A 2-approximation algorithm for interval data minmax regret sequencing problems with the total flow time criterion

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Abstract

In this paper we discuss a minmax regret version of the single machine scheduling problem with the total flow time criterion. Uncertain processing times are modeled by closed intervals. We show that if the deterministic problem is polynomially solvable, then its minmax regret version is approximable within 2.

Keywords: Scheduling; Interval data; Minmax regret; Approximation algorithm

1 Preliminaries

We are given a set of jobs $J = \{J_1, \dots, J_n\}$ to be processed on a single machine. For the sake of simplicity, we will identify every job J_i with its index i. A *schedule* is a permutation $\pi = (\pi(1), \dots, \pi(n))$ of jobs. The set of jobs may be partially ordered by some *precedence constraints*, namely if $i \to j$, then job j must be processed after job i. A schedule π is feasible if it satisfies all the precedence constraints. We denote by Π the set of all feasible schedules. Consider the case in which the processing time of job i is only known to belong to a closed interval $\tilde{p}_i = [\underline{p}_i, \overline{p}_i]$ for all $i \in J$. A particular realization of the processing times $S = (p_i^S)_{i\in J}$, such that $p_i \in \tilde{p}_i$ for all $i \in J$, is called a *scenario*. The set of all scenarios, being the Cartesian product of all $\tilde{p}_i$, is denoted by Γ. Now $C_i(\pi, S)$ is the completion time of job i in schedule π under a given scenario $S \in \Gamma$ and $F(\pi, S) = \sum_{i\in J} C_i(\pi, S)$ is the *total flow time* in π under S. Let $F^*(S) = \min_{\pi\in\Pi} F(\pi, S)$ be the total flow time in an optimal schedule under scenario S. The *maximal regret* of $\pi \in \Pi$ is defined as follows:

$$Z(\pi) = \max_{S\in\Gamma}\{F(\pi, S) - F^*(S)\}. \tag{1}$$

A scenario S_π that maximizes the right hand side of (1) is called the *worst case scenario* for π. We seek a feasible schedule that minimizes the maximal regret. We thus consider the minmax regret version of the deterministic $1|prec|\sum C_i$ problem. The general deterministic $1|prec|\sum C_i$ problem is strongly NP-hard [5, 7]. It is, however, polynomially solvable for some

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particular structure of the precedence constraints [1, 5, 9]. In literature only the simplest case of the minmax regret problem, that is the one with no precedence constraints, was discussed. It was first studied in [2] and it has been recently proved to be NP-hard in [6]. In [2, 4, 8] some exact algorithms and heuristics to solve this problem were proposed.

In this paper, we show that solving a deterministic counterpart of the problem under a midpoint scenario leads to a schedule π such that $Z(\pi) \leq 2OPT$, where OPT is the maximal regret of an optimal minmax regret schedule. This method has been already used as a heuristic in [2, 4]. However, the authors in [2, 4] did not prove the bound of 2 and our result is valid for the problem with arbitrary precedence constraints. A similar result was obtained for a wide class of combinatorial optimization problems in [3]. However, it cannot be applied to the considered sequencing problem. The reason is that we are not able to represent the sequencing problem as a combinatorial optimization problem discussed in [3], in which all costs are independent (the deterministic $1||\sum C_i$ problem can be represented as an assignment problem, but the same parameter appears multiple times in the objective function). In consequence, we cannot use a very simple characterization of the worst case scenario, which is true for the class of problems discussed in [3].

2 The main result

Let us denote by i_π the position occupying by job i in schedule π. For any two schedules π, σ and scenario $S \in \Gamma$ the following equality is true:

$$F(\pi, S) - F(\sigma, S) = \sum_{i \in J} (i_\sigma - i_\pi) p_i^S. \tag{2}$$

Using (2) and the definition of the maximal regret (1) one can easily prove that for any two feasible schedules π and σ the following inequality holds:

$$Z(\pi) \geq \sum_{\{i: i_\sigma > i_\pi\}} (i_\sigma - i_\pi)\overline{p}_i + \sum_{\{i: i_\sigma < i_\pi\}} (i_\sigma - i_\pi)\underline{p}_i. \tag{3}$$

We now show that any feasible schedules π and σ satisfy the inequality:

$$Z(\sigma) \leq Z(\pi) + \sum_{\{i: i_\pi > i_\sigma\}} (i_\pi - i_\sigma)\overline{p}_i + \sum_{\{i: i_\pi < i_\sigma\}} (i_\pi - i_\sigma)\underline{p}_i. \tag{4}$$

Indeed, the following inequality follows from (2):

$$F(\sigma, S_\sigma) \leq F(\pi, S_\sigma) + \sum_{\{i: i_\pi > i_\sigma\}} (i_\pi - i_\sigma)\overline{p}_i + \sum_{\{i: i_\pi < i_\sigma\}} (i_\pi - i_\sigma)\underline{p}_i, \tag{5}$$

where S_σ is the worst case scenario for σ. Subtracting $F^*(S_\sigma)$ from both sides of (5) yields $Z(\sigma) \leq F(\pi, S_\sigma) - F^*(S_\sigma) + \sum_{\{i: i_\pi > i_\sigma\}} (i_\pi - i_\sigma)\overline{p}_i + \sum_{\{i: i_\pi < i_\sigma\}} (i_\pi - i_\sigma)\underline{p}_i$, which, together with $Z(\pi) \geq F(\pi, S_\sigma) - F^*(S_\sigma)$, gives (4). We are ready to prove our main result.

Theorem 1. *Let S be the midpoint scenario, i.e. $p_i^S = \frac{1}{2}(\underline{p}_i + \overline{p}_i)$ for all $i \in J$, and let σ be an optimal schedule under S. Then for every feasible schedule π it holds $Z(\sigma) \leq 2Z(\pi)$.*

Proof. Since σ is an optimal schedule under the midpoint scenario S, we have $\sum_{i\in J}(i_\sigma - i_\pi)(\underline{p}_i + \overline{p}_i) \geq 0$ (see (2)), which is equivalent to the following inequality:

$$\sum_{\{i:i_\sigma>i_\pi\}} (i_\sigma - i_\pi)\overline{p}_i + \sum_{\{i:i_\sigma<i_\pi\}} (i_\sigma - i_\pi)\underline{p}_i \geq \sum_{\{i:i_\pi>i_\sigma\}} (i_\pi - i_\sigma)\overline{p}_i + \sum_{\{i:i_\pi<i_\sigma\}} (i_\pi - i_\sigma)\underline{p}_i. \quad (6)$$

Now applying formula (3) to (6) we obtain

$$Z(\pi) \geq \sum_{\{i:i_\pi>i_\sigma\}} (i_\pi - i_\sigma)\overline{p}_i + \sum_{\{i:i_\pi<i_\sigma\}} (i_\pi - i_\sigma)\underline{p}_i. \quad (7)$$

Now inequalities (4) and (7) yield $Z(\sigma) \leq Z(\pi) + Z(\pi) = 2Z(\pi)$. □

The following corollary is an immediate consequence of Theorem 1:

Corollary 1. *If the deterministic $1|prec|\sum C_i$ problem, for some particular structure of the precedence constraints, is polynomially solvable, then the minmax regret version of the problem with interval procesing times is approximable within 2.*

It can be easily shown that the bound of 2 obtained in Corollary 1 is tight. Consider a problem with three jobs and with no precedence constraints. The interval processing times of jobs are $\tilde{p}_1 = [0, 2]$, $\tilde{p}_2 = [1, 1]$ and $\tilde{p}_3 = [1, 1]$. The midpoint scenario assigns to all jobs the processing times equal to 1. An easy computation shows that schedule $\sigma = (1, 3, 2)$ has the maximal regret equal to 2, while the optimal minmax regret schedule $\pi = (3, 1, 2)$ has the maximal regret equal to 1, which implies $Z(\sigma)/Z(\pi) = 2$.

References

[1] Brucker P. (1998) Scheduling algorithms. Springer-Verlag, Berlin.

[2] Daniels R.L., Kouvelis P. (1995). Robust scheduling to hedge against processing time uncertainty in single-stage production. Manage. Sci. 41: 363–376.

[3] Kasperski A., Zieliński P. (2006) An approximation algorithm for interval data minmax regret combinatorial optimization problems. Inform.Process. Lett. 97: 177–180.

[4] Kouvelis P., Yu G. (1997) Robust discrete optimization and its applications. Kluwer Academic Publishers, Boston.

[5] Lawler E.L. (1978) Sequencing jobs to minimize total weighted completion time subject to precedence constraints. Ann. Discrete Math. 2: 75–90.

[6] Lebedev V., Averbakh I. (2006) Complexity of minimizing the total flow time with interval data and minmax regret criterion. Discrete. Appl. Math. 154: 2167–2177.

[7] Lenstra J.K., Rinnooy Kan A.H.G. (1978) Complexity of scheduling under precedence constraints. Oper. Res. 26: 22–35.

[8] Montemanni R. (2006) A mixed integer programming formulation for a single machine robust scheduling with interval data. Journal of Mathematical Modelling and Algorithms 6: 287–296

[9] Smith W.E. (1956) Various optimizers for single-stage production. Nav. Res. Log. 3: 59–66.