $\tilde{T}$ which is now defined in the following way: $\tilde{T} : A \to F(\mathbb{R}_+)$, where $F(\mathbb{R}_+)$ is the set of nonnegative fuzzy numbers. $\tilde{T}(i,j) \overset{\text{def}}{=} \tilde{T}_{ij}, (i,j) \in A$. Fuzzy number $\tilde{T}_{ij}$ imprecisely determines a duration time of activity $(i,j) \in A$. The membership function $\mu_{\tilde{T}_{ij}}$ generates a possibility distribution for the duration time of activity $(i,j) \in A$ with respect to the formula (4).

Let T be a configuration of activity duration times in the network with activity times $t_{ij} \in \mathbb{R}_+, (i,j) \in A$. The (joint) possibility distribution over configurations, induced by the $\tilde{T}_{ij}$'s is $\pi(T) = \min_{(i,j) \in A} \mu_{\tilde{T}_{ij}}(t_{ij}), T \in \mathbb{R}_+^m$.

Now we introduce the criticality notions for a network with fuzzy activity duration times. The first two notions have been proposed in [7].

Definition 15: The following formula determines the possibility that a path $p \in P$ is critical in S:

$$\text{Poss}(p \text{ is critical}) = \sup_{T: p \text{ is critical in } T} \pi(T).$$

Definition 16: The following formula determines the possibility that an activity $(k,l) \in A$ is critical in S

$$\text{Poss}((k,l) \text{ is critical}) = \sup_{T:(k,l) \text{ is critical in } T} \pi(T).$$

Definition 17: The following formula determines the possibility that a path $p \in P$ is noncritical in S

$$\text{Poss}(p \text{ is noncritical}) = \sup_{T:p \text{ is not critical in } T} \pi(T).$$

Definition 18: The following formula determines the possibility that an activity $(k,l) \in A$ is noncritical in S

$$\text{Poss}((k,l) \text{ is noncritical}) = \sup_{T:(k,l) \text{ is not critical in } T} \pi(T).$$

Due to possibility-necessity relations (see [12]) we can obtain a necessity measure of criticality.

Definition 19: The following formula determines the necessity that a path $p \in P$ is noncritical in S

$$\begin{aligned}\text{Nec}(p \text{ is noncritical}) &= 1 - \text{Poss}(p \text{ is critical}) \\ &= \inf_{T:p \text{ is critical in } T} (1 - \pi(T)).\end{aligned}$$

Definition 20: The following formula determines the necessity that an activity $(k,l) \in A$ is noncritical in S

$$\text{Nec}((k,l) \text{ is noncritical}) = \inf_{T:(k,l) \text{ is critical in } T} (1 - \pi(T)).$$

Definition 21: The following formula determines the necessity that a path $p \in P$ is critical in S

$$\begin{aligned}\text{Nec}(p \text{ is critical}) &= 1 - \text{Poss}(p \text{ is noncritical}) \\ &= \inf_{T:p \text{ is not critical in } T} (1 - \pi(T)).\end{aligned}$$

Definition 22: The following formula determines the necessity that an activity $(k,l) \in A$ is critical in S

$$\text{Nec}((k,l) \text{ is critical}) = \inf_{T:(k,l) \text{ is not critical in } T} (1 - \pi(T)).$$

In [7] two effective methods of calculating the value of index Poss(p is critical) have been presented. The first one is adapted to fuzzy activity times given in a general form and the second one, based on linear programming, is valid only for fuzzy activity times determined by fuzzy numbers of the same $L - L$ type, i.e., each fuzzy activity time $\tilde{T}_{ij}, (i,j) \in A$, is given by means of fuzzy number of the $L - R$ type (see Definition 13), in which the left shape function L_{ij} is equal to the right shape function R_{ij} and additionally the left shape function L is the same for all activity duration times, $L = L_{ij} = R_{ij}, (i,j) \in A$. Naturally, those methods may be also applied to calculating the index Nec(p is noncritical) determined in Definition 19.

In the next section we present specific methods for calculating the value of index Nec(p is critical). Of course, these methods can be also applied, by the relation given in Definition 21, to calculating the index Poss(p is noncritical).

C. Calculating the Degree of Necessity That a Path Is Critical

In this section, we present two effective methods of determining the necessity degree that a path is critical. The first method enables to determine the necessity degree in a general case while the second one needs special assumptions on the membership functions of the fuzzy activity times $\tilde{T}_{ij}, (i,j) \in A$ i.e., activity times should be determined by means of fuzzy numbers of the same $L - L$ type.

Before presenting the methods we formulate some theorems, which form the theoretical basis for these methods. These theorems determine relationships between indices Nec(p is critical) and Poss(p is noncritical) and the notions of necessary criticality and possible noncriticality (see Definitions 7 and 9).

Let us denote by S^λ the λ-cut of the network S, i.e., the network S with the interval activity times $T_{ij} = \tilde{T}_{ij}^\lambda = [\underline{t}_{ij}^\lambda, \bar{t}_{ij}^\lambda], (i,j) \in A$, and by $S_c^\lambda = \langle A_c, V_c, \tilde{T}^\lambda \rangle$ a subnetwork composed of all necessarily critical paths in $S^\lambda, \lambda \in (0,1]$.

Theorem 1: For a path $p \in P$ the following equivalence holds for a fixed $\lambda \in (0,1]$

$$\text{Poss}(p \text{ is noncritical}) \geq \lambda \iff p \text{ is possibly noncritical in } S^\lambda.$$

Proof: It follows directly from Definition 9, Definition 17, and the fact that if $\alpha < \beta$ then $\tilde{T}_{ij}^\beta \subseteq \tilde{T}_{ij}^\alpha, (i,j) \in A$. ■

Theorem 2: For a path $p \in P$ the following equivalence holds for a fixed $\lambda \in (0,1]$

$$\text{Poss}(p \text{ is noncritical}) < \lambda \iff p \text{ is necessarily critical in} S^\lambda.$$

Proof: It follows directly from Theorem 1 and the fact that the notion of possible noncriticality is complementary to the notion of necessary criticality, and reversely. ■

Theorem 3: For a path $p \in P$ the following equivalence holds for a fixed $\lambda \in [0,1)$

$$\text{Nec}(p \text{ is critical}) > \lambda \iff p \text{ is necessarily critical in } S^{1-\lambda}. \quad (8)$$

Proof: From Definition 21 we have:

$$\begin{aligned}\text{Nec}(p \text{ is critical}) > \lambda \iff& \text{Poss}(p \text{ is noncritical}) \\ &< 1 - \lambda, \quad \lambda \in [0,1), \quad (9)\end{aligned}$$

and from Theorem 2:

$$\begin{aligned}&\text{Poss}(p \text{ is noncritical}) < 1 - \lambda \\ &\quad \iff p \text{ is necessarily critical in } S^{1-\lambda}, \quad \lambda \in [0,1). \quad (10)\end{aligned}$$

And finally, we obtain (8) directly from conditions (9) and (10). ■

The next theorem provides a tool for calculating the Poss(p is noncritical) and Nec(p is critical) indices.

Theorem 4: For a path $p \in P$ the following equalities hold:

$$\text{Poss}(p \text{ is noncritical}) = \sup\{\lambda \mid p \text{ is possibly noncritical in } S^\lambda\}, \quad (11)$$

$$\text{Nec}(p \text{ is critical}) = \sup\{\lambda \mid p \text{ is necessarily critical in } S^{1-\lambda}\}. \quad (12)$$

Proof: Obvious. The equalities (11) and (12) follow directly from Theorems 1 and 3, Definitions 9 and 7 and the fact that if $\alpha < \beta$ then $\tilde{T}_{ij}^\beta \subseteq \tilde{T}_{ij}^\alpha, (i,j) \in A$. ■

The above theorem may be formulated in another, clearer, form.

Theorem 5: Nec(p is critical) $= \lambda_o$ (and Poss(p is noncritical) $= 1 - \lambda_o$,) $0 \leq \lambda_o \leq 1$, if and only if p is necessarily critical for each $\lambda > 1 - \lambda_o$ and possibly noncritical in S^λ for each $\lambda \le 1 - \lambda_o$.

1) General Case: Now we propose the algorithm (Algorithm 3) for calculating the necessity degree that a fixed path $p \in P$ is critical (Definition 21) for a general case, i.e., the case when the activity times are fuzzy numbers in a sense of Definition 12. The algorithm is based on the idea of bisection of the unit interval of possible values of λ to compute the optimal value in (12) or, in other words, to find the value λ_o fulfilling the conditions of Theorem 5. This value is computed with accuracy equal to $\varepsilon = 10^{-N}$.

Algorithm 3 Calculation of Nec(p is critical)

Require: $S = \langle V, A, \tilde{T}\rangle$, a fixed path $p \in P$, absolute error of computation $\varepsilon = 10^{-N}, K \ge N/\log_{10} 2$.

Ensure: Nec(p is critical).

Step 1 Assign $k := 0$.

Step 2 Test if p is necessarily critical in S^1.

If it is not then assign $\lambda_{\max} := 0$ and go to Step 6.

Step 3 Assign $\lambda_k := 1$ and test if p is necessarily critical in S^ε.

If it is then assign $\lambda_{\max} := 1$ and go to Step 6.

Step 4 Assign $k := k + 1$,

$$\lambda_k := \begin{cases} \lambda_{k-1} + 1/2^k & \text{if } p \text{ is necessarily critical in } S^{1-\lambda_{k-1}}, \\ \lambda_{k-1} - 1/2^k & \text{otherwise} \end{cases}$$

Test if p is necessarily critical in $S^{1-\lambda_k}$.

If it is then assign $\lambda_{\max} := \lambda_k$.

Step 5 If $k < K$ then go to Step 4.

Step 6 Assign Nec(p is critical): $= \lambda_{\max}$, stop.

At each iteration k in the algorithm, we test if p is necessarily critical in $S^{1-\lambda_k}$. The testing can be reduced to applying the classical CPM method to the network $S^{1-\lambda_k}$, after replacing the interval times $\tilde{T}_{ij}^{1-\lambda_k}$ with the exact values t_{ij} determined as in Lemma 1.

2) Linear Programming Approach: Another approach to the problem of determining the necessity degree that a path is critical is based on linear programming. We show that this problem can be reduced, under certain assumptions about membership functions of fuzzy activities duration times $\tilde{T}_{ij}$, to that of determining the optimal solution of a classical linear programming problem. From Lemma 1 it follows that stating if p is necessarily critical in $S^{1-\lambda}$, for a fixed $\lambda \in [0, 1)$, can be reduced to checking if the following system of equalities and inequalities has a solution:

$$\begin{aligned} & t_j - t_i - \underline{t}_{ij}^{1-\lambda} = 0, \quad (i,j) \in p, \\ & t_j - t_i - \bar{t}_{ij}^{1-\lambda} \ge 0, \quad (i,j) \notin p, \\ & t_1 = 0, \\ & t_i \ge 0, \quad (i = 2, \ldots, n), \end{aligned} \tag{13}$$

where variables t_i denote moments of occurrence of the events $i \in V$ in network $S^{1-\lambda}$. Hence, the determination of the necessity degree of criticality of a path $p \in P$, Nec(p is critical), can be reduced, according to Theorem 4, to the following mathematical programming problem:

$$\begin{aligned} & \lambda \to \max, \\ & t_j - t_i - \underline{t}_{ij}^{1-\lambda} = 0, \quad (i,j) \in p, \\ & t_j - t_i - \bar{t}_{ij}^{1-\lambda} \ge 0, \quad (i,j) \notin p, \\ & t_1 = 0, \quad 0 \le \lambda \le 1, \\ & t_i \ge 0, \quad (i = 2, \ldots, n), \end{aligned} \tag{14}$$

If $\lambda_{\max}$ is the optimal objective value of (14) then Nec(p is critical)$= \lambda_{\max}$. If the problem (14) is infeasible then we have the case Nec(p is critical)$= 0$.

Let us assume that fuzzy activities duration times $\tilde{T}_{ij}, (i,j) \in A$, are given by means of fuzzy numbers of the same $L - L$ type (see Definition 3, $L = L_{ij} = R_{ij}$), i.e., $\tilde{T}_{ij} = (\underline{t}_{ij}, \bar{t}_{ij}, \alpha_{ij}, \beta_{ij})_{L-L}$. In this case $(1-\lambda)$-cuts of a fuzzy number $\tilde{T}_{ij}, \lambda \in [0, 1)$, have the form

$$\begin{aligned} \tilde{T}_{ij}^{1-\lambda} &= [\underline{t}_{ij}^{1-\lambda}, \bar{t}_{ij}^{1-\lambda}] \\ &= [\underline{t}_{ij} - L^{-1}(1-\lambda)\alpha_{ij}, \bar{t}_{ij} + L^{-1}(1-\lambda)\beta_{ij}]. \end{aligned}$$

Putting $\Theta = L^{-1}(1-\lambda)$, and taking advantage of the fact that the function $L^{-1}(y)$ is decreasing in the $[0, 1]$ interval, we can transform the problem (14) to the following linear programming problem:

$$\begin{aligned} & \Theta \to \max, \\ & t_j - t_i - \underline{t}_{ij} + \alpha_{ij}\Theta = 0, \quad (i,j) \in p, \\ & t_j - t_i - \bar{t}_{ij} - \beta_{ij}\Theta \ge 0, \quad (i,j) \notin p, \\ & t_1 = 0, \quad \underline{\Theta} \le \Theta \le \bar{\Theta}, \\ & t_i \ge 0 \quad (i = 2, \ldots, n), \end{aligned} \tag{15}$$

where $\underline{\Theta} = L^{-1}(1), \bar{\Theta} = L^{-1}(0)$. If $\Theta_{\max}$ is the optimal objective value of (15) then Nec(p is critical)$= 1 - L(\Theta_{\max})$. If the problem is infeasible then Nec(p is critical)$= 0$.

D. Determining Paths With Maximal Necessity Degree of Criticality

Now we consider the problem of determining paths with the maximum necessity degree of criticality. Let us denote by $S_{c\max}$ a subnetwork of S composed of all paths with maximal necessity degree of criticality $\mathrm{NECC}_{\max}$, where $\mathrm{NECC}_{\max} = \max_{p \in P}$ Nec(p is critical).

It is easy to prove the following theorem, similar to Theorem 5 (in fact this theorem follows directly from Theorem 5).

Theorem 6: $\mathrm{NECC}_{\max} = \lambda_0$, $0 \le \lambda_0 \le 1$, if and only if for each $\lambda > 1 - \lambda_0$ there exists a necessarily critical path in S^λ (i.e., there exists S_c^λ in S^λ) and for each $\lambda \le 1 - \lambda_0$ there does not exist a necessarily critical path in S^λ (i.e., there does not exist S_c^λ in S^λ).

Let us recall that S_c^λ denotes the subnetwork of S^λ composed of all necessarily critical paths in S^λ.

Depending on assumptions on the duration times of activities, we have two cases:

Case 1: Activity times are given by means of fuzzy numbers in general form (see Definition 12). In this case, if $\mathrm{NECC}_{\max} > 0$, then there may exist paths in S with different necessity degree of criticality. In fact it may happen only in the case when there exist activities with partially precise durations, determined by

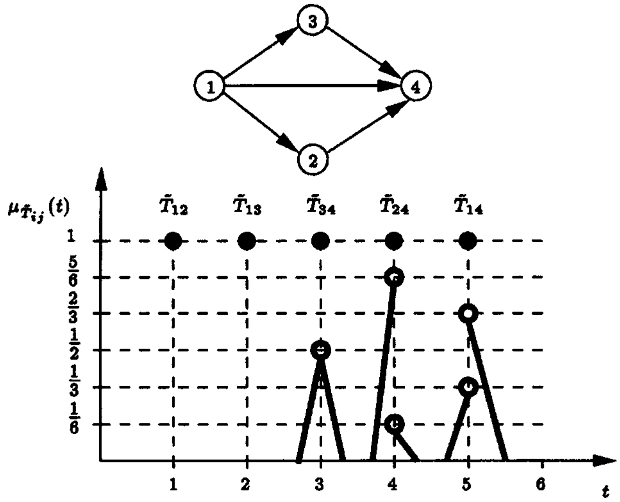

Fig. 5. A network containing paths with different necessity degree of criticality.

membership functions of the form (2). An example of such a network is given in Fig. 5. There are three paths with different necessity degree of criticality. Paths $1 \to 2 \to 4, 1 \to 3 \to 4$ and $1 \to 4$ have the degrees of criticality $1/6, 1/3$ and $1/2$, respectively.

To find $S_{c\max}$ we propose Algorithm 4. Like in Algorithm 3, the idea of this algorithm is based on bisection of the unit interval to find the value λ_0 fulfilling the conditions of Theorem 6. This value is computed with accuracy equal to $\epsilon = 10^{-N}$. In this algorithm at each iteration k, we find a subnetwork $S_c^{1-\lambda_k}$ composed of all necessarily critical paths in $S^{1-\lambda_k}$ by applying Algorithm 1 to $S^{1-\lambda_k}$.

Algorithm 4 Determining all paths with maximal necessity degree of criticality

Require: $S =< V, A, \tilde{T} >$, absolute error of computation $\epsilon = 10^{-N}$, $K \geq N/\log_{10} 2$.

Ensure: $S_{c\max}$ a subnetwork composed of all paths with maximal necessity degree of criticality $\mathrm{NECC}_{\max}$—if such a subnetwork exists.

Step 1 Assign $k := 0$.

Step 2 Apply Algorithm 1 to S^1 for finding a subnetwork S_c^1.

If S_c^1 does not exist then assign $\lambda_{\max} := 0$ and go to Step 6.

Step 3 Assign $\lambda_k := 1$ and apply Algorithm 1 to S^c for finding a subnetwork S_c^c.

If S_c^c exists then assign $\lambda_{\max} := 1$ and go to Step 6.

Step 4 Assign $k := k + 1$,

$$\lambda_k := \begin{cases} \lambda_{k-1} + 1/2^k & \text{if there exists } S_c^{1-\lambda_{k-1}} \text{ in } S^{1-\lambda_{k-1}}, \\ \lambda_{k-1} - 1/2^k & \text{otherwise} \end{cases}$$

Apply Algorithm 1 to $S^{1-\lambda_k}$ for finding a subnetwork $S_c^{1-\lambda_k}$.

If $S_c^{1-\lambda_k}$ exists then assign $\lambda_{\max} := \lambda_k$.

Step 5 If $k < K$ then go to Step 4.

Step 6 Assign $\mathrm{NECC}_{\max} := \lambda_{\max}, S_{c\max} := S_c^{1-\lambda_{\max}}$, stop.

The correctness of Algorithm 4 follows directly from Theorem 6 and the fact that if $\alpha < \beta$ then $S_c^{1-\beta} \subseteq S_c^{1-\alpha}$.

Case 2: Activity times are given by means of fuzzy numbers $\tilde{T}_{ij}$ of $L_{ij} - R_{ij}$ type $(i,j) \in A$ (see Definition 13). In this case, if there exists any path $p' \in P$ with Nec(p' is critical)> 0, we have $\mathrm{NECC}_{\max} = \mathrm{Nec}(p'$ is critical$)$ and there are two groups of paths in S, i.e., paths with the necessity degree of criticality equal to zero and paths with the same necessity degree of criticality equal to Nec(p' is critical). In other words, for each $\alpha, \beta > \mathrm{NECC}_{\max}$ we have $S_c^\beta = S_c^\alpha = S_{c\max}$. Thus, determining $S_{c\max}$ reduces itself to applying Algorithm 1 to S^1 for finding a subnetwork S_c^1 and computing the necessity degree of criticality Nec(p' is critical) of any path $p' \in P$ composed of activities $(i,j),(i,j) \in A_c^1$, by applying Algorithm 3—if S_c^1 exists. If S_c^1 does not exist, then $\mathrm{NECC}_{\max} = 0$.

E. Degree of Necessity of Criticality for Activities

At the end of this section we briefly discuss the problem of calculating the necessity degree of criticality of a fixed activity. Generally the problem does not seem to be easy, since it is more general than the problem, in the interval case, of stating if an activity is necessarily critical.

Actually, the following theorem, similar to Theorem 4 for a path, may be formulated.

Theorem 7: For an activity $(k,l) \in A$ the following equality holds: Nec((k,l) is critical$) = \sup\{\lambda \,|\, (k,l)$ necessarily critical in $S^{1-\lambda}\}$.

Proof: Similar to the proof of Theorem 4. ■

However, in some situations we may evaluate the degrees of necessary criticality of certain activities by using information provided by Algorithm 4. Exactly, if $\mathrm{NECC}_{\max} > 0$, then these degrees may be calculated, both in Case 1 and 2, by applying Theorems 8 and 9.

The first theorem concerns Case 1 (general fuzzy numbers).

Theorem 8: If (k,l) does not belong to $S_{c\max}$, then

$$\mathrm{Nec}((k,l) \text{ is critical}) = \sup\{\lambda \,|\, (k,l) \text{ belongs to } S_c^{1-\lambda}, \; 0 \leq \lambda < \mathrm{NECC}_{\max}\}, \quad (16)$$

otherwise

$$\mathrm{NECC}_{\max} \leq \mathrm{Nec}((k,l) \text{ is critical}) \leq 1. \quad (17)$$

Proof: Formula (16) follows from Lemma 5, Theorem 7 and the fact that if $\alpha < \beta$ then $S_c^{1-\beta} \subseteq S_c^{1-\alpha}$.

The lower and upper bounds (17) are obvious. ■

According to the formula (16) one may construct an algorithm, similar to Algorithms 3 and 4, for computing the necessity degrees of criticality of some activities. For other activities one can only give lower and upper bounds of these degrees.

The second theorem concerns Case 2 ($L-R$ fuzzy numbers).

Theorem 9: If (k,l) does not belong to $S_{c\max}$, then

$$\mathrm{Nec}((k,l) \text{ is critical}) = 0, \quad (18)$$

otherwise

$$\mathrm{NECC}_{\max} \leq \mathrm{Nec}((k,l) \text{ is critical}) \leq 1. \quad (19)$$

Proof: Equality (18) follows from Lemma 5, Theorem 7 and the fact that for each $\alpha, \beta > \mathrm{NECC}_{\max}$ the following equality holds $S_c^\beta = S_c^\alpha = S_{c\max}$.

The lower and upper bounds (19) are obvious. ■

V. Final Remarks

In the paper the notion of necessary criticality (both with respect to path and to activity) in a network with imprecisely defined (by means of intervals or fuzzy interval numbers) activity duration times is introduced and analyzed.

In the interval case we show that both the problem of stating whether a given path is necessarily critical and the problem of determining an arbitrary necessarily critical path (more exactly, a subnetwork covering all the necessarily critical paths) are easy. We give corresponding solution algorithms. Unfortunately, the problem of evaluating whether a given isolated activity is necessarily critical does not seem to be straightforward. In the paper we formulated a hypothesis (see Conjectures 1 and 2) that this problem is hard. The question of proving this fact is still open. It is interesting that the attempt of applying a similar proof method to that used in the case of the corresponding problem of evaluating the possible criticality (see [6]) has failed. The reason is maybe that the problem considered here is only apparently similar to that solved successfully in [6]. In reality the two problems differ substantially one from another. A possibly critical activity always belongs to a possibly critical path, i.e., no isolated possibly critical activity can ever exist in a network. In each network there is always at least one possibly critical path. The situation in the case of necessarily critical activities is different. It is possible (and this case is not very rare) to have a network without any necessarily critical path, but there may be isolated necessarily critical activities. And these activities are just those difficult to identify. The problem is easy to solve when a necessarily critical path exists, since all the necessarily critical activities belong then to the necessarily critical paths, which are easy to determine.

In the paper we formulated conditions which in some situations allow evaluating the necessary criticality of activities also in case when there is no necessarily critical path. However, they do not cover all the possible situations, so do not solve the problem completely.

We generalized the results obtained for networks with interval activity duration times to the case of networks with fuzzy activity duration times. We proposed effective algorithms of calculating the degree of the necessary criticality of a path, as well as an algorithm of determining the paths that are necessarily critical to the maximal degree. Unfortunately, we have not been able to propose such algorithms for evaluating the degree of the necessary criticality of an activity. It will be not easy if the hypotheses formulated in Sect. III.C, concerning the computational complexity of the corresponding interval problems, are true.

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