Recoverable robust combinatorial optimization problems

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Abstract This paper deals with two Recoverable Robust (RR) models for combinatorial optimization problems with uncertain costs. These models were originally proposed by Büsing (2012) for the shortest path problem with uncertain costs. In this paper, we generalize the RR models to a class of combinatorial optimization problems with uncertain costs and provide new positive and negative complexity results in this area.

1 Introduction

Let $E = \{e_1, \ldots, e_n\}$ be a finite ground set and let $\Phi \subseteq 2^E$ be a set of subsets of E called the set of the *feasible solutions*. A nonnegative cost c_e is given for each element $e \in E$. A *combinatorial optimization problem* $\mathscr{P}$ with a linear objective function consists in finding a feasible solution X whose total cost, $C(X) = \sum_{e\in X} c_e$, is minimal, namely:

$$\mathscr{P}: \quad \min_{X\in\Phi} C(X). \tag{1}$$

Formulation (1) encompasses a large variety of the classical combinatorial optimization problems. In practice, the precise values of the element costs c_e, $e \in E$, in (1) may be ill-known. This uncertainty can be modeled by specifying a set of all possible realizations of the element costs (states of the world) called *scenarios*. We denote by $\mathscr{S}$ the set of all scenarios. Formally, a scenario is a vector $S = (c_e^S)_{e\in E}$,

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that represents an assignment of costs to the elements of E. Let $C^S(A)=\sum_{e\in A} c_e^S$, where $A\subseteq E$. A popular approach to combinatorial optimization problems $\mathscr{P}$ for hedging against the uncertainty of the element costs, modeled by scenarios, is a *robust approach*, in which we seek a solution minimizing a worst case performance over all scenarios (see, e.g. [9]):

$$\text{ROB } \mathscr{P}: \quad OPT_{Rob}=\min_{X\in\Phi} C_{Rob}(X)=\min_{X\in\Phi}\max_{S\in\mathscr{S}} C^S(X). \tag{2}$$

In this paper, we investigate two *Recoverable Robust* (RR) models for combinatorial optimization problems with uncertain element costs under the scenario uncertainty representation. These models were originally proposed in [4] for the shortest path problem. Here, we generalize them to the combinatorial optimization problem (1).

In the *Rent-Recoverable Robust* model, we are given a *rental factor* $\alpha\in(0,1)$ and an *inflation factor* $\beta\geq 0$. Let $C_R^S(X)=\alpha C^S(X)$ be the *rent cost* of solution $X\in\Phi$ under scenario S and $C_I^S(X)=\min_{Y\in\Phi}\{(1-\alpha)C^S(Y)+(\alpha+\beta)C^S(Y\setminus X)\}$ be the *implementation cost* of solution $X\in\Phi$ under scenario S. Define $C_{Rent}(X)=\max_{S\in\mathscr{S}}\{C_R^S(X)+C_I^S(X)\}$. In the RENT-RR $\mathscr{P}$ problem we wish to find a solution $X\in\Phi$ minimizing $C_{Rent}(X)$, namely:

$$\text{RENT-RR } \mathscr{P}: \quad OPT_{Rent}=\min_{X\in\Phi} C_{Rent}(X)=\min_{X\in\Phi}\max_{S\in\mathscr{S}}\{C_R^S(X)+C_I^S(X)\}. \tag{3}$$

In the *k-Distance-Recoverable Robust* model, we are given the *first stage element costs* c_e^1, $e\in E$, and a *recovery parameter* $k\in\mathbb{N}$. For a given $X\in\Phi$ and k, we will denote by Φ_X^k the set of feasible solutions Y such that $|Y\setminus X|\leq k$. Let $C^1(X)=\sum_{e\in X} c_e^1$ and $C_{Rec}(X)=\max_{S\in\mathscr{S}}\min_{Y\in\Phi_X^k} C^S(Y)$ be the first stage and recovery costs, respectively. Define $C_{Dist}(X)=C^1(X)+C_{Rec}(X)$. In the k-DIST-RR $\mathscr{P}$ problem we seek a solution $X\in\Phi$ minimizing $C_{Dist}(X)$, namely:

$$k\text{-DIST-RR } \mathscr{P}: \quad OPT_{Dist}=\min_{X\in\Phi} C_{Dist}(X)=\min_{X\in\Phi}\{C^1(X)+C_{Rec}(X)\}. \tag{4}$$

In this paper we consider two methods of describing the set of scenarios $\mathscr{S}$. In the *discrete scenario uncertainty representation*, the scenario set, denoted by $\mathscr{S}_D$, is defined by explicitly listing all possible scenarios. So, $\mathscr{S}_D=\{S_1,\ldots,S_K\}$ is finite and contains exactly $K\geq 1$ scenarios. We distinguish the *bounded case*, where the number of scenarios K is bounded by a constant and the *unbounded case*, where the number of scenarios K is a part of the input. In the *interval uncertainty representation* the element costs are only known to belong to closed intervals $[\underline{c}_e,\overline{c}_e]$. Thus, the set of scenarios, denoted by $\mathscr{S}_I$, is the Cartesian product of these intervals, i.e. $\mathscr{S}_I=\times_{e\in E}[\underline{c}_e,\overline{c}_e]$.

2 Rent-RR combinatorial optimization problems

In this section we discuss the RENT-RR $\mathscr{P}$ problem. We provide some new complexity and approximation results for various problems $\mathscr{P}$. We now focus on the discrete scenario uncertainty representation. Consider first the case when $\mathscr{P}$ is the MINIMUM SPANNING TREE. Then E is the set of edges of a given undirected graph $G=(V,E)$ and Φ contains all spanning trees of G (a spanning tree is a subset of exactly $|V|-1$ edges that form an acyclic subgraph of G).

Proposition 1. *There is a polynomial time approximation preserving reduction from* ROB MINIMUM SPANNING TREE *to* RENT-RR MINIMUM SPANNING TREE.

Proof. Let $(G=(V,E),\mathscr{S}_D=\{S_1,\ldots,S_K\})$ be an instance of ROB MINIMUM SPANNING TREE. We build a graph $G'=(V',E')$ by adding an additional node v' to V and additional parallel edges $e_v^1,\ldots,e_v^K$ of the form $\{v',v\}$ for each node $v\in V$. We form the scenario set $\mathscr{S}_D^{'}=\{S_1^{'},\ldots,S_K^{'}\}$ as follows. If $e\in E$, then the cost of e under $S_k^{'}$ is the same as under S_k. The cost of additional edge e_v^j, $v\in V,\ j\in[K]$, under $S_k^{'}$ equals 0 if $j=k$ and M otherwise, where $M=|E|\max_{e\in E}\max_{S\in\mathscr{S}_D}c_e^S$. Finally, we add the *distinguished edge*, denoted by f, that connects v' with any node of V. The edge f has zero costs under all scenarios $S\in\mathscr{S}_D^{'}$. Now it is easy to check that every solution $X^{'}$, to the RENT-RR MINIMUM SPANNING TREE in graph G' and with scenario set $\mathscr{S}_D^{'}$, whose cost is $C_{Rent}(X^{'})<\alpha M$ (at least one such solution always exists) is of the form $X^{'}=X\cup\{f\}$, where X is a spanning tree in G (X is a solution to ROB MINIMUM SPANNING TREE). Furthermore $C_I^S(X^{'})=0$ for all $S\in\mathscr{S}_D^{'}$. So, $C_{Rent}(X^{'})=\alpha\max_{S\in\mathscr{S}'_D}C^S(X\cup\{f\})=\alpha\max_{S\in\mathscr{S}_D}C^S(X)=\alpha C_{Rob}(X)$. Therefore, it is evident that the reduction becomes approximation preserving one. □

We now examine the case when $\mathscr{P}$ is MINIMUM S-T CUT. We are given a graph $G=(V,E)$ with distinguished two nodes s and t and Φ consists of all s-t-cuts in G, that is the subset of the edges whose removal disconnects s and t.

Proposition 2. *There is a polynomial time approximation preserving reduction from* ROB MINIMUM S-T CUT *to* RENT-RR MINIMUM S-T CUT.

Proof. Let $(G=(V,E),\mathscr{S}_D=\{S_1,\ldots,S_K\},s,t)$ be an instance of ROB MINIMUM S-T CUT. We form graph $G'=(V',E')$ by adding to V additional nodes $v^1,\ldots,v^K$ and edges $e^1=\{t,v^1\},e^2=\{v^1,v^2\},\ldots,e^K=\{v^{K-1},v^K\}$. Furthermore $s'=s$ and $t'=v^K$. We form the scenario set $\mathscr{S}_D^{'}=\{S_1^{'},\ldots,S_K^{'}\}$ in the following way. If $e\in E$, then the cost of e under $S_k^{'}$ is the same as under S_k. The cost of additional edge e^j, $j\in[K]$, under $S_k^{'}$ equals 0 if $j=k$ and M otherwise, where $M=|E|\max_{e\in E}\max_{S\in\mathscr{S}_D}c_e^S$. The rest of the proof runs similarly as the one of Proposition 1. □

Assume now that $\mathscr{P}$ is MINIMUM SELECTING ITEMS, where E is a set of n items and $\Phi=\{X\subseteq E:\ |X|=p\}$, where p is a given integer between 1 and n.

Proposition 3. *There is a polynomial time approximation preserving reduction from* ROB MINIMUM SELECTING ITEMS *to* RENT-RR MINIMUM SELECTING ITEMS.

Proof. Given an instance $(E, \mathscr{S}_D = \{S_1, \ldots, S_K\}, p)$ of ROB MINIMUM SELECTING ITEMS, we form E' by adding to E additional items $e_1^j, \ldots, e_p^j$ for each $j \in [K]$. We form the scenario set $\mathscr{S}_D' = \{S_1', \ldots, S_K'\}$ in the following way. If $e \in E$, then the cost of e under S_k' is the same as under S_k. The cost of additional item e_i^j, $i \in [p]$, $j \in [K]$, under S_k' equals 0 if $j = k$ and M otherwise, where $M = |E| \max_{e \in E} \max_{S \in \mathscr{S}_D} c_e^S$. The reasoning is then similar to that in the proof of Proposition 1. □

Assume now that $\mathscr{P}$ is MINIMUM ASSIGNMENT, so we are given a bipartite graph $G = (V, E)$ and Φ consists of all perfect matchings in G.

Proposition 4. *There is a polynomial time approximation preserving reduction from* RENT-RR SHORTEST PATH *with a discrete scenario set to* RENT-RR MINIMUM ASSIGNMENT *with a discrete scenario set.*

Proof. In [1] it has been proposed an approximation preserving reduction from ROB SHORTEST PATH with $\mathscr{S}_D$ to ROB MINIMUM ASSIGNMENT with $\mathscr{S}_D$. A reduction from RENT-RR SHORTEST PATH to RENT-RR MINIMUM ASSIGNMENT is almost the same. □

From some complexity results for the robust versions of the problems under consideration with a discrete scenario set [2, 3, 6–9] and Propositions 1-4, we obtain the following two theorems:

Theorem 1. *For the bounded case,* RENT-RR MINIMUM SPANNING TREE, RENT-RR MINIMUM ASSIGNMENT *and* RENT-RR MINIMUM SELECTING ITEMS *are weakly NP-hard,* RENT-RR MINIMUM S-T CUT *is strongly NP-hard even for two scenarios.*

Theorem 2. *For the unbounded case,* RENT-RR MINIMUM S-T CUT *and* RENT-RR MINIMUM ASSIGNMENT *are not approximable within* $\log^{1-\varepsilon} K$ *for any* $\varepsilon > 0$*, unless NP* $\subseteq$ *DTIME*$(n^{\mathrm{poly}\log n})$, RENT-RR MINIMUM SPANNING TREE *is not approximable within* $O(\log^{1-\varepsilon} n)$ *for any* $\varepsilon > 0$*, where n is the input size, unless NP* $\subseteq$ *DTIME*$(n^{\mathrm{poly}\log n})$ *and* RENT-RR MINIMUM SELECTING ITEMS *is not approximable within constant factor* $\gamma > 1$*, unless P=NP.*

We now show some positive results, which are generalizations of the results given in [4], for the shortest path problem. We consider first the ROB $\mathscr{P}$ and RENT RR $\mathscr{P}$ problems with the same discrete scenario set $\mathscr{S}_D$.

Theorem 3. *Suppose that there exists an approximation algorithm for* ROB $\mathscr{P}$ *with a performance ratio of* γ*. Let* $X_{Rob} \in \Phi$ *be a solution constructed by this algorithm. Then* $C_{Rent}(X_{Rob}) \leq \min\{\gamma + 1 + \beta, \gamma/\alpha\} \cdot OPT_{Rent}$.

Proof. The following bounds can be concluded directly from (2) and (3):

$$OPT_{Rent} \geq \alpha \min_{X \in \Phi} \max_{S \in \mathscr{S}_D} C^S(X) = \alpha OPT_{Rob}, \quad OPT_{Rent} \geq \max_{S \in \mathscr{S}_D} \min_{Y \in \Phi} C^S(Y), \tag{5}$$

$$\begin{aligned} C_{Rent}(X) &= \max_{S \in \mathscr{S}_D} \{\alpha C^S(X) + \min_{Y \in \Phi}\{(1-\alpha)C^S(Y) + (\alpha+\beta)C^S(Y \setminus X)\}\} \\ &\leq \max_{S \in \mathscr{S}_D} \{\alpha C^S(X) + (1-\alpha)C^S(X)\} = C_{Rob}(X) \text{ for all } X \in \Phi. \end{aligned} \tag{6}$$

$C_{Rent}(X_{Rob}) = \max_{S\in\mathscr{S}_D}\{\alpha C^S(X_{Rob}) + \min_{Y\in\Phi}\{(1-\alpha)C^S(Y) + (\alpha+\beta)C^S(Y\setminus X_{Rob})\}\} \leq$

$\gamma\alpha\, OPT_{Rob} + \max_{S\in\mathscr{S}_D}\min_{Y\in\Phi}\{(1-\alpha)C^S(Y) + (\alpha+\beta)C^S(Y)\} \overset{(5)}{\leq} (\gamma+1+\beta)\, OPT_{Rent}$.

$C_{Rent}(X_{Rob}) \overset{(6)}{\leq} C_{Rob}(X_{Rob}) \leq \gamma OPT_{Rob} \overset{(5)}{\leq} (\gamma/\alpha)\, OPT_{Rent}$. □

We now consider the interval uncertainty representation.

Theorem 4. *An optimal solution to* RENT-RR $\mathscr{P}$ *with scenario set* $\mathscr{S}_I$ *can be obtained by computing an optimal solution of its deterministic counterpart* $\mathscr{P}$ *with the costs* $\overline{c}_e$, $e \in E$.

Proof. Let $\overline{X} \in \Phi$ be an optimal solution for $\mathscr{P}$ with the costs $\overline{c}_e$, $e \in E$. Then for every $X \in \Phi$ it holds: $C_{Rent}(X) = \max_{S\in\mathscr{S}_I}\{\alpha\sum_{e\in X} c_e^S + \min_{Y\in\Phi}\{(1-\alpha)\sum_{e\in Y} c_e^S + (\alpha+\beta)\sum_{e\in Y\setminus X} c_e^S\}\} \geq \alpha\sum_{e\in X}\overline{c}_e + (1-\alpha)\min_{Y\in\Phi}\sum_{e\in Y}\overline{c}_e \geq \min_{Y\in\Phi}\sum_{e\in Y}\overline{c}_e = \sum_{e\in\overline{X}}\overline{c}_e$. A trivial verification shows that $C_{Rent}(\overline{X}) = \sum_{e\in\overline{X}}\overline{c}_e$. □

3 *k*-Dist-RR Spanning tree problem

In this section, we prove hardness and inapproximability results for *k*-Dist-RR MINIMUM SPANNING TREE with scenario set $\mathscr{S}_D$.

Theorem 5. *The k-Dist-RR* MINIMUM SPANNING TREE *problem with scenario set* $\mathscr{S}_D$ *is weakly NP-hard in series-parallel graphs, even for two scenarios and any constant k.*

Proof. Consider an instance of 2-PARTITION [5] in which we are given a set $A = \{a_1,\ldots,a_n\}$ and an integer size $s(a)$ for each $a \in A$ such that $\sum_{a\in A} s(a) = 2b$. We ask if there is a subset $A' \subset A$ such that $\sum_{a\in A'} s(a) = \sum_{a\in A\setminus A'} s(a)$. We construct an instance of *k*-Dist-RR MINIMUM SPANNING TREE as follows: graph $G = (V,E)$ is a series composition of $n+k$, 4-edge subgraphs, $G_1,\ldots,G_n,G'_1,\ldots,G'_k$, where G_i corresponds to element $a_i \in A$ and k is a constant. The costs of each edge $e \in E$ are given by a triple $(c_e^1, c_e^{S_1}, c_e^{S_2})$, where c_e^1 is the first stage cost and $c_e^{S_1}$ and $c_e^{S_2}$ are the costs under scenarios S_1 and S_2, respectively. The reduction is depicted in Fig. 1, $M > 2b$.

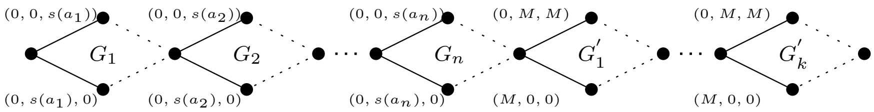

Fig. 1 A reduction from 2-PARTITION to *k*-Dist-RR MINIMUM SPANNING TREE. All the dummy edges (the dashed edges) have costs $(0,0,0)$.

It is not difficult to show that a 2-partition exists if and only if there exists an optimal spanning tree X in G such that $C_{Dist}(X) = b$ (see Fig. 1). □

Theorem 6. *For the unbounded case, the k-Dist-RR* MINIMUM SPANNING TREE *with scenario set $\mathscr{S}_D$ is strongly NP-hard and not at all approximable unless P=NP.*

Proof. We show a gap-introducing reduction from a decision version of MINIMUM DEGREE SPANNING TREE [5]. We are given a graph $G=(V,E)$ and $d \in \mathbb{N}$, $d<|V|$. We ask if there is a spanning tree in G such that its maximum node degree is not greater than d. For each $e=\{i,j\} \in E$, we add to E a *recovery edge*, $e^r=\{i,j\}^r$, that connects nodes i and j. The resulting graph $G^{'}=(V,E^{'})$ is a multigraph such that $|E^{'}|=2|E|$. All the edges in E have zero first stage costs and all the recovery edges have the first stage costs equal to $|V|$. The scenario set $\mathscr{S}_D=\{S_1,\dots,S_{|V|}\}$. The cost of edge $\{u,v\} \in E$ under scenario S_j equals 1 if $u=j$ or $v=j$ and 0 otherwise; the cost of recovery edge $\{u,v\}^r$ under scenario S_j equals 0 if $u=j$ or $v=j$ and $|V|$ otherwise. Finally, we set $k=d$. Suppose that the maximum node degree of some spanning tree X of G is at most d. Clearly, X is also a spanning tree of $G^{'}$ and does not use any recovery edge. Under each scenario $S_j \in \mathscr{S}_D$, we can decrease the cost of X to zero be replacing at most $k=d$ edges incident to node j with their recovery counterparts. Thus, $C_{Dist}(X)=0$. On the other hand, if $C_{Dist}(X)=0$, then the spanning tree X of G' cannot use any recovery edge (because its first stage cost is positive) and at most d edges incident to each node j, which can be replaced by at most $k=d$ recovery edges. Thus X is a spanning tree of G with the maximum node degree at most d. □

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