Robust discrete optimization under discrete and interval uncertainty - a survey

Adam Kasperski and Paweł Zieliński

Abstract In this chapter a review of recent results on robust discrete optimization is presented. The most popular discrete and interval uncertainty representations are discussed. Various robust concepts are presented, namely the traditional minmax (regret) approach with some of its recent extensions, and several two-stage concepts. A special attention is paid to the computational properties of the robust problems considered.

1 Introduction

In this chapter we will be concerned with a class of discrete optimization problems defined as follows. We are given a finite set of elements $E = \{e_1, \ldots, e_n\}$ and a set of feasible solutions $\Phi \subseteq 2^E$. Each element $e_i \in E$ has a nonnegative cost c_i and we seek a feasible solution $X \in \Phi$ which minimizes the total cost $f(X) = \sum_{e_i \in X} c_i$. This traditional deterministic discrete optimization problem will be denoted by $\mathscr{P}$. The above formulation encompasses, for instance, an important class of network problems. Namely, E can be identified with the set of arcs of a network $G = (V, E)$ and Φ contains some objects in G such as $s-t$ paths, spanning trees, $s-t$ cuts, perfect matchings, or Hamiltonian cycles. We thus get the well known and basic problems such as SHORTEST PATH, MINIMUM SPANNING TREE, MINIMUM S-T CUT, MINIMUM ASSIGNMENT, or TRAVELING SALESPERSON, respectively. A comprehensive description of the class of deterministic network problems can be found, for example, in books [1, 79].

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In most cases, $\mathscr{P}$ can be alternatively formulated as a 0-1 programming problem. Indeed, a binary variable $x_i \in \{0,1\}$ is associated with element $e_i \in E$ and so $\mathscr{P}$ has the following formulation:

$$\begin{array}{ll} \min \sum_{i=1}^n c_i x_i \\ \text{s.t.} \;\; (x_1,\dots,x_n) \in ch(\Phi), \end{array}$$

where $ch(\Phi)$ is the set of characteristic vectors of Φ, described in a compact form by a system of constraints involving $x_1,\dots,x_n$. For example, when we have one constraint of the form $\sum_{i=1}^n w_i x_i \geq p$, we obtain the KNAPSACK problem. If, additionally, $w_i = 1$ for each $i \in [n]$ and p is an integer in $[n]$ ($[n]$ denotes the set $\{1,\dots,n\}$), then we get the SELECTION problem. An optimal solution to this problem can be computed in $O(n)$ time by choosing p elements out of E of the smallest costs. In this chapter, we will also discuss the following REPRESENTATIVES SELECTION problem (it is also called WEIGHTED HITTING DISJOINT SET, see, e.g., [17]). Let us partition the set $[n]$ into u disjoint subsets $T_1,\dots,T_u$. Then $ch(\Phi)$ is described by a system of constraints of the form $\sum_{i\in T_j} x_i = 1$ for each $j \in [u]$. Hence, each feasible solution is composed of exactly one element e_j from each T_j. An important characteristic of this problem is the value of $r_{\max} = \max_{j\in[u]} |T_j|$. An optimal solution to this problem is composed of elements of the smallest costs from each T_j. Both SELECTION and REPRESENTATIVES SELECTION problems become nontrivial under uncertainty. We will discuss them later in detail as they allow us to obtain strong negative complexity results for many robust versions of discrete optimization problems.

In many practical applications the element costs are often uncertain, which means that their precise values are not known before computing a solution. In this case a *scenario set* $\mathscr{U}$, containing all possible realizations of the element costs, is a part of input. Each particular cost realization $(c_1^S,\dots,c_n^S) \in \mathscr{U}$ is called a *scenario*. Then $f(X,S) = \sum_{e_i\in X} c_i^S$ is the cost of solution X under scenario S. In this chapter we will focus on two popular methods of defining set $\mathscr{U}$ - *discrete and interval uncertainty representations*. For the *discrete uncertainty representation* [62], scenario set, denoted by $\mathscr{U}_D$, contains K explicitly listed scenarios. This uncertainty representation is appropriate when each scenario corresponds to an event which globally influences the element costs. For example, an uncertain weather forecast can globally change a system environment, and these uncertain weather conditions can be modeled by discrete scenarios. For the *interval uncertainty representation* [15], scenario set, denoted by $\mathscr{U}_I^\ell$ is defined as follows. We assume that the cost of element e_i can take any value within the interval $[c_i, c_i + d_i]$, where c_i is a nominal cost and d_i is the maximum deviation of the value of the cost from its nominal value. Then $\mathscr{U}_I^\ell$ is a subset of the Cartesian product of these intervals, under the additional assumption that in each scenario in $\mathscr{U}_I^\ell$, the costs of at most ℓ elements can be greater than their nominal values. The value of $\ell \in [0,n]$ is fixed and allows us to control the degree of uncertainty. When $\ell = 0$, then we get a deterministic problem with one scenario. On the other hand, when $\ell = n$, then we get the traditional interval uncertainty representation [62], in which scenario set is equal to the Cartesian product of all the uncertainty intervals. We will denote this particular special case of scenario set by $\mathscr{U}_I = \mathscr{U}_I^n$. The scenario set $\mathscr{U}_I$ models a local uncertainty, i.e. we assume that the cost of each element may vary independently on the costs of the remaining elements. For instance, a traveling time of some link is often uncertain and can be modeled by a closed interval which provides us a bound on the minimum and the maximum possible value of the traveling time. It is often not possible to measure some costs precisely and the measurement error can also be expressed as a closed interval.

In mathematical programming problems some other types of scenario sets, in particular the *ellipsoidal uncertainty* or the *column-wise uncertainty*, are also used. In general $\mathscr{U}$ can be any set, typically assumed to be convex [14]. In this chapter we will not be concerned with such more general scenario sets. Some discussion on them can be found in the recent survey [39]. In robust optimization, also the set of feasible solutions can be uncertain and may depend on a scenario (see, e.g. [66]). In the class of problems discussed in this chapter the set of feasible solutions Φ is deterministic, i.e. it remains the same for each scenario in $\mathscr{U}$. Under this assumption, the discrete and interval uncertainty representations are the easiest and, in many cases, possess sufficient expressive power.

If no additional information for $\mathscr{U}$ (such as a probability distribution) is provided, then we face a decision problem under *uncertainty*. In order to choose a solution we can use some well known criteria used in decision theory under uncertainty (see e.g. [64]). Among them there are the minmax and minmax regret criteria, which assume that the decision maker is risk averse and seeks a solution minimizing the cost or opportunity loss in a worst case, i.e under a worst scenario which may occur. By using the minmax (regret) criterion we obtain the *robust minmax (regret) optimization problem*. This traditional robust approach to discrete optimization has some well known drawbacks, which we will discuss in more detail in Sect. 2. By applying the minmax (regret) criterion we may sometimes get unreasonable solutions (we will show some examples in Sect. 2). Furthermore, it is not true that decision makers are always extremely risk averse. Hence, there is a need to soften the very conservative minmax (regret) criterion. Also, in many practical applications decision makers have some additional information provided with $\mathscr{U}$. For example, a probability distribution in $\mathscr{U}$ or its estimation may be available. This information should be taken into account while computing a solution. In Sect. 3 we will present some recent extensions of the robust approach which take into account both an attitude of decision makers towards a risk and an information about the probability distribution in $\mathscr{U}$.

The minmax approach can be generalized by considering the *robust optimization problem with incremental recourse* [73]. This problem can be seen as a zero-sum game against the nature with the following rules. The decision maker chooses first a solution X whose cost $f(X)$ is precisely known. Then nature picks a scenario S from $\mathscr{U}$ and the decision maker, chooses the next solution Y after observing S. The solution Y has the cost $f(Y,S)$ and must be of some predefined distance from X. The decision maker wants to minimize the total cost $f(X)+f(Y,S)$ while the nature aims to maximize this total cost, i.e. it always picks the worst scenario for solution X. It is easily seen that the robust optimization problem with incremental recourse contains the minmax problem as a special case. Indeed, by assuming that the initial cost of X is always 0 and Y must be the same as X (no modification of X is allowed) we arrive to the minmax problem. The robust optimization with incremental recourse is similar to *robust recoverable* optimization [17, 18, 63], because a limited recovery action is allowed after observing which scenario has occurred. We will study the robust optimization problems with incremental recourse in Sect. 4.

The traditional min-max (regret) approach is a one-stage decision problem, i.e. a complete solution must be computed before a true scenario reveals. However, many practical problems have a two-stage nature. Namely, a partial solution is formed in the first stage, when the costs are precisely known and then it is completed optimally when a true cost scenario from $\mathscr{U}$ occurs. We seek a solution whose maximum total cost in both stages is minimum. We will discuss the class of robust two-stage problems in Sect. 5.

The aim of this chapter is to present and compare various concepts used in robust discrete optimization under the discrete and interval uncertainty representations. A survey of the results in the area of robust minmax (regret) optimization up to 2009 can be found in [62, 41, 5]. In this chapter we present new results and concepts which have recently appeared in the literature. We will pay a special attention to the computational properties of the problems under study (a recent survey from the algorithmic perspective can be found in [39]). In Sect. 2 we present the traditional minmax (regret) approach. We also show, in Sect. 3, some of its extensions which allow decision makers to model their attitude towards risk and exploit scenario probabilities. In Sect. 4 we examine the robust optimization problems with incremental recourse. Finally, in Sect. 5 we describe the class of robust two-stage problems.

The class of problems considered in this chapter is rather broad. However, it does not cover an important class of sequencing problems in which a feasible solution is represented by a permutation of the elements (typically called jobs). A recent survey of the results for the minmax (regret) sequencing problems can be found in [55]. Another class of problems, which is not discussed in detail, contains the ones with the bottleneck cost function. The minmax (regret) versions of such problems were investigated in [8], where it was shown that their complexity is nearly the same as the complexity of their deterministic counterparts. An extension of the minmax bottleneck problems has been discussed in [54]. We also do not mention about the maximum relative regret criterion. Some properties of this criterion, in particular its connections with the maximum regret, can be found in [10, 62].

2 Robust min-max (regret) problems

In this section we discus the traditional robust approach to deal with discrete optimization problems with uncertain costs. We describe the minmax and minmax regret criteria, which are typically used in the robust optimization framework. We present the known complexity results for basic problems and show some drawbacks of the minmax (regret) approach.

2.1 Using the minmax criterion

This section is devoted to the study of the following *minmax problem*:

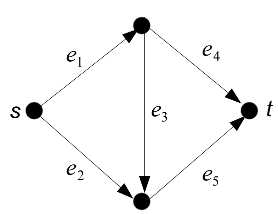
$$\text{MIN-MAX } \mathscr{P}: \min_{X \in \Phi} \max_{S \in \mathscr{U}} f(X,S).$$

We thus seek a solution minimizing the maximum cost over all scenarios. Minmax is the most popular criterion used in robust optimization [62, 14]. The minmax problem can be alternatively stated as follows:

$$\begin{array}{ll} \min t & \\ \text{s.t. } \sum_{i\in[n]} c_i^S x_i \le t & \forall S \in \mathscr{U} \\ \quad (x_1,\dots,x_n) \in ch(\Phi) & \\ \quad t \ge 0 & \end{array} \tag{1}$$

The minmax criterion can be extremely conservative and it will be used by pessimistic decision makers, or in situations in which it is very important to avoid bad scenarios. Perhaps, the most serious drawback of the minmax approach is that it may lead to solutions which are not Pareto optimal. Consider two sample MIN-MAX SHORTEST PATH problems, shown in Fig. 1. All the three paths, depicted in Fig. 1a, have the same maximum cost equal to 16. Hence, we can choose the path $\{e_2, e_5\}$ which is weakly dominated by the remaining two paths. When the number of scenarios becomes large, then the so-called *drowning effect* may occur [33], i.e. only one bad scenario is taken into account while choosing a solution and the information associated with the remaining scenarios is ignored. A similar situation occurs for the interval uncertainty representation and it is shown in the sample problem in Fig. 1b. Path $\{e_1\}$ is almost always better than path $\{e_2\}$, but both can be chosen after applying the minmax criterion. Also, an optimal minmax solution which is Pareto optimal, can be a questionable choice. Consider again the sample problem presented in Fig. 1a, and change the cost of arc e_2 under S_3 to 7. The path $\{e_2, e_5\}$ is then an optimal minmax solution which is also Pareto optimal. However, this path is only slightly better than $\{e_1, e_4\}$ under S_3 and much worse under S_1 and S_2.

a)



	S_1	S_2	S_3
$\{e_1, e_4\}$	3	6	16
$\{e_1, e_3, e_5\}$	6	2	16
$\{e_2, e_5\}$	11	12	16

b)

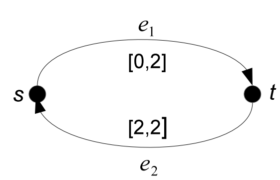


Fig. 1 a) A sample MIN-MAX SHORTEST PATH problem with three scenarios $S_1 = (2,10,3,1,1)$, $S_2 = (1,11,0,5,1)$, $S_3 = (8,8,0,8,8)$. b) A sample MIN-MAX SHORTEST PATH problem with interval costs.

The examples given in Fig. 1 show that there is a need of modification of the minmax criterion. If the decision maker is interested in minimizing the total cost, then a chosen solution should always be Pareto optimal. Furthermore, an attitude of decision makers towards a risk should be taken into account, because not all decision makers are extremely risk averse. In Sect. 3.1 we will suggest a criterion which allows us to overcome both these drawbacks. In the next section, we will discuss all the known complexity results for MIN-MAX $\mathscr{P}$.

2.1.1 Discrete uncertainty representation

Consider the discrete uncertainty representation, i.e. when $\mathscr{U} = \mathscr{U}_D$. The known complexity results for some basic minmax problems are shown in Table 1.

Table 1 Complexity results for various MIN-MAX $\mathscr{P}$ problems with scenario set $\mathscr{U}_D$.

MIN-MAX $\mathscr{P}$	constant K	unbounded K
SHORTEST PATH	NP-hard for $K = 2$ [89], FPTAS [3]	strongly NP-hard [89], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [50], appr. within K [5]
MINIMUM SPANNING TREE	NP-hard for $K = 2$ [88, 62], FPTAS [3]	strongly NP-hard [88, 62], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [53], appr. within $O(\log^2 n)$ with high probability [53]
MINIMUM S-T CUT	strongly NP-hard for $K = 2$ [4]	strongly NP-hard [4], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [50], appr. within K [5]
MINIMUM ASSIGNMENT	strongly NP-hard for $K = 2$ [84, 90]	strongly NP-hard [2, 84, 90], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [50], appr. within K [5]
SELECTION	NP-hard for $K = 2$ [9], FPTAS [3]	strongly NP-hard [51], not appr. within any const. $\gamma > 0$ [42], appr. within $O(\log K / \log\log K)$ [31]
REPR. SELECTION	NP-hard for $K = 2$ [32], FPTAS [32]	strongly NP-hard [32], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [44], not appr. within $2 - \varepsilon$ when $r_{\max} = 2$ for any $\varepsilon > 0$ [29], appr. within $\min\{K, r_{\max}\}$ [44]
KNAPSACK	NP-hard for $K = 1$ [38], FPTAS [3]	strongly NP-hard [87, 51], not appr. within any const. $\gamma > 0$ [42]

Observe that all these problems become NP-hard or strongly NP-hard, even when the number of scenarios equals 2. However, if the number of scenarios is constant then some of them can be solved in pseudopolynomial time (typically a dynamic programming method is applied) and admit a fully polynomial time approximation scheme (FPTAS). We should point out, here, that the running times of the pseudopolynomial algorithms and the FPTAS's proposed in the literature are exponential in K and, in consequence, the practical applicability of them is rather limited. The complexity of the problems become worse when the number of scenarios is a part of input. In particular, the network problems are then hard to approximate within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [50, 53]. A similar result holds for the MIN-MAX REPRESENTATIVES SELECTION problem [44]. The MIN-MAX SELECTION and MIN-MAX KNAPSACK problems are then hard to approximate within any constant factor $\gamma > 0$ [42].

If the underlying deterministic problem $\mathscr{P}$ is polynomially solvable, then MIN-MAX $\mathscr{P}$ is approximable within K. It is enough to solve the deterministic problem $\mathscr{P}$ for the aggregated costs $\hat{c}_i = \max_{S\in\mathscr{U}_D} c_i^S$ (or $\hat{c}_i = \sum_{S\in\mathscr{U}_D} c_i^S$), $i \in [n]$. A straightforward proof of this fact can be found, for instance, in [5]. This approximation ratio has been improved for two particular problems. For MIN-MAX MINIMUM SPANNING TREE a randomized $O(\log^2 n)$-approximation algorithm was constructed in [53] and for the MIN-MAX SELECTION problem a deterministic $O(\log K/\log\log K)$-approximation algorithm was proposed in [31]. These algorithms are based on the idea of randomized rounding of linear programming programs, which seems to be a promising tool for establishing stronger approximation results for the minmax problems, when the number of scenarios is a part of input.

The MIN-MAX $\mathscr{P}$ problem can be solved exactly by applying the formulation (1). After replacing $(x_1,\ldots,x_n) \in ch(\Phi)$ with a system of linear constraints, we obtain a compact MIP formulation for the problem. Other exact methods for this problem, such as branch and bound algorithms, can be found in [62].

In some cases, the underlying deterministic problem $\mathscr{P}$ is a maximization problem, i.e. we seek a solution which maximizes the total cost. It is then natural to study the symmetric MAX-MIN $\mathscr{P}$ problem, in which we wish to find a solution maximizing the minimum cost over all scenarios, i.e. $\max_{X\in\Phi}\min_{S\in\mathscr{U}} f(X,S)$. Interestingly, for scenario set $\mathscr{U}_D$, MAX-MIN $\mathscr{P}$ seems to be harder than the corresponding MIN-MAX $\mathscr{P}$ problem. In [57] it has been shown that MAX-MIN INDEPENDENT SET problem in interval graphs (this problem was first discussed in [76]), whose deterministic version is polynomially solvable, is not at all approximable when K is a part of input. A similar fact was observed for MAX-MIN KNAPSACK in [3] (see also [75]).

2.1.2 Interval uncertainty representation

Let us address the interval uncertainty representation, i.e. when $\mathscr{U} = \mathscr{U}_I^\ell$. We first discuss the case $\mathscr{U}_I = \mathscr{U}_I^n$. It is easy to check that the complexity of MIN-MAX $\mathscr{P}$ is then almost the same as $\mathscr{P}$, because it is sufficient to solve the deterministic problem $\mathscr{P}$ for scenario $(c_1+d_1,\ldots,c_n+d_n)$. Consequently, when $\mathscr{P}$ is solvable in $O(T(n))$ time, then MIN-MAX $\mathscr{P}$ is solvable in $O(n+T(n))$ time. The problem is more challenging when $\mathscr{U} = \mathscr{U}_I^\ell$ for a fixed $\ell \in [0,n]$. An algorithm for this case was proposed in [15]. We now briefly describe it. Let us number the elements so that $d_1 \geq d_2 \geq \cdots \geq d_n$ and define $d_{n+1} = 0$. Define scenario S^j under which the cost of e_i is equal to $c_i + (d_i - d_j)$ if $i \leq j$ and c_i otherwise, where $j \in [n+1]$. In [15] it has been shown that MIN-MAX $\mathscr{P}$ with scenario set $\mathscr{U}_I^\ell$ is equivalent to the following problem:

$$\min_{j\in[n+1]}(\ell d_j + \min_{X\in\Phi} f(X,S^j)). \tag{2}$$

Observe that (2) reduces to solving $n+1$ deterministic problems $\mathscr{P}$ for the costs specified in scenarios $S^1,\ldots,S^{n+1}$ and, in consequence, when $\mathscr{P}$ is solvable in $O(T(n))$ time, then MIN-MAX $\mathscr{P}$ is solvable in $O(nT(n))$ time. We thus get a tractable class of problems under uncertainty. Furthermore, it has been observed in [15] that this algorithm can be extended to problems $\mathscr{P}$ which are NP-hard but admit an α-approximation algorithm. In this case, that approximation algorithm can be used to solve the inner problem $\min_{X\in\Phi} f(X,S^j)$ and the minmax problem is also approximable within α.

2.2 Using the minmax regret criterion

In this section we treat the following *minmax regret problem*:

$$\text{MIN-MAX REGRET } \mathscr{P}: \min_{X\in\Phi}\max_{S\in\mathscr{U}}(f(X,S) - f^*(S)),$$

where $f^*(S)$ is the cost of an optimal solution under scenario S. The quantity $f(X,S) - f^*(S)$ is called a *regret* of X under S and it expresses a deviation of solution X from the optimum under S. We thus seek a solution which minimizes the maximum regret over all scenarios. The maximum regret criterion is also called *Savage criterion* or *maximum opportunity loss*. The minmax regret problem can be alternatively stated as follows:

$$\begin{array}{ll} \min & t \\ \text{s.t.} & \sum_{i\in[n]} c_i^S x_i \leq t + t^S \quad \forall S \in \mathscr{U} \\ & (x_1,\ldots,x_n) \in ch(\Phi) \\ & t \geq 0, \end{array} \tag{3}$$

where t^S is the cost of an optimal solution under scenario S. If we apply the minmax regret criterion to the sample problem presented in Fig. 1, then we get the reasonable paths $\{e_1,e_3,e_5\}$ in Fig. 1a and $\{e_1\}$ in Fig. 1b. Observe that the maximum regret of $\{e_1\}$ equals 0 which means that this path is optimal under each scenario.

It is important to realize that the maximum regret is quite different quantity than the maximum cost. In the former, the decision maker aims to minimize the opportunity loss, i.e. the cost of a solution is compared ex-post to the cost of the best solution which could be chosen. Consider the sample MIN-MAX REGRET SHORTEST PATH problem depicted in Fig. 2a. Both paths $\{e_1\}$ and $\{e_2\}$ have the same maximum regret equal to 1. However, the maximum cost of path $\{e_1\}$ is twice the maximum cost of $\{e_2\}$. Hence, a solution with small maximum regret may have a large maximum cost in comparison with other solutions. Decision makers who just want to minimize the solution cost should be careful while using the minmax regret criterion.

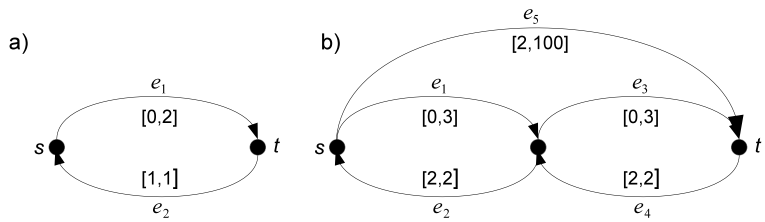


Fig. 2 Two sample MIN-MAX REGRET SHORTEST PATH problems with scenarios sets $\mathscr{U}_I$.

Another drawback of the minmax regret criterion is shown in Fig. 2b. It is easy to check that path $\{e_1, e_4\}$ has the smallest maximum regret equal to 3 and there is no path with smaller maximum regret. Suppose that we remove the path (a single arc) $\{e_5\}$ from the network. Then path $\{e_1, e_3\}$ has the smallest maximum regret equal to 2 and no other path has smaller maximum regret. Observe that the path $\{e_5\}$ is never optimal since its regret is very large. This example shows that the maximum regret criterion does not satisfy the property of *independency of irrelevant alternatives* [64], i.e. adding a non-optimal (and thus irrelevant) solution to the problem can make the optimal solution nonoptimal and vice versa.

It is evident that MIN-MAX REGRET $\mathscr{P}$ is NP-hard and not at all approximable when $\mathscr{P}$ is NP-hard. This is true even in the deterministic case when $K = 1$. It follows from the fact that it is then NP-hard to compute a solution of the maximum regret equal to 0. Also, computing the maximum regret of a given solution is, in this case, NP-hard. This implies, in particular, that MIN-MAX REGRET KNAPSACK is not at all approximable under both discrete and interval uncertainty representations.

In [65] the following randomized version of MIN-MAX REGRET $\mathscr{P}$ has been proposed. That is, instead of choosing a single solution, a probability distribution over all solutions is computed and we seek a probability distribution which minimizes the maximum expected regret. This problem can be seen as a game, in which the decision maker chooses a probability distribution and an adversary chooses then the worst scenario, knowing this probability distribution. Notice that in the traditional minmax regret problem, the decision maker is restricted to choose the solution deterministically. Interestingly, the best probability distribution can be computed in polynomial time if $\mathscr{P}$ is polynomially solvable. This holds for both discrete and interval uncertainty representations (see [65] for details).

2.2.1 Discrete uncertainty representation

Let us now discuss the discrete uncertainty representation, i.e. the case when $\mathscr{U} = \mathscr{U}_D$. The known complexity results for the minmax regret versions of some basic problems $\mathscr{P}$ are shown in Table 2.

Table 2 Complexity results for various MIN-MAX REGRET $\mathscr{P}$ problems with scenario set $\mathscr{U}_D$.

MIN-MAX REGRET $\mathscr{P}$	constant K	unbounded K
SHORTEST PATH	NP-hard for $K=2$ [89], FPTAS [3]	strongly NP-hard [89], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [50], appr. within K [5]
MINIMUM SPANNING TREE	NP-hard for $K=2$ [88, 62], FPTAS [3]	strongly NP-hard [88, 62], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [53], appr. within K [5]
MINIMUM S-T CUT	strongly NP-hard for $K=2$ [4]	strongly NP-hard [4], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [50], appr. within K [5]
MINIMUM ASSIGNMENT	strongly NP-hard for $K=2$ [84, 90]	strongly NP-hard [2, 84, 90], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [50], appr. within K [5]
SELECTION	NP-hard for $K=2$ [9], FPTAS [3]	strongly NP-hard [51], not appr. within any const. $\gamma > 0$ [42], appr. within K [5]
REPR. SELECTION	NP-hard for $K=2$ [32], FPTAS [32]	strongly NP-hard [32], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [44], not appr. within $2-\varepsilon$ when $r_{\max}=2$ for any $\varepsilon > 0$ [29], appr. within K [5]
KNAPSACK	NP-hard for $K=1$ [38], not at all appr.	strongly NP-hard [87, 51], not at all appr.

Similarly to MIN-MAX $\mathscr{P}$, MIN-MAX REGRET $\mathscr{P}$ becomes NP-hard or strongly NP-hard when the number of scenarios equals 2. These negative results can be strengthen when the number of scenarios is a part of input and they are the same as for MIN-MAX $\mathscr{P}$ (see Table 1). In fact, the proof showing the hardness of MIN-MAX $\mathscr{P}$ can be, in most cases, easily modified to show the same hardness result for the minmax regret version of $\mathscr{P}$. Typically, it suffices to add a number of dummy elements and scenarios to the constructed instance.

Observe that there is lack of stronger positive results when the number of scenarios is a part of input. The only known and general result states that when $\mathscr{P}$ is polynomially solvable, then MIN-MAX REGRET $\mathscr{P}$ is approximable within K. The idea is to solve the deterministic problem for the average costs $\hat{c}_i = \frac{1}{K}\sum_{S\in\mathscr{U}_D} c_i^S$, $i \in [n]$. A straightforward proof of this fact can be found in [5].

We can use (3) to construct a compact MIP formulation for the minmax regret problem. However, the underlying problem $\mathscr{P}$ must by polynomially solvable, since we need the costs of the optimal solutions t^S for each $S \in \mathscr{U}_D$. Other exact methods for solving the problem can be found in [62].

2.2.2 Interval uncertainty representation

We now turn to case when $\mathscr{U} = \mathscr{U}_I$, i.e. the interval uncertainty representation. In the existing literature the problem with scenario set $\mathscr{U}_I$ has been extensively studied. To the best of our knowledge, more general scenario set $\mathscr{U}_I^\ell$ has been not yet investigated. The known complexity results and solution methods for various problems, under scenario set $\mathscr{U}_I$, are shown in Table 3.

Table 3 Complexity results and solutions methods for various MIN-MAX REGRET $\mathscr{P}$ problems with scenario set $\mathscr{U}_I$.

MIN-MAX REGRET $\mathscr{P}$	Complexity	Solution methods
SHORTEST PATH	strongly NP-hard [11], appr. within 2 [46], NP-hard for planar graphs [91], NP-hard for sp-graphs [47], FPTAS for sp-graphs [49]	MIP [40], B&B [72, 23], Benders [71], Enumeration [69], Other methods [34]
MINIMUM SPANNING TREE	strongly NP-hard [11, 6], appr. within 2 [46]	MIP [86], B&B [70, 7], B&Cut [82], Benders [67], Tabu Search [45], Simulated Annealing [74], Other methods [34]
MINIMUM S-T CUT	strongly NP-hard [4], appr. within 2 [46], NP-hard for sp-graphs [47, 49], FPTAS for sp-graphs [49]	MIP [41]
MINIMUM ASSIGNMENT	strongly NP-hard [2], appr. within 2 [46]	MIP [41], Benders [80], Local Search [80], GA [80]
SELECTION	solv. in $O(n \cdot \min\{n, n-p\})$ time [24]	
REPR. SELECTION	solv. in $O(n^2)$ time [32]	
KNAPSACK	Σ_2^p-hard [28], not at all appr.	MIP [37], B&Cut [37], Local Search [37]

The number of scenarios in $\mathscr{U}_I$ is infinite. It is, however, easy to show that we can replace $\mathscr{U}_I$ with the set of *extreme scenarios*, which is the Cartesian product $\Pi_{i\in[n]}\{c_i, c_i+d_i\}$. It is also not difficult to show (see, e.g. [41, 52]) that the maximum regret of X equals $f(X, S_X) - f^*(S_X)$, where S_X is the extreme scenario under which the costs of $e_i \in X$ equal $c_i + d_i$ and the costs of $e_i \notin X$ are equal to c_i. Consequently, the maximum regret of a given solution X can be computed in polynomial time if the underlying deterministic problem $\mathscr{P}$ is polynomially solvable. Remarkable, this is not the case for the minmax regret version of the linear programming problem with interval objective function coefficients, since it has been shown in [12] that computing the maximum regret of a given solution is strongly NP-hard.

It turns out (see [11]) that in order to compute an optimal minmax regret solution it is enough to compute an optimal solution for each extreme scenario and choose the best one. Consequently, if the number of *nondegenerate* cost intervals, i.e. such that $d_i > 0$, is bounded by $r \cdot \log n$, then it is sufficient to enumerate at most n^r solutions. This yields a polynomial method for constant r. Obviously, this method is exponential in general case.

Let us now discuss some general properties of MIN-MAX REGRET $\mathscr{P}$. A solution X is called *possibly optimal* if it is optimal under at least one scenario in $\mathscr{U}$ and X is called *necessarily optimal* if it is optimal under all scenarios in $\mathscr{U}$. Similarly, an element e_i is *possibly optimal* if it is a part of an optimal solution under at least one scenario and it is *necessarily optimal* if it is a part of an optimal solution under all scenarios. It turns out that under scenario set $\mathscr{U}_I$, each optimal minmax regret solution is possibly optimal and is entirely composed of possibly optimal elements. This fact was first observed for the minmax regret versions of SHORTEST PATH and MINIMUM SPANNING TREE in [40, 86] and it was generalized to all problems $\mathscr{P}$ in [52]. Notice, that this is not the case for scenario set $\mathscr{U}_D$, where it is easy to construct a sample problem whose optimal minmax regret solution is not optimal under any scenario. On the other hand, the maximum regret of a necessarily optimal solution equals 0, so it must be the optimal minmax regret solution (this is true for any scenario set $\mathscr{U}$). It was shown in [52] that when all cost intervals are nondegenerate, i.e. $d_i > 0$ for all $i \in [n]$, then there is an optimal minmax regret solution containing all necessarily optimal elements.

The notions of possibly and necessarily optimal elements can be very useful, as they allow us to reduce the size of a problem instance before a solution is computed, for example, by using a MIP formulation or a branch and bound algorithm. Namely, all non-possibly optimal elements can be removed from E and, under the absence of degeneracy, all necessarily elements can be automatically added to the solution constructed. Some computational tests (see. e.g. [40, 45]) suggest that for many instances more than 50% elements are non-possibly optimal. One can also expect several elements to be necessarily optimal in each instance. Hence, a partial solution can be formed before a more complex algorithm is executed. Unfortunately, detecting possibly and necessarily optimal elements is not an easy task in general. In particular, the problem of checking whether a given element is possibly optimal, is strongly NP-hard for the SHORTEST PATH, MINIMUM ASSIGNMENT and MINIMUM S-T CUT problems [21, 52]. All possibly and necessarily optimal elements can be detected in polynomial time if $\mathscr{P}$ is a matroidal problem [48], in particular, when $\mathscr{P}$ is SELECTION or MINIMUM SPANNING TREE. For the SHORTEST PATH problem a subset of possibly optimal elements (arcs) can be detected by efficient algorithms proposed in [40, 20]. Also, when the network is acyclic all necessarily optimal arcs can be detected in polynomial time [36]

As we can see in Table 3, the minmax regret versions of all the basic network problems are strongly NP-hard for general graphs. Two special cases, namely MIN-

MAX REGRET SHORTEST PATH and MIN-MAX REGRET MINIMUM S-T CUT in series-parallel multidigraphs can be solved in pseudopolynomial time and admit an FPTAS [47, 49]. Fortunately, the following positive and general result is known for all polynomially solvable problems $\mathscr{P}$. Let S^M be the *midpoint scenario*, under which the cost of e_i is equal to $c_i + 0.5d_i$, $i \in [n]$. In [46] it has been shown that if X^* is an optimal solution under scenario S^M, then the maximum regret of X^* is at most twice the maximum regret of an optimal minmax regret solution. Consequently, if $\mathscr{P}$ is polynomially solvable, then MIN-MAX REGRET $\mathscr{P}$ is approximable within 2. The 2-approximation algorithm has been extended to a wider class of minmax regret problems in [25]. We do not know whether there exists an approximation algorithm with a performance ratio better than 2 (except for some very special cases). Also, no negative approximation result for MIN-MAX REGRET $\mathscr{P}$ is known, when $\mathscr{P}$ is polynomially solvable. So, the existence of an PTAS in this case cannot be excluded. Observe that this result allows us to detect efficiently a solution with the maximum regret equal to 0 (i.e. a necessarily optimal solution).

The computational tests (see, e.g. [45, 41]) suggest that the approximation algorithm behaves well in practice. It is often profitable to modify it by considering two solutions: an optimal solution under the midpoint scenario and an optimal solution under the pessimistic scenario $(c_1 + d_1, \ldots, c_n + d_n)$. The approximation algorithm, denoted as AMU, returns the better of these two solutions. Algorithm AMU seems to perform well except for some rather artificial instances [45]. However, it is also only a 2-approximation algorithm and a sample worst case instance for it (a MIN-MAX REGRET SHORTEST PATH instance) is depicted in Fig. 3. Note that algorithm AMU may return any of the three possible paths. But the maximum regret of the best path equals 1, while the maximum regret of the worst path equals 2. A similar example for MIN-MAX REGRET MINIMUM SPANNING TREE can be found in [45].

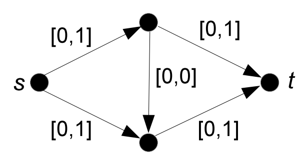


Fig. 3 A worst case instance for algorithm AMU.

In the following we give a brief exposition of the known approaches to deal with the NP-hard minmax regret problems (see also Table 3). For the class of network problems there exists a compact mixed integer programming (MIP) formulation [5, 41], which can be solved by means of some available software such as CPLEX. Another popular approach is to apply the Benders decomposition technique or a specialized branch and bound (cut) method. The detailed description of the computational tests for various instances can be found in the references given in Table 3.

The exact methods seem to be particularly efficient for the minmax regret version of the SHORTEST PATH problem, as they allow us to solve large problems in reasonable time (see e.g. [71]). The exact methods perform much worse for the minmax regret version of MINIMUM SPANNING TREE, which is a very interesting problem still requiring more deep investigation. The largest instances which can be solved to optimality are composed of networks having up to 40 nodes [45, 82]. For this problem a local search method seems to be more efficient. There is a very natural definition of a neighborhood of a given spanning tree. Namely, we get a neighbor X' of a spanning tree X by performing the operation $X' = X \cup \{e\} \setminus \{f\}$, where $e \in E \setminus X$ and $f \in X$. We can then apply a simple iterative improvement or more sophisticated tabu search algorithm to compute a solution. The computational tests performed in [45] suggest that the obtained solutions are close to the optimum even for large instances. Interestingly, a local minimum with respect to the specified neighborhood can also be a factor of 2 away from the global minimum, even when one starts from a solution computed by AMU (see [45]).

For the minmax regret version of MINIMUM SPANNING TREE another interesting result has been established in [34]. It turns out that the problem complexity depends on the number of intersecting intervals. Indeed, an optimal minmax regret spanning tree can be found in $O(2^k n \log n)$ time, where k is the maximum number of intervals that intersect at least one other interval. So, from this point of view, the hardest instances are the ones in which all the cost intervals are the same, for instance equal to [0,1] (a MIP approach is very poor in this case [45]). This special case is equivalent to the strongly NP-hard CENTRAL SPANNING TREE problem [16, 6]. Observe that for this problem algorithm AMU may return any solution and designing an approximation algorithm with a performance ratio better than 2 is an interesting and important open problem.

In the existing literature some other problems have been also investigated and, in the following, we briefly describe them. In [68] the minmax regret version of the TRAVELING SALESPERSON problem with interval costs and in [81] the minmax regret version of the SET COVERING problem with interval costs have been studied. Both problems are quite challenging as their deterministic versions are strongly NP-hard. The solution methods proposed in [68, 81] (a branch and cut algorithm, Benders decomposition and some heuristics) are general and can easily be extended to other minmax regret problems with interval data, whose deterministic counterparts are NP-hard. In [26] the minmax regret version of the MINIMUM SPANNING ARBORESCENCE problem with interval costs has been examined. An *arborescence* is a subgraph of a given graph G in which there is exactly one path from a given root node r to any other node of G. For undirected graphs the problem is equivalent to MINIMUM SPANNING TREE, so its minmax regret version is strongly NP-hard. However, for acyclic directed graphs this problem can be solved in polynomial time [26].

3 Extensions of the minmax approach

In this section we introduce several extensions of the traditional minmax approach presented in the previous section. These extensions allow us to overcome some drawbacks of this approach. Namely, we will be able to model an attitude of decision makers towards risk and take additional information associated with scenario set into account.

3.1 Using the OWA criterion

In decision making under uncertainty some other criteria for choosing a solution, such as minmin, Hurwicz, or Laplace (the average), are also used. For an excellent discussion on their various properties we refer the reader to [64]. It turns out that most of them are special cases of the criterion called *Weighted Ordering Averaging* aggregation (OWA for short) proposed by Yager in [85]. We will now show how to apply the OWA criterion to problem $\mathscr{P}$ under scenario set $\mathscr{U}_D$.

Let $\boldsymbol{v} = (v_1, \ldots, v_K)$ be a vector of weights, where $v_j \in [0,1]$ for each $j \in [K]$ and $v_1 + \cdots + v_K = 1$. Given a feasible solution $X \in \Phi$, let σ be a permutation of $[K]$ such that $f(X, S_{\sigma(1)}) \geq f(X, S_{\sigma(2)}) \geq \cdots \geq f(X, S_{\sigma(K)})$. The OWA aggregation criterion is defined as follows:

$$\mathrm{OWA}(X) = \sum_{j \in [K]} v_j f(X, S_{\sigma(j)}).$$

Observe that $\mathrm{OWA}(X)$ is a convex combination of the costs $f(X, S_1), \ldots, f(X, S_K)$. Hence its value is always between the minimum and the maximum cost of X over scenarios in $\mathscr{U}_D$. In this section we assume that a vector of weights $\boldsymbol{v}$ is specified for scenario set $\mathscr{U}_D$ and we consider the following problem:

$$\text{MIN-OWA } \mathscr{P} : \min_{X \in \Phi} \mathrm{OWA}(X).$$

By fixing the weights $\boldsymbol{v}$ we get some special cases of MIN-OWA $\mathscr{P}$ which are listed in Table 4. Observe that OWA generalizes all the basic criteria used in decision making under uncertainty except for the minmax regret (Savage) criterion.

Table 4 Special cases of MIN-OWA $\mathscr{P}$.

Problem	weights $\boldsymbol{v}$
MIN-MAX $\mathscr{P}$	$v_1 = 1$, $v_j = 0$ for $j \neq 1$
MIN-MIN $\mathscr{P}$	$v_K = 1$, $v_j = 0$ for $j \neq K$
MIN-AVERAGE $\mathscr{P}$	$v_j = \frac{1}{K}$ for all $j \in [K]$
MIN-QUANT(k) $\mathscr{P}$	$v_k = 1$, $v_j = 0$ for $j \neq k$
MIN-MEDIAN $\mathscr{P}$	$v_{\lfloor K/2 \rfloor + 1} = 1$, $v_j = 0$ for $j \neq \lfloor K/2 \rfloor + 1$
MIN-HURWICZ $\mathscr{P}$	$v_1 = \alpha$, $v_K = 1 - \alpha$, $v_j = 0$ for $j \neq 1, K$, $\alpha \in [0,1]$

Since MIN-MAX $\mathscr{P}$ is a special case of MIN-OWA $\mathscr{P}$, all the negative results presented in Table 1 remain valid for MIN-OWA $\mathscr{P}$. However, the computational properties of MIN-OWA $\mathscr{P}$ strongly depend on the weight distribution in $\boldsymbol{v}$. For example, it is easily seen that MIN-MIN $\mathscr{P}$ and MIN-AVERAGE $\mathscr{P}$ are polynomially solvable if $\mathscr{P}$ is polynomially solvable. On the other hand, MIN-HURWICZ $\mathscr{P}$ is at least as hard as MIN-MAX $\mathscr{P}$, because it generalizes the latter problem. Table 5 summarizes all the known results for the MIN-OWA SHORTEST PATH problem. Most of these results remain valid for any problem $\mathscr{P}$ (see [56] for details).

Table 5 Summary of results for the MIN-OWA SHORTEST PATH problem.

Problem	$K=2$	$K \geq 3$ constant	K unbounded
MIN-OWA	equivalent to MIN-HURWICZ $\mathscr{P}$	NP-hard, FPTAS	strongly NP-hard, appr. within $v_1 K$ if the weights are nonincreasing, not at all appr. if the weights are nondecreasing
MIN-MAX	NP-hard, FPTAS	NP-hard, FPTAS	strongly NP-hard, appr. within K, not appr. within $O(\log^{1-\varepsilon} K)$, $\varepsilon > 0$
MIN-MIN	poly. solvable	poly. solvable	poly. solvable
MIN-AVER.	poly. solvable	poly. solvable	poly. solvable
MIN-HURWICZ	poly. solv. if $\alpha \in [0, \frac{1}{2})$ NP-hard if $\alpha \in (\frac{1}{2}, 1]$ FPTAS if $\alpha \in (\frac{1}{2}, 1]$	NP-hard if $\alpha \in (0,1]$ FPTAS	strongly NP-hard if $\alpha \in (0,1]$, appr. within $\alpha K + (1-\alpha)(K-2)$ if $\alpha \in [\frac{1}{2}, 1]$ $\frac{K}{\alpha}$ if $\alpha \in (0, \frac{1}{2})$, not appr. within $O(\log^{1-\varepsilon} K)$, $\varepsilon > 0$
MIN-QUANT(k)	poly. solvable if $k=2$ NP-hard if $k=1$ FPTAS	poly. solvable for $k=K$, NP-hard for constant $k \in [K-1]$, FPTAS	strongly NP-hard for any $k \in [K-1]$, approx. within K for constant k, not at all appr. if $k = \lfloor \frac{K}{2} \rfloor + 1$

The complexity of MIN-OWA $\mathscr{P}$ depends on the weight distribution in $\boldsymbol{v}$. In particular, if the weights model the minimum or the average, then the problem is polynomially solvable when $\mathscr{P}$ is polynomially solvable. On the other hand, when $\boldsymbol{v}$ models the median, MIN-OWA SHORTEST PATH is not at all approximable. Fortunately there is a positive approximation result for the problem when the weights are nonincreasing, i.e. $v_1 \geq v_2 \geq \cdots \geq v_K$. This important special case will be described in more detail in the next section.

3.1.1 OWA criterion and the robust approach

The maximum and the average (Laplace) criteria are special cases of OWA. They form two boundary cases of nonincreasing weights, i.e. when $v_1 \geq v_2 \geq \cdots \geq v_K$. We get the maximum when $v_1 = 1$, $v_j = 0$ for $j \neq 1$, and the average when $v_j = 1/K$

for all $j \in [K]$. It turns out that for nonincreasing weights a general positive approximation result holds. Namely, if $\mathscr{P}$ is polynomially solvable, then MIN-OWA $\mathscr{P}$ is approximable within $v_1 K$ [56]. The idea of the approximation algorithm is to solve problem $\mathscr{P}$ for the aggregated costs $\hat{c}_i = \mathrm{owa}_{\boldsymbol{v}}(c_i^{S_1}, \ldots, c_i^{S_K})$, $i \in [n]$. So, it generalizes the K-approximation algorithm, well known for the MIN-MAX $\mathscr{P}$ problem. Note that $v_1 \in [1/K, 1]$, so we get the worst approximation ratio when OWA is the maximum. On the other hand, when $v_1 = 1/K$, i.e. when OWA is the average, we obtain a polynomial algorithm for the problem. The assumption of nonincreasing weights allows us also to construct more efficient MIP formulations for MIN-OWA $\mathscr{P}$ (see [77, 22, 35]), which makes the problem more tractable.

The nonincreasing weights are compatible with the robust approach, because larger weights are assigned to larger solution costs. Furthermore, the weights allow risk-averse decision makers to model their attitude towards a risk. The more uniform is the weight distribution the less risk averse the decision maker is. In particular, extremely risk averse the decision maker will choose $v_1 = 1$, which leads to the maximum criterion and the minmax problem discussed in Sect. 2. Using the OWA criterion allows us to overcome another drawback of the minmax approach. When all the weights in $\boldsymbol{v}$ are positive, then the obtained solution must be Pareto optimal. We can thus reject such solutions as the one presented in Fig. 1a, by choosing positive (even very small) weights.

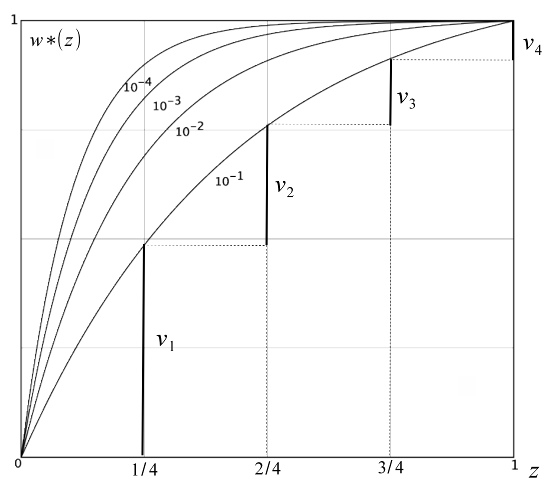


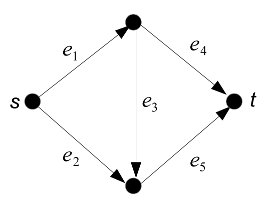
Fig. 4 A sample interpolation function $w^*(z) = \frac{1}{(1-\alpha)}(1-\alpha^z)$ for $\alpha \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$. The weights $v_1, \ldots, v_4$ for $\alpha = 10^{-1}$.

The weights $v_1, \ldots, v_K$ can be specified explicitly. However, it may be convenient to obtain them by using an *interpolation function* $w^* : [0,1] \to [0,1]$, which is assumed to be concave, nondecreasing and satisfies $w^*(0) = 0$, $w^*(1) = 1$. Having w^* we get $v_j = w^*(j/K) - w^*((j-1)/K)$ for $j \in [K]$. A sample interpolation function $w^*(z) = \frac{1}{(1-\alpha)}(1-\alpha^z)$, $\alpha \in (0,1)$, is shown in Fig. 4. Observe that when α tends to 0, the OWA tends to the maximum. On the other hand, when α tends to 1, the

OWA tends to the average. Thus the decision maker can adjust his attitude towards a risk by fixing a single value of $\alpha \in (0,1)$. The weights obtained for $\alpha = 10^{-1}$ and $K = 4$ are also shown in Fig. 4. Observe that the more concave is w^* the less uniform is the weight distribution in $\boldsymbol{v}$. When w^* is a straight line, then $v_1 = \cdots = v_K$ and OWA is the average. The interpolation function $w^*(z)$ can also be defined for explicitly listed weights $v_1 \geq v_2 \geq \cdots \geq v_K$. It is enough to assume that $w^*(z)$ is the linear interpolation of the points $(0,0)$ and $(j/K, \sum_{i \leq j} v_i)$ for $j \in [K]$. Then $w^*(z)$ is a concave piecewise linear function. We use this fact in the next section, in which we describe a generalization of the OWA criterion.

3.2 Using the WOWA criterion

One drawback of the OWA criterion is that it does not allow us to exploit any additional information associated with scenarios. One such important information is a probability distribution over the set $\mathscr{U}_D$. Assume that such a probability distribution is available and let p_j be the probability that scenario $S_j \in \mathscr{U}_D$ will occur, $j \in [K]$. Let us examine a sample SHORTEST PATH problem with scenario set $\mathscr{U}_D$, shown in Fig. 5.



p_j	0.5	0.2	0.2	0.1	
	S_1	S_2	S_3	S_4	Exp.
$X_1 = \{e_1, e_4\}$	10	1	1	2	**5.6**
$X_2 = \{e_1, e_3, e_5\}$	5	5	7	8	**5.7**
$X_3 = \{e_2, e_5\}$	6	6	6	6	**6**

Fig. 5 A sample Shortest Path problem with four scenarios $S_1 = (5,6,0,5,0)$, $S_2 = (1,6,4,0,0)$, $S_3 = (1,6,6,0,0)$, and $S_4 = (2,6,6,0,0)$. The costs of all three paths under all scenarios are shown in the table.

A natural approach to solve this problem is to choose a solution with the minimum expected cost. Hence, the path $X_1 = \{e_1, e_4\}$ is then the best choice. However, X_1 may be unreasonable for some risk averse decision makers. Observe that the probability that the path X_1 will have a large cost equal to 10 is equal to 0.5. This choice may be questionable if path X_1 is to be used only once, i.e. when a decision is not repetitious in the same environment. On the other hand, path $X_3 = \{e_2, e_5\}$ has the smallest maximum cost and should be chosen when the minmax criterion is used and the probabilities of scenarios are ignored. Notice that the path X_3 has a deterministic cost equal to 6. However, some decision makers may feel that path $X_2 = \{e_1, e_3, e_5\}$ is better, since the probability that the cost of X_2 will be less than 6 equals 0.7 and the probability that X_2 will have a large cost, equal to 8, is only 0.1.

The sample problem demonstrates that there is a need of criterion which establishes a link between the stochastic and robust approach when scenario probabilities are available. Such a criterion can be proposed by a generalization of OWA. In order to introduce this criterion it is convenient to use the interpolation function $w^*(z)$ defined in the previous section. Consider a solution X and let σ be a permutation of $[K]$ such that $f(X,S_{\sigma(1)}) \geq \cdots \geq f(X,S_{\sigma(K)})$. The permutation σ defines also the order of scenario probabilities $p_{\sigma(1)} \geq \cdots \geq p_{\sigma(K)}$. In particular $p_{\sigma(1)}$ is the probability that the worst scenario will occur and $p_{\sigma(K)}$ is the probability that the best scenario will occur for X. Define now the weights $\omega_j = w^*(\sum_{i\leq j} p_{\sigma(i)}) - w^*(\sum_{i<j} p_{\sigma(i)})$, $j \in [K]$ and let

$$\text{WOWA}(X) = \sum_{j\in[K]} \omega_j f(X,S_{\sigma(j)}).$$

We have thus obtained the Weighted OWA criterion (WOWA for short), first proposed in [83]. A trivial verification shows that $\omega_j \in [0,1]$ for all $j \in [K]$ and $\omega_1 + \cdots + \omega_K = 1$. The value of ω_j can be seen as a *distorted probability* of scenario $S_{\sigma(j)}$. The value of ω_j depends on p_j and the rank position of scenario $S_{\sigma(j)}$ for solution X, so it is solution dependent. Hence, WOWA(X) can be seen as the expected cost of solution X with respect to the distorted probabilities. For a more detailed interpretation of this expectation we refer the reader to [30].

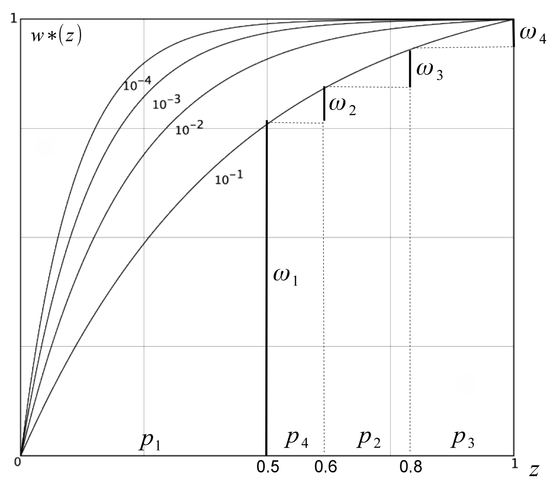


Fig. 6 Computing the weights $\omega_1,\dots,\omega_4$ for path $\{e_1,e_4\}$ in Fig. 4.

Let us look at the sample problem in Fig. 5 again. For path $X_1 = \{e_1,e_4\}$ we have $\sigma = (1,4,2,3)$. The computations of the weights $\omega_1,\dots,\omega_4$ is shown in Fig. 6. One can see in Fig. 6 how scenario probabilities are distorted. For example, $\omega_1 > p_1$ and $\omega_4 < p_3$, so we assign larger probability to the worst scenario S_1 and smaller to good scenario S_3. If the probability distribution in $\mathscr{U}_D$ is uniform, then WOWA becomes OWA, because $\omega_j = v_j$ for each $j \in [K]$. The uniform probability distribution results from applying the *principle of insufficient reason*, i.e. in a situation under uncertainty, when it is not possible to distinguish more or less probable scenarios [64]. If $w^*(z)$ is a straight line, or equivalently, $v_1 = v_2 = \cdots = v_K$, then WOWA becomes the expected value, because $\omega_j = p_{\sigma(j)}$ is then just the scenario probability. We thus can see that WOWA is a very general criterion. It contains both OWA and the expected value as special cases. It allows us to establish a link between the robust and stochastic approaches.

Since MIN-MAX $\mathscr{P}$ is a special case of MIN-WOWA $\mathscr{P}$, all the negative results shown in Table 1 remain true for the latter problem. Fortunately, when $w^*(z)$ is the linear interpolation function for the nonincreasing weights $v_1 \geq v_2 \geq \cdots \geq v_K$ (see Sect. 3.1), then the problem is approximable within $v_1 K$ if $\mathscr{P}$ is polynomially solvable [59]. An idea is to solve $\mathscr{P}$ for the aggregated cots $\hat{c}_i = \mathrm{wowa}_{\boldsymbol{v},\boldsymbol{p}}(c_i^{S_1}, \ldots, c_i^{S_K})$, $i \in [n]$. Note that this approximation ratio is the same as for MIN-OWA $\mathscr{P}$. For the linear interpolation function $w^*(z)$ a compact MIP formulation for the problem can also be constructed [59, 78]. Some computational tests for the MIP formulation and the approximation algorithm were performed in [59].

4 Robust optimization with incremental recourse

In this section we address the adjustable approach to combinatorial optimization problems with uncertain element costs, introduced in [73], called the *robust optimization with incremental recourse*. It extends the concept of robustness to deal with uncertainties by incorporating adjustable actions, after an element cost scenario is realized. Namely, the decision maker chooses first the best initial solution, taking into account that a worst scenario can happen (the first stage). Then he makes some incremental changes in the initial solution chosen, subject to a given distance measure, in order to obtain another one (the incremental recourse stage).

Formally, the robust optimization problem $\mathscr{P}$ with incremental recourse can be stated as follows:

$$\text{ROIR } \mathscr{P}: \min_{X \in \Phi} (f(X) + \max_{S \in \mathscr{U}} \min_{Y \in \Phi_X^k} f(Y,S)),$$

where $f(X) = \sum_{e_i \in X} C_i$ is the cost of an initial solution X and $\Phi_X^k = \{Y \in \Phi : d(X,Y) \leq k\}$ is the *incremental set*, i.e. the set of possible solutions in the incremental recourse stage, where $d(X,Y)$ is a fixed measure of the distance between the initial solution X and the incremental solution Y. The distance $d(X,Y)$ is also called an *incremental function* bounded by a specified parameter k. Finally, $f(Y,S) = \sum_{e_i \in Y} c_i^S$ is the cost of solution Y under scenario S. The most popular distance measures $d(X,Y)$, proposed in literature [17, 18, 73, 27], are: the *element inclusion distance* $d(X,Y) = |Y \setminus X|$, the *element exclusion distance* $d(X,Y) = |X \setminus Y|$, and the *element symmetric difference distance* $d(X,Y) = |X \oplus Y|$. It is worthwhile to mention that for MINIMUM SPANNING TREE the above distance measures are equivalent from the computational point of view.

The concept of the robustness with incremental recourse is similar in spirit to the one of the *recoverable robustness*, proposed in [63] for linear programing under uncertainty. In the recoverable approach limited recovery actions are permitted after uncertain parameters reveal. Later, in [17, 18, 19] the recoverable robustness, called k-distance recoverable robustness, has been applied to some classical combinatorial optimization problems. Yet another interesting concept of recoverable robustness, proposed in [17, 18], is the *rent recoverable robustness* in which, in the second recoverable stage, the number of elements that can be replaced is not limited, but deviating from previous choice comes at extra cost.

The ROIR $\mathscr{P}$ problem contains the following three inner problems. The first one is the *incremental problem*:

$$\text{INC}(X,S^*)\ \mathscr{P}:\ \min_{Y\in\Phi_X^k} f(Y,S^*),$$

where we are given an initial solution X and a cost scenario S^* revealed. We wish to make incremental changes in X, subject to the constraint $d(X,Y)\le k$, which lead to the maximum improvement in the objective function. This problem is a special case of ROIR $\mathscr{P}$. Indeed, it is sufficient to set $\mathscr{U}=\{S^*\}$, the initial costs $C_i=0$ if $e_i\in X$ and M otherwise, where M is a sufficiently large number, for example $M\ge nC, C=max_{e_i\in E}\{c_i^{S^*}\}$. Several incremental versions of network problems have been investigated in [27]. The second inner problem is the *adversarial* one:

$$\text{ADV}(X)\ \mathscr{P}:\ \max_{S\in\mathscr{U}}\min_{Y\in\Phi_X^k} f(Y,S).$$

In this problem we seek a scenario $S\in\mathscr{U}$ that maximizes $\text{INC}(X,S)$ with respect to a given solution X. The problem ROIR $\mathscr{P}$ reduces to $\text{ADV}(X)$ $\mathscr{P}$ when we set $C_i=0$ if $e_i\in X$ and $C_i=M$, otherwise. The last inner problem is MIN-MAX $\mathscr{P}$. We get this problem after fixing $C_i=0$ for each $e_i\in E$ and $k=0$, which implies $\Phi_X^k=\{X\}$. We thus can see that ROIR $\mathscr{P}$ generalizes MIN-MAX $\mathscr{P}$. In consequence, all the negative results for MIN-MAX $\mathscr{P}$ remain valid for the robust incremental version of $\mathscr{P}$.

In the next two sections we will present the known results on ROIR $\mathscr{P}$. We will show that the complexity of this problem highly depends on both the uncertainty representation and the distance measure. As we will see, there are a lot of things to do in this area. In particular, there is lack of approximation algorithms for the considered problem. The most of presented results are negative ones.

4.1 Discrete uncertainty representation

We now give a brief summary of the complexity results on ROIR $\mathscr{P}$ under the discrete scenario uncertainty, i.e. when $\mathscr{U}=\mathscr{U}_D$. In this case, the only complexity results that exist in the literature are for the element inclusion distance $d(X,Y)=|Y\setminus X|$. Unfortunately, all of them are negative ones, except for the robust incremental problems with one scenario (see Table 6).

Table 6 Complexity results for ROIR $\mathscr{P}$ with scenario set $\mathscr{U}_D$ and the element inclusion distance.

ROIR $\mathscr{P}$	constant K	unbounded K
SHORTEST PATH	strongly NP-hard, not at all appr. for $K = 1$ and $k \geq 2$ [18]	
MINIMUM SPANNING TREE	NP-hard in sp-graphs for $K = 2$ and constant k [43]	strongly NP-hard, not at all appr. for unbounded k [43]
SELECTION	NP-hard for $K = 2$ and $k \geq 1$, solv. in $O((p-k+1)n^2)$ for $K = 1$ [58]	strongly NP-hard, not at all appr. for any const. $k \geq 1$ [58]
MINIMUM MATROID BASE	poly. solvable for constant k and for $K = 1$ [17]	

Since MIN-MAX $\mathscr{P}$ is a special case of ROIR $\mathscr{P}$, all negative results presented in Table 1 for MIN-MAX $\mathscr{P}$ are still true for ROIR $\mathscr{P}$. Hence, Table 6 can be completed by complexity results for other combinatorial problem. One can observe (see Table 6) that ROIR $\mathscr{P}$ can be much harder than its minmax counterpart. For instance, the ROIR SHORTEST PATH problem is strongly NP-hard and not at all approximable even for one scenario. Notice that MIN-MAX SHORTEST PATH in this case is a deterministic problem and so it is polynomially solvable. Let us also mention a more general result on KNAPSACK that has been examined in [19], i. e. the problem under the discrete uncertainty in the objective and the constraint with a distance measure that takes into account the element inclusion and exclusion distances. The problem turned out to be inapproximable for unbounded K, but pseudopolynomially solvable for constant K.

We now look into the adversarial problem with scenario set $\mathscr{U}_D$. It easily seen that the complexity of this problem highly relies on the complexity of the incremental problem. Indeed, solving ADV(X) $\mathscr{P}$, for a given initial solution X, boils down to solving INC(X,S) $\mathscr{P}$ for every $S \in \mathscr{U}_D$ and choosing a scenario which results in the maximum cost. It turns out (see [27]) that the incremental versions of SHORTEST PATH and MINIMUM SPANNING TREE, with the element inclusion distance function, are polynomially solvable, and the incremental version of MINIMUM ASSIGNMENT can be solved in random polynomial time. Unfortunately the incremental versions of MINIMUM S-T CUT and SHORTEST PATH, with the element symmetric difference distance function, $d(X,Y) = |X \oplus Y|$, and the incremental version of SHORTEST PATH, with the element exclusion distance, $d(X,Y) = |X \setminus Y|$, are NP-hard [27, 73]. In consequence, their adversarial and robust incremental versions are also NP-hard.

4.2 Interval uncertainty representation

In this section we are concerned with ROIR $\mathscr{P}$ under the interval uncertainty representation and with three distance measures. We start by showing a few complexity results for the case $\mathscr{U} = \mathscr{U}_I$ and the element inclusion distance $d(X,Y) = |Y \setminus X|$

(see Table 7). To the authors' knowledge, nothing more has been recorded in the literature on the robust incremental optimization with recourse under the scenario set $\mathscr{U}_I$.

Table 7 Complexity results for ROIR $\mathscr{P}$ problems with scenario set $\mathscr{U}_I$ and the element inclusion distance.

ROIR $\mathscr{P}$	Complexity
SHORTEST PATH	strongly NP-hard, not at all appr., poly. solvable in sp-graphs [18]
SELECTION	solvable in $O((p-k+1)n^2)$ time [58]
MINIMUM MATROID BASE	poly. solvable for constant k [17]

In the robust incremental optimization with recourse there is a link between the interval uncertainty representation $\mathscr{U}_I$ and the discrete one $\mathscr{U}_D$, namely, the ROIR $\mathscr{P}$ problem with scenario set $\mathscr{U}_I$ can be rewritten as follows:

$$\min_{X\in\Phi}(\sum_{e_i\in X} C_i + \max_{S\in\mathscr{U}_I}\min_{Y\in\Phi_X^k}\sum_{e_i\in Y} c_i^S) = \min_{X\in\Phi}(\sum_{e_i\in X} C_i + \min_{Y\in\Phi_X^k}\sum_{e_i\in Y}(c_i+d_i))$$
$$= \min_{X\in\Phi}(\sum_{e_i\in X} C_i + \mathrm{INC}\left(X,(c_i+d_i)_{i\in[n]}\right)).$$

From the above it follows that ROIR $\mathscr{P}$ with scenario set $\mathscr{U}_I$ is equivalent to ROIR $\mathscr{P}$ with only one scenario $S=(c_i+d_i)_{i\in[n]}$. This property has been exploited in construction of the algorithms for the SHORTEST PATH, SELECTION and MINIMUM MATROID BASE problems with the element inclusion distance [58, 17, 18]. Moreover, using it we can conclude that ROIR KNAPSACK is at least NP-hard. One can also deduce from the NP-hardness of the incremental versions of SHORTEST PATH and MINIMUM S-T CUT with the element symmetric difference and exclusion distances the NP-hardness of their robust incremental counterparts [27, 73] (see Sect. 4.1).

Let us now discuss the interval uncertainty representation which allows us to control the amount of uncertainty. Following [73], we define two scenario sets. Namely, given $\ell > 0$

$$\mathscr{U}_{I1}^{\ell} = \{S=(c_i^S)_{i\in[n]} : c_i^S = c_i+\delta_i, 0\le\delta_i\le d_i, \sum_{i\in[n]}\delta_i\le\ell\},$$
$$\mathscr{U}_{I2}^{\ell} = \{S=(c_i^S)_{i\in[n]} : c_i^S = c_i+\delta_i d_i, \delta_i\in\{0,1\}, \sum_{i\in[n]}\delta_i\le\ell\}.$$

It easy to see that $\mathscr{U}_{I1}^{\ell}$ models the situation where the total amount of deviation in the element costs is bounded by a specified ℓ. The set $\mathscr{U}_{I2}^{\ell}$ is the set of extreme points (scenarios) of $\mathscr{U}_I^{\ell}$, here $\ell\in[0,n]$ (for $\ell=n$ it is the set of extreme points of $\mathscr{U}_I$).

We first review of the complexity results on the robust incremental optimization problems with recourse under the scenario set $\mathscr{U}_{I1}^{\ell}$. In this case the adversarial version of SHORTEST PATH with the element inclusion distance can be formulated as a linear program and, in consequence, is solvable in polynomial time [73]. Unfortunately, the robust incremental version of SHORTEST PATH with the same distance is strongly NP-hard and not approximable within a factor of 2 even for $\ell = k = 1$. The rest of the hardness results for the adversarial and robust incremental versions of SHORTEST PATH with the element symmetric difference and exclusion distances presented in [73] follow from the NP-hardness of its incremental versions [27] (see Sect. 4.1). The adversarial version of MINIMUM SPANNING TREE with the element inclusion distance is polynomially solvable since it can be also modeled as a linear program [73]. However, the complexity status of the robust incremental version of MINIMUM SPANNING TREE remains open.

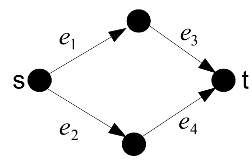
The complexity situation under the scenario set $\mathscr{U}_{I2}^{\ell}$ is much worse that under the set $\mathscr{U}_{I1}^{\ell}$. The adversarial version of SHORTEST PATH with all three distance measures considered is NP-hard and not approximable within a factor of 2 [73]. The robust incremental version of SHORTEST PATH with the element inclusion distance is strongly NP-hard and not approximable within a factor of 2 even for $\ell = k = 1$ [73]. This result improves the one shown in [18], where it has been proved that the problem is NP-hard for constant $k \geq 1$. For the remaining two distance measures, the robust incremental version of SHORTEST PATH is NP-hard and not approximable within a factor of 2, due to the hardness of the adversarial counterpart [73]. The adversarial version of MINIMUM SPANNING TREE with the element inclusion distance is NP-hard and hence its robust incremental version is NP-hard as well [73].

5 Robust two-stage problems

In many applications, the discrete optimization problem has a two-stage nature. Namely, a partial solution is formed in the first stage, when the element costs are precisely known. This partial solution is then completed optimally after a true scenario reveals. Let C_i be the deterministic, first stage cost of element $e_i \in E$ and let c_i^S be the second stage cost of element e_i under scenario $S \in \mathscr{U}$. In this section we study the following problem:

$$\text{TWO-STAGE } \mathscr{P} : \min_{X \subseteq E} \Big(\sum_{e_i \in X} C_i + \max_{S \in \mathscr{U}} \min_{\{Y \subseteq E :\, X \cup Y \in \Phi\}} \sum_{e_i \in Y} c_i^S \Big).$$

Note that a solution to this problem is determined by a subset X of the elements, chosen in the first stage. Given X and scenario S we compute Y such that $X \cup Y \in \Phi$. It may happen that X cannot be completed to any solution from Φ. In this case we assume that the cost of X is infinite. Given Φ, let us define set Φ' in the following way: $X' \in \Phi'$ if there is $X \in \Phi$ such that $X \subseteq X'$. Hence $\Phi \subseteq \Phi'$ and Φ' contains all solutions from Φ and all the supersets of these solutions. In the one-stage robust problems described in Sect. 2, an optimal solution is the same when we replace Φ with Φ'. However, for the examined two-stage model the problems with Φ and Φ' may be quite different. Consider the sample TWO-STAGE SHORTEST PATH problem with two scenarios, shown in Figure 7. If Φ contains two paths $\{e_1, e_3\}$ and $\{e_2, e_4\}$, then we can choose either e_1 or e_2 in the first stage. In both cases the maximum cost of the obtained path after the second stage equals $M+1$. However, if we replace Φ with Φ', then we can choose both e_1 and e_2 in the first stage and the maximum cost of the obtained solution after the second stage equals only 3. This example demonstrates that it may be profitable to add some redundant elements in the first stage. It is thus justified to explore the complexity of the problem with both Φ and Φ'.



	C_i	S_1	S_2
e_1	1	M	M
e_2	1	M	M
e_3	M	M	1
e_4	M	1	M

Fig. 7 A sample TWO-STAGE SHORTEST PATH problem with two scenarios.

5.1 Discrete uncertainty representation

The known complexity results for the TWO-STAGE $\mathscr{P}$ problem, when $\mathscr{U} = \mathscr{U}_D$ are shown in Table 8. All the basic problems are NP-hard even for two scenarios. One exception is the two-stage version of MINIMUM SPANNING TREE, for which no negative result for constant K has appeared in the literature yet. As usual, the problems become more complex when the number of scenarios is a part of input (see Table 8). The negative approximation results for the two-stage versions of SHORTEST PATH, MINIMUM S-T CUT and MINIMUM ASSIGNMENT have been established in [60] by showing a cost preserving reduction from the MIN-MAX REPRESENTATIVES SELECTION problem. For the two-stage versions of MINIMUM SPANNING TREE, SELECTION, and MINIMUM ASSIGNMENT some approximation algorithms have been recently proposed [53, 58, 61]. They are based on randomized rounding of LP programs, which is a promising technique to construct approximation algorithms for robust problems with discrete scenario sets.

5.2 Interval uncertainty representation

Let us first deal with scenario set $\mathscr{U}_I = \Pi_{i\in[n]}[c_i, c_i + d_i]$. In this case, the two-stage problem can be rewritten as follows:

$$\min_{\{X,Y\subseteq E:\, X\cup Y\in\Phi\}} \Big(\sum_{e_i\in X} C_i + \sum_{e_i\in Y} (c_i + d_i)\Big). \tag{4}$$

Table 8 Complexity results for various TWO-STAGE $\mathscr{P}$ problems with scenario set $\mathscr{U}_D$. The symbol * means that the negative result holds only for Φ (we do not know if it holds for Φ').

TWO-STAGE $\mathscr{P}$	constant K	unbounded K
SHORTEST PATH	NP-hard for $K = 2$ [60]	strongly NP-hard [60], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [60]*,
MINIMUM SPANNING TREE		strongly NP-hard [53], not appr. within $(1-\varepsilon)\log n$ for any $\varepsilon > 0$ [53], appr. within $O(\log^2 n)$ with high probability [53]
MINIMUM S-T CUT	NP-hard for $K = 2$ [60]	strongly NP-hard [60], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [60]*,
MINIMUM ASSIGNMENT	NP-hard for $K = 2$ [60]	strongly NP-hard [60], not appr. within $O(\log^{1-\varepsilon} K)$ for any $\varepsilon > 0$ [60]*, appr. within $1/\beta$, $\beta \in (0,1)$ to match at least $n(1-\beta)$ nodes [61]
SELECTION	NP-hard for $K = 2$ [13]	strongly NP-hard [58], not appr. within $(1-\varepsilon)\log n$ for any $\varepsilon > 0$ [58], appr. within $O(\log K + \log n)$ with high probability [58]

It follows easily that an optimal solution to (4) can be obtained by solving the deterministic problem $\mathscr{P}$ for the element costs $\hat{c}_i = \min\{C_i, c_i + d_i\}$, $i \in [n]$. If Z is an optimal solution to this problem, then for each $e_i \in Z$ we choose e_i in the first stage when $C_i \leq c_i + d_i$ and we choose e_i in the second stage otherwise.

Consider now the more general scenario set $\mathscr{U}_I^\ell$. If all the first stage costs C_i, $i \in [n]$, are large enough, then $X = \emptyset$ and the two-stage problem reduces to a special case of the adversarial problem, where $\Phi_X^k = \Phi$, (see Sect. 4.1), namely:

$$\max_{S \in \mathscr{U}_I^\ell} \min_{Y \in \Phi} \sum_{e_i \in Y} c_i^S. \tag{5}$$

It turns out (see [60]) that (5) is strongly NP-hard when $\mathscr{P}$ is MINIMUM SPANNING TREE or SHORTEST PATH. This fact immediately implies that the two-stage versions of both problems, under scenario set $\mathscr{U}_I^\ell$, are strongly NP-hard. It is worth pointing out that the corresponding MIN-MAX $\mathscr{P}$ problem, which can be obtained by interchanging the min and max operators in (5), is polynomially solvable (see Sect. 2.1.2).

There is a number of interesting open questions related to TWO-STAGE $\mathscr{P}$ with scenario set $\mathscr{U}_I^\ell$. We do not know if the problem is NP-hard when ℓ is constant (only the boundary cases $\ell = 1$ and $\ell = n$ are known to be polynomially solvable). There is also lack of positive results for this problem, in particular approximation algorithms with some guaranteed worst case ratio.

6 Conclusions

In this chapter we have described a class of robust discrete optimization problems with uncertain costs. We have discussed two most popular methods of modeling the uncertainty, namely the discrete and interval uncertainty representations. A lot of results and new concepts in this area have appeared in literature since 1997, when the book [62] has been published. In particular, the complexity of basic minmax (regret) problems, described in [62], has been explored more deeply. Unfortunately, with a few exceptions, all these problems are NP-hard and solving them is often a challenging task. There is still a number of important open problems in this area. One of them is to decide whether the approximation ratio of 2 is the best possible for the minmax regret problems with interval data. There is also lack of positive results for the minmax regret problems under the discrete uncertainty representation. The minmax approach has been recently generalized by using the OWA and WOWA criteria, which allow us to take both the attitude of decision makers towards a risk and scenario probabilities into account. Some special cases, for instance the problems with the Hurwicz criterion, still require more deep investigation.

In this chapter we have also reviewed some recent extensions of the minmax approach, in which computing an optimal solution is a two-stage process. In the robust incremental recourse approach an initial solution can be modified to some extent after observing a true scenario. In the two-stage approach a solution is built in two stages. A part of this solution is constructed in the first stage and the rest is constructed after a true scenario reveals. Most results, known on both approaches, are negative and there are many open questions related to their complexity and approximability.

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