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Approximation by Matrices with Prescribed Spectrum

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Outline

1. Matrix nearness problems
2. Examples of matrix approximation problems
3. Approximation of normal matrices by normal matrices with the spectrum in a strip
4. Khalil and Maher's generalization of the Halmos problem
5. Algorithm for Khalil and Maher's approximant
6. Numerical tests
7. Future generalizations – numerical range approximation

Matrix nearness problem

$$d(A) = \min\left\{ \|E\| : A + E \in S \text{ has property P } \right\}, \quad A \in S$$

P: e.g. symmetry, positive semi-definiteness, unitarity

TASKS:

- determine formula for *d(A)*
- determine $X=A+E_{min}$
- is *X* unique?
- develop algorithms for computing / estimating *d(A)* and *X*

Higham (1989)

Notation

- unitarily invariant norm

The norm is unitarily invariant iff $\| A \| = \| UAV \|$ for all A

and for all unitary matrices U, V

$$\|X\|_p = \left(\sum_{j=1}^{\infty} \sigma_j(A) \right)^{1/p}$$

$p = 2$ – Frobenius norm

$p = \infty$ – spectral norm

- normal matrices: $XX^H = X^H X$
- spectrum of the matrix X: the set of its eigenvalues - $\sigma(X)$

Notation – cont.

$\mathbb{X}(\mathbb{S})$ – the set of all complex matrices of order n with spectrum in $\mathbb{S}$

$\mathbb{X}\mathbb{N}(\mathbb{S})$ – the set of all normal matrices of order n with spectrum in $\mathbb{S}$

Cartesian decomposition

$$A = B + iC$$

A - arbitrary

$$B = \mathrm{Re}(A) = \frac{A + A^H}{2} \qquad C = \mathrm{Im}(A) = \frac{A - A^H}{2i}$$

$$B^{(+)} = Q diag(a_j^+) Q^H$$

$\lambda_j = a_j + ib_j$

Q – unitary

$a_j^+ = \max\{0, a_j\}$

Examples of matrix approximation problems

Normal approximation

$$\left\|A - B^{(+)}\right\| \le \left\|A - P\right\|$$

$P>0$

$A=B+iC$, normal

Halmos (1972), operator norm

Bhatia, Kittaneh (1992), unitarily invariant norms

$$\left\|A - F(A)\right\| \le \left\|A - N\right\|$$

$N \in \mathbb{X}\mathbb{N}(\mathbb{S})$

A - normal

$$A = Q diag(\lambda_j) Q^H$$

$$F(A) = Q diag(F(\lambda_j)) Q^H$$

$$\left|z - F(z)\right| \le \left|z - s\right|, s \in \mathbb{S}$$

Halmos(1974), operator norm

Bouldin (1980), c_p norms, $p \ge 2$

Bhatia (1987), unitarily invariant norms

Examples of matrix approximation problems

Spectral approximation

$$\left\| A - P^{(hs)} \right\| \leq \left\| A - P \right\|$$

$$A = B + iC$$
$$P \geq 0$$

$$P^{(hs)} = B + \left(\eta(A)^2 I - C^2 \right)^{1/2}$$

Halmos approximant

$$\eta(A) = \inf \left\{ r : B + \left(r^2 I - C^2 \right)^{1/2} \text{ and } r^2 I - C^2 \text{ Hermitian psd} \right\}$$

Halmos (1972), operator norm

Examples of matrix approximation problems

$$\| A-(B^{(+)}+iC) \| \leq \|A-X\|$$

accretive approximant

Re(X)>0

$$A=B+iC$$

Halmos (1972), operator norm
Bhatia, Kittaneh (1992),unitarily invariant norms

Examples of $\mathbb{S}$

$$\mathbb{E}_j = [a,b] \text{ or } \mathbb{E}_j = [0,\infty)$$

$$\mathbb{S} = \mathbb{E}_1 \times \mathbb{E}_2$$

$$\mathbb{S} = [0,\infty) \times \{0\}$$

$$\mathbb{S} = \mathbb{E}_a = [0,\infty) \times [0,a]$$ *'the strip'*

$$\mathbb{S} = \mathbb{E}_1 \times \mathbb{E}_2 = [0,\infty) \times [0,\infty)$$ *quarter*

$$\mathbb{S} = \mathbb{E}_1 \times \mathbb{E}_2 = [a,b] \times [c,d]$$ *rectangular*

$$F : R \to [a,b]$$

$$F(x) = \begin{cases} a & x \le a, \\ x & x \in (a,b), \\ b & x \ge b. \end{cases}$$

Problems that are considered

Spectral approximation

$$\min_{X \in \mathbb{X}(\mathbb{E}_a)} \|A - X\|$$

A – arbitrary

$\|\cdot\|$ - spectral norm

Normal approximation

$$\min_{X \in \mathbb{XN}(\mathbb{E}_a)} \|A - X\|$$

A - normal

$\|\cdot\|$ - unitarily invariant norm

Normal approximation: case $\mathbb{S}=\mathbb{E}_a$

$$\mathbb{E}_a = \{x+iy:\ 0 \le x,\ 0 \le y \le a\}$$

$$\|A - F(A)\| \le \|A - N\|$$

$$F(A) = Q\,diag(F(\lambda_j))Q^H$$
$$N \in \mathbb{X}\mathbb{N}(\mathbb{E}_a)$$

$$F(z) = (\mathrm{Re}\, z)^+ + (\mathrm{Im}\, z)^+ i - \left[(\mathrm{Im}\, z)^+ - a\right]^+ i$$

$$X^{(nl)} = F(A) = B^{(+)} + C^{(+)} i - (C^{(+)} - aI)^{(+)} i$$

Khalil, Maher (2000)
– incorrect retraction

Spectral approximation: case $\mathbb{S}=\mathbb{E}_a$

$$\mathbb{E}_a=\{x+iy:\ 0\le x,\ 0\le y\le a\}$$

$$\min_{X\in\mathbb{X}(\mathbb{E}_a)} \|A-X\|$$

A – arbitrary

$\|\cdot\|$ - spectral norm

Khalil, Maher (2000)

generalization of the Halmos problem

$$\min_{Y\in\mathbb{Y}(\mathbb{E}_a)} \|A-Y\|$$

$Y = Y_1 + iY_2$

Y_1 - Hermitian psd

Y_2 - Hermitian, $\sigma(Y_2)\in[0,a]$

PROBLEMS...

- for X from $\mathbb{X}(\mathbb{E}_a)$ it is not guaranteed that $\sigma(\mathrm{Im}(X)) \in [0, a]$:

Example:

$$A = B + iC = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} + i \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\sigma(A) \in [0, \infty) \times [0, 5.0076i]$$

$$\lambda_{\mathrm{max}}(C) = 5.4142 \notin [0, a] \text{ for } a = 5.0076$$

We do not know the spectrum of *Re(A)* and *Im(A)* even if we know the spectrum of *A.*

PROBLEMS...

- for X from $\mathbb{X}(\mathbb{E}_a)$ it is not guaranteed that $Re(X)$ is psd

Example:

$$A = \begin{bmatrix} 10 & 1-i & 1-i \\ -12+30i & 70i & -10+26i \\ 12-31i & 12-32i & 50 \end{bmatrix}$$

Eigenvalues: 9.8526 + 0.5593i, 0.4906 +57.4117i, 49.6568 +12.0290i

$$B = \frac{A + A^H}{2}$$ is not psd

Eigenvalues of B: -22.3348, 14.0459, 68.2889

Spectral approximation: case $\mathbb{S}=\mathbb{E}_a$

$\mathbb{E}_a=\{x+iy:\ 0\le x,\ 0\le y\le a\}$

$$\delta(A)=\min_{Y\in\mathbb{Y}(\mathbb{E}_a)}|||A-Y|||$$

$Y=Y_1+iY_2$

Y_1 - Hermitian psd

Y_2 - Hermitian, $\sigma(Y_2)\in[0,a]$

$$\mathbb{H}(A)=\left\{\ r: r^2I-(C-\tilde{C})^2\ge 0,\ \ B+\left(r^2I-(C-\tilde{C})^2\right)^{1/2}\ge 0,\ \text{for some } \tilde{C}\in\mathbb{X}([0,a])\right\}$$

$$\mathbb{D}(A)=\left\{|||A-Y|||: Y\in\ \mathbb{Y}(\mathbb{E}_a)\right\}$$

$$\delta(A)=\inf\mathbb{D}(A)=\inf\ \mathbb{H}(A)$$

Khalil, Maher (2000)

their proof is valid on the assumption that Y_1,Y_2 *- Hermitian psd*, $\sigma(Y_2)\in[0,a]$

Spectral approximation: case $\mathbb{S}=\mathbb{E}_a$

$\mathbb{E}_a = \{x+iy:\ 0 \le x,\ 0 \le y \le a\}$

$$\left\| A - X^{(km)} \right\| \le \left\| A - X \right\|$$

$$X = X_1 + iX_2$$

X_1 - Hermitian psd

X_2 - Hermitian, $\sigma(X_2) \in [0, a]$

$$X^{(km)} = B + \left[\delta(A)^2 I - \left(C - \hat{C}\right)^2 \right]^{1/2} + i\hat{C}$$

$X^{(km)} \in \mathbb{X}(\mathbb{E}_a)$

$\hat{C} \in \mathbb{X}([0, a])$, Hermitian

Khalil, Maher (2000)

their proof is valid on the assumption that X_1, X_2 *- Hermitian psd,* $\sigma(X_2) \in [0, a]$

$\hat{C}$ *was not determined directly*

PROBLEMS...

- $\left\| A - X^{(km)} \right\| \leq \left\| A - X \right\|$ does not hold for all $X \in \mathbb{X}(\mathbb{E}_a)$

Example:

$$X = A = B + iC = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} + i \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$a = \max \mathrm{Im}\left(\lambda_j (\mathrm{A})\right) = 5.0076$$

$$\left\| A - X \right\| < \left\| A - X^{(km)} \right\| = 4.0666 \times 10^{-01}$$

Finding $X^{(km)}$

- computing $\hat{C} = F(C)$
- computing $\tilde{P}^{(hs)}$ for $A - i\hat{C}$ ⟶ Higham's algorithm
- computing $X^{(km)} = \tilde{P}^{(hs)} + i\hat{C}$

$\hat{C}$ - best approx. of C by Hermitian psd with spectrum in $[0,a]$

Algorithm for spectral approximation: case $\mathbb{S}=\mathbb{E}_a$

modified algorithm of Higham (1988)

Input: $A \in \mathbb{C}^{n\times n}$, $f < 1$ – relative error tolerance, *tol* – absolute error tolerance

Output: $\alpha, \beta \geq 0$ such that $\alpha \leq \delta(A) \leq \beta \leq \alpha + 2\max\{f\alpha, tol\}$

$X \in \mathbb{E}_a$ such that $|||A - X||| = \beta$

Algorithm for spectral approximation: case $\mathbb{S}=\mathbb{E}_a$

$B := \left(A + A^H\right)/2; \quad C = \left(A - A^H\right)/(2i);$

$C := U\Lambda U^H;$

$\hat{\Lambda} := F(\Lambda); \quad \Lambda := \Lambda - \hat{\Lambda}; \quad \Lambda := \Lambda^2;$

$B := U^H B U;$

form $[\alpha, \beta]$;

if $\mathrm{B} + \left(\alpha^2 I - \Lambda\right)^{1/2}$ is psd then $\beta := \alpha$ and goto

while $(\beta - \alpha)/2 > \max\{f\alpha, tol\}$ do

$r := (\alpha + \beta)/2;$

$G := B + \left(r^2 I - \Lambda\right)^{1/2};$

if G is psd then $\beta := r$ else $\alpha :=$ r;

$$X = U\left(B + \left(\beta^2 I - \Lambda\right)^{1/2} + i\hat{\Lambda}\right)U^H$$

Example: $A \in \mathbb{C}^{10\times 10}$ – random matrix

$$f = 10^{-14}$$

$$a \to 0$$

for *a=0:*

$$\min_{X \in \mathbb{X}(\mathbb{E}_a)} \left\| A - X \right\| = \left\| A - P^{(hs)} \right\|$$

$$P^{(hs)} = B + \left(\eta(A)^2 I - C^2 \right)^{1/2}$$

$$\eta(A) = \left\| A - P^{(hs)} \right\| = 1.0440537$$

a	$\left\| A - X^{(km)} \right\|$
10^{-1}	1.0198997
10^{-2}	1.0409322
10^{-3}	1.0437334
10^{-4}	1.0440216
10^{-5}	1.0440505
10^{-6}	1.0440534
10^{-8}	1.0440537

Example $A \in \mathbb{C}^{5\times5}$ – random normal matrix, random eig.

$f = 10^{-14}$
$a = 3$

$$X^{(km)} = B + \left[\delta(A)^2 I - \left(C - \hat{C}\right)^2\right]^{1/2} + i\hat{C}$$

$$X^{(nl)} = Q\,diag\left(a_j^+ + b_j^+ i - (b_j^+ - a)^+ i\right) Q^H$$

$$\left\|A - X^{(nl)}\right\| = \left\|A - X^{(km)}\right\|$$

$$X^{(nl)} \neq X^{(km)}$$

$\lambda_j(X^{(km)})$	$\lambda_j(X^{(nl)})$
$4.3432\times10^{+00} + 1.8386\times10^{-16} i$	$7.3647\times10^{+00} + 4.4483\times10^{-16} i$
$1.0877\times10^{+01} - 5.1693\times10^{-16} i$	$-5.2480\times10^{-16} - 2.0208\times10^{-16} i$
$8.6561\times10^{+00} + 8.0708\times10^{-01} i$	$5.4284\times10^{-01} + 8.0708\times10^{-01} i$
$9.9185\times10^{+00} + 6.2966\times10^{-16} i$	$2.7973\times10^{+00} - 3.4515\times10^{-16} i$
$6.0693\times10^{+00} + 2.2479\times10^{-15} i$	$6.0693\times10^{+00} - 5.9814\times10^{-16} i$

Remark:

- algorithm can be applied for

$$\min_{X \in \mathbb{Y}(\mathbb{S})} \left\| A - X \right\|$$

$$\mathbb{S} = \mathbb{E}_1 \times \mathbb{E}_2 = [0, \infty) \times [0, \infty)$$

$$X = X_1 + iX_2$$

X_1, X_2 - Hermitian psd

Future generalization – numerical range approximation

$\mathbb{W}(\mathbb{S})$ – the set of all complex matrices of order *n* with their numerical range in $\mathbb{S}$

$\mathbb{WN}(\mathbb{S})$ – the set of all normal matrices of order *n* with their numerical range in $\mathbb{S}$

$$\inf_{X \in \mathbb{W}(\mathbb{S})} \|A - X\|$$

A - arbitrary

$$\inf_{X \in \mathbb{NW}(\mathbb{S})} \|A - X\|$$

A - normal

Summary

1. 2 problems of approximation of matrices were considered:
   - Approximation of normal matrices by normal matrices with the spectrum in a strip
   - Approximation of matrices by matrices $X = X_1 + iX_2,$ $X_1 \geq 0,\ \sigma(X_2) \in [0, a]$
2. Results of Khalil and Maher were corrected and completed
3. Algorithm for computing Khalil and Maher's approximant was presented

Thank you for your attention